

Linear Wavelet-Based Estimation for Derivative of a Density under Random Censorship

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Abstract. In this paper we consider estimation of the derivative of a density based on wavelets methods using randomly right censored data. We extend the results regarding the asymptotic convergence rates due to Prakasa Rao (1996) and Chaubey *et al.* (2008) under random censorship model. Our treatment is facilitated by results of Stute (1995) and Li (2003) that enable us in demonstrating that the same convergence rates are achieved as in Prakasa Rao (1996) and Chaubey *et al.* (2008).

Key words and phrases: Besove space, censored data, nonparametric estimation of derivative of a density, wavelets.

1 Introduction

Survival data appear in medical research, industrial life-testing, and other studies, where the main focus is on the time to occurrence of a particular event. Examples of such an event are the failure of an electric equipment, the recurrence of a particular disease, etc. These survival data are typically observed in an incomplete way, due to the presence of a number of events which potentially censor the event of interest. Withdrawals from a clinical trial, deaths unrelated to the disease under study, individuals still alive at the end of a follow-up period, and so on, are examples of censoring issues. In the right censored data, cases report incomplete observations of the survival time in the sense that in such cases we only know that the time-to-event is greater than an observed (censoring) time. In this paper, we consider the random censorship model from the right, where two sequences of random variables, X_1, X_2, \dots and Y_1, Y_2, \dots are considered. We regard X_1, X_2, \dots as survival times (or failure times), having a common unknown distribution function $F(\cdot)$ and density function $f(\cdot)$. Let the survival times X_i be censored from the right by the censoring times Y_i , with a common distribution function $G(\cdot)$, then we observe

$$Z_i = \min(Y_i, X_i) := Y_i \wedge X_i \quad \text{and} \quad \delta_i = I(X_i \leq Y_i),$$

where $I(\cdot)$ denotes the indicator function. In this random censorship model, we assume that the survival times $\{X_i\}$ are independent of the censoring times $\{Y_i\}$. Following the convention in the survival analysis literature, we assume that both X_i and Y_i are nonnegative random variables. In contrast to statistics for complete data where the whole sample X_1, X_2, \dots is available, the estimators of relevant quantities for censored data are based on the pairs of observations $(Z_1, \delta_1), (Z_2, \delta_2), \dots, (Z_n, \delta_n)$. For example, the distribution function F may be estimated by using the Kaplan-Meier estimator

$$\hat{F}_n(x) = 1 - \prod_{i=1}^n \left[1 - \frac{\delta_{(i)}}{n - i + 1} \right]^{I(Z_{(i)} \leq x)}, \quad (1.1)$$

where $Z_{(1)} \leq Z_{(2)} \leq \dots \leq Z_{(n)}$ denote the order statistics of Z_1, Z_2, \dots, Z_n , and is the concomitant of $Z_{(i)}$, i.e., $\delta_{(m)} = \delta_k$ if $Z_{(m)} = Z_k$. The Kaplan-Meier estimator of the censoring distribution may similarly

be given by

$$\hat{G}_n(x) = 1 - \prod_{i=1}^n \left[1 - \frac{1 - \delta_{(i)}}{n - i + 1} \right]^{I(Z_{(i)} \leq x)}. \tag{1.2}$$

Note that $\delta_k/n(1 - \hat{G}(Z_m^-))$ is the jump of the Kaplan-Meier estimator \hat{F}_n at Z_m . Here our interest is in estimating $f^{(d)}$, the d^{th} derivative of f based on $(Z_i, \delta_i), i = 1, 2, \dots, n$.

There is an extensive literature on the right censorship model with independent failure and censoring times. density estimation was studied by Hall *et al.* (1999), Antoniadis *et al.*(1999), Cai (1999) and Li (2003). Nonparametric regression function estimation in this context is discussed by Dabrowska (1995), Heuchenne *et al.*(2007) and Lopez and Patilea (2009). The objective of this paper is to propose wavelet based method for estimating $d^{\text{th}}, d \geq 0$ derivative of a density that belongs to a Besov space using randomly right censored data and investigate the asymptotic convergence rate of the resulting estimator. We show that the proposed estimator attains the same optimal rates of convergence as obtained in Prakasa Rao (1996) and Chaubey *et al.* (2008).

The rest of the paper is organized as follows. In section 2 we describe preliminaries of Besov spaces and wavelet transform and provide the linear wavelet estimators. The main results are described in Section 3 and Section 4 is devoted to the proofs.

2 Preliminaries

We recall that in the random censorship model we observe $Z_m = \min(Y_m, X_m)$, and $\delta_m = I(X_m \leq Y_m), m = 1, 2, \dots, n$. Let $T < \tau_H$ be a fixed constant, where $\tau_H = \inf\{x : H(x) = 1\} \leq \infty$ is the least upper bound for the support of H , the distribution function of Z_1 and $f_1(x) = f(x)I(x \leq T)$. Here we estimate $f_1(x)$, for $x \in (-\infty, T)$, that in turn provides the estimate of $f(x)$ over the interval $x \in (-\infty, T)$. To motivate the estimator, we write a formal expansion for any function $f_1 \in L_2(R)$ (see Daubechies (1992)):

$$f_1 = \sum_{k \in Z} \alpha_{j_0, k} \phi_{j_0, k} + \sum_{j \geq j_0} \sum_{k \in Z} \delta_{j, k} \psi_{j, k} = P_{j_0} f_1 + \sum_{j \geq j_0} D_j f_1 \tag{2.1}$$

where the functions

$$\phi_{j_0, k}(x) = 2^{j_0/2} \phi(2^{j_0} x - k) \tag{2.2}$$

and

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^jx - k) \tag{2.3}$$

constitute an (inhomogeneous) orthonormal basis of $L_2(R)$. Here $\phi(x)$ and $\psi(x)$ are the scale function and the orthogonal wavelet, respectively. Wavelet coefficients in (2.1) are given by the integrals

$$\alpha_{j_0,k} = \int f(x)I(x \leq T)\phi_{j_0,k}(x)dx, \quad \delta_{j,k} = \int f(x)I(x \leq T)\psi_{j,k}(x)dx. \tag{2.4}$$

We suppose that both ϕ and $\psi \in \mathbf{C}^r$, (space of functions with r continuous derivatives), r being a positive integer, and have compact supports included in $[-\beta, \beta]$, for some $\beta > 0$. Note that, by Corollary 5.5.2 in Daubechies (1988), ψ is orthogonal to polynomials of degree $\leq r$, i.e.

$$\int \psi(x)x^l dx = 0, \forall l = 0, 1, \dots, r. \tag{2.5}$$

We suppose that f belongs to the Besov class (see Meyer (1990), §VI.10), $F_{s,p,q} = \{f \in B_{p,q}^s, \|f\|_{B_{p,q}^s} \leq M\}$ for some $0 \leq s \leq r + 1, p \geq 1$ and $q \geq 1$, where

$$\|f\|_{B_{p,q}^s} = \|P_{j_0}f\|_p + \left(\sum_{j \geq j_0} (\|D_j f\|_p 2^{js})^q\right)^{1/q}$$

We may also say $f \in B_{p,q}^s$ if and only if

$$\|\alpha_{j_0,\cdot}\|_{l_p} < \infty, \quad \text{and} \quad \left(\sum_{j \geq j_0} (\|\delta_{j,\cdot}\|_{l_p} 2^{j(s+1/2-1/p)})^q\right)^{1/q} < \infty \tag{2.6}$$

where $\|\gamma_{j,\cdot}\|_{l_p} = (\sum_{k \in Z} \gamma_{j,k}^p)^{1/p}$. We consider Besov spaces essentially because of their executional expressive power [see Triebel (1992) and the discussion in Donoho *et al.* (1996)].

A wavelet based density estimator may be motivated from the expansion in Eq.(2.1) (see Li (2003)) as given by

$$\hat{f}_1 = \sum_{k \in K_{j_0}} \hat{\alpha}_{j_0,k} \phi_{j_0,k}, \tag{2.7}$$

with

$$\hat{\alpha}_{j_0,k} = \int \phi_{j_0,k}(x)I(x \leq T)d\hat{F}_n x = \frac{1}{n} \sum_{i=1}^n \frac{\delta_i I(Z_i \leq T)\phi_{j_0,k}(Z_i)}{1 - \hat{G}(Z_i^-)}, \tag{2.8}$$

where K_{j_0} is the set of k such that $\text{supp}(f) \cap \text{supp}(\phi_{j_0,k}) \neq \emptyset$. The fact that ϕ has a compact support implies that K_{j_0} is finite and $\text{card}(K_{j_0}) = O(2^{j_0})$. Wavelet density estimators aroused much interest in the recent literature, see Donoho *et al.* (1996). In the case of independent samples, the properties of the linear estimator (2.11) have been studied for a variety of error measures and density classes [see Kerkyacharian and Picard (1992), Leblanc (1996) and Tribouley (1995)]. In the setup considered by Prakasa Rao (1996), we assume that ϕ is a scaling function generating an r -regular multiresolution analysis and $f^{(d)} \in L_2(\mathbb{R})$, for some $r \geq (d + 1)$. Furthermore, we assume that there exists $C_m \geq 0$ and $\beta_m \geq 0$ such that

$$|f^{(m)}(x)| \leq C_m(1 + |x|)^{-\beta_m}, 0 \leq m \leq r. \tag{2.9}$$

Prakasa Rao (1996) showed that the projection of $f_1^{(d)}$ on V_{j_0} is

$$f_{1j_0}^{(d)}(x) = \sum_{k \in K_{j_0}} a_{j_0,k} \phi_{j_0,k}(x), \tag{2.10}$$

where

$$a_{j_0,k} = (-1)^d \int \phi_{j_0,k}^{(d)}(x) f_1(x) dx,$$

hence, an estimator of $f_1^{(d)}(x)$ may be proposed by replacing $a_{j_0,k}$ in (2.10) by its estimator, that is,

$$\hat{f}_1^{(d)}(x) = \sum_{k \in K_{j_0}} \hat{a}_{j_0,k} \phi_{j_0,k}(x), \tag{2.11}$$

where

$$\hat{a}_{j_0,k} = \frac{(-1)^d}{n} \sum_{i=1}^n \frac{\delta_i I(Z_i \leq T) \phi_{j_0,k}^{(d)}(Z_i)}{1 - \hat{G}(Z_i^-)}. \tag{2.12}$$

The estimator in Eq. (2.11) will be used as an estimator for $f_1^{(d)}(x)$.

3 Main Results

In this section, we discuss asymptotic properties of our proposed estimator. Below, we extend the results of Prakasa Rao (1996) given in the following theorems for the expected loss $\mathbf{E} \|\hat{f}_{1j_0}^{(d)} - f_1^{(d)}\|_2^2$.

Theorem 3.1. Assume that wavelet ϕ is $(r + d)$ -regular and has d bounded derivatives. Let $\hat{f}_1^{(d)} \in F_{p,q}^s$ be the wavelet based estimator given in (2.11) with $1/p < s < r + d$, $s > d$ and $q \in [1, \infty]$. Then for $p \in [2, \infty]$, there exists a constant C such that for all $M, L \in (0, \infty)$:

$$\mathbf{E} \|\hat{f}_{1j_0}^{(d)} - f_1^{(d)}\|_2^2 \leq Cn^{-2(s-d)/(1+2s)}, \quad (3.1)$$

where $2^{j_0} = n^{\frac{1}{1+2s}}$.

Theorem 3.2. Assume that the wavelet ϕ is $(r + d)$ -regular and has d bounded derivatives, such that $d > 1/p - 1/2$. Let $\hat{f}_1^{(d)} \in F_{p,q}^s$ be the wavelet based estimator (2.11) with $1/p < s < r + d$, $s > d$ and $q \in [1, \infty]$. Then for $1 < p \leq 2$, there exists a constant C such that for all $M, L \in (0, \infty)$:

$$\mathbf{E} \|\hat{f}_{1j_0}^{(d)} - f_1^{(d)}\|_2^2 \leq Cn^{-2(s-d)/(1+2s)}$$

where $2^{j_0} = n^{\frac{1}{1+2s}}$.

4 Proofs

The method of proof for the above theorems follows along the lines of Li (2003). The key part in this proof is to approximate the empirical coefficients \hat{a}_{j_0k} with an average of *i.i.d.* random variables with a sufficiently small rate similar to the construction in Stute (1995) for approximating the Kaplan-Meier integrals. The following lemmas are acquired from Li (2007) that are used in the proofs. Lemma 4.1 is the same as Lemma 4.1 of Li (2007), except for a small notational change and Lemma 4.2 is exactly the same as Lemma 4.2 of Li (2007).

Lemma 4.1. Let \hat{a}_{j_0k} be defined as in equations (2.12). Also, let

$$\varphi_{j_0k}^{(s)}(x) = \phi_{j_0k}^{(s)}(x)I(x \leq T) \quad j \in K_{j_0}, \quad (4.1)$$

$$\bar{a}_{j_0k} = \frac{(-1)^d}{n} \sum_{m=1}^n \frac{\delta_m \varphi_{j_0k}^{(d)}(Z_m)}{1 - \hat{G}(Z_m)}, \quad j \in K_{j_0}, \quad (4.2)$$

Then the following equations hold:

$$\hat{a}_{j_0k} = \bar{a}_{j_0k} + \bar{W}_{j_0k} + R_{n,j_0k}, \quad E(R_{n,j_0k}^2) = O\left(\frac{1}{n^2}\right) \int \varphi_{j_0k}^{2(d)} dF. \quad (4.3)$$

where

$$W_{j_0k}(Z_m) = U_{j_0k}(Z_m) - V_{j_0k}(Z_m), \quad \bar{W}_{j_0k} = n^{-1} \sum_{m=1}^n W_{j_0k}(Z_m) \quad (4.4)$$

$$U_{j_0k}(Z_m) = \frac{(-1)^d(1 - \delta_{(m)})}{1 - H(Z_m)} \int_{Z_m}^{\tau_H} \varphi_{j_0k}^{(d)}(\omega) F(d\omega), \quad (4.5)$$

$$V_{j_0k}(Z_m) = (-1)^d \int_{-L}^{\tau_H} \int_{-L}^{\tau_H} \frac{\varphi_{j_0k}^{(d)}(\omega) I(\nu < Z_m \wedge \omega)}{(1 - H(\nu))(1 - G(\nu))} G(d\nu) F(d\omega). \quad (4.6)$$

Lemma 4.2. *Let $u \in R^n$, $\|u\|_{l_p} = (\sum_i |u_i|^p)^{1/p}$ and $0 < p_1 \leq p_2 \leq \infty$. Then the following inequalities hold:*

$$\|u\|_{l_{p_2}} \leq \|u\|_{l_{p_1}} \leq n^{\frac{1}{p_1} - \frac{1}{p_2}} \|u\|_{l_{p_2}}.$$

Proof of Theorem 3.1. First, we decompose $\mathbf{E}\|\hat{f}_{1j_0}^{(d)}(x) - f_1^{(d)}(x)\|_2^2$ into a bias term and stochastic term:

$$\mathbf{E}\|\hat{f}_{1j_0}^{(d)}(x) - f_1^{(d)}(x)\|_2^2 \leq 2(\|f_1^{(d)} - f_{1j_0}^{(d)}\|_2^2 + \mathbf{E}\|\hat{f}_{1j_0}^{(d)} - f_{1j_0}^{(d)}\|_2^2) = 2(T_1 + T_2) \quad (4.7)$$

where

$$T_1 = \|f_1^{(d)} - f_{1j_0}^{(d)}\|_2^2 \quad \text{and} \quad T_2 = \mathbf{E}\|\hat{f}_{1j_0}^{(d)} - f_{1j_0}^{(d)}\|_2^2.$$

Next, we obtain upper bounds for T_1 and T_2 . Note that

$$T_1 = \left\| \sum_{j \geq j_0} D_j f_1^{(d)} \right\|_2^2 = \sum_{j \geq j_0} \sum_k \delta_{jk}^2,$$

hence, by using Lemma 4.2 and (2.6) for $p \geq 2$, we have $\|\delta_j\|_2 \leq (C2^j)^{\frac{1}{2} - \frac{1}{p}} \|\delta_j\|_p \leq C2^{-js}$. Thus, we get

$$\sum_k \delta_{jk}^2 \leq C2^{-2js}.$$

Hence we have

$$T_1 \leq C \sum_{j \geq j_0} 2^{-2js} = C2^{-2j_0s} 2^{-2s} (1 - 2^{-2s})^{-1} \leq C2^{-2j_0s},$$

On the basis of orthogonality of wavelets ϕ , we have

$$T_2 = \sum_{k \in K_{j_0}} E(\hat{a}_{j_0k} - a_{j_0k})^2.$$

Now, from Lemma 4.1 and (4.3) we have

$$\begin{aligned} T_2 &\leq 3\left\{ \sum_{k \in K_{j_0}} E(\bar{a}_{j_0k} - a_{j_0k})^2 + \sum_{k \in K_{j_0}} E\bar{W}_{j_0k}^2 + \sum_{k \in K_{j_0}} ER_{n,j_0k}^2 \right\} \\ &= 3(T_{21} + T_{22} + T_{23}). \end{aligned}$$

Noting that

$$\begin{aligned} E(\bar{a}_{j_0k} - a_{j_0k})^2 &= E\{(-1)^d 2^{j_0(d+1/2)} n^{-1} \\ &\quad \sum_{m=1}^n \frac{\delta_m \varphi^{(d)}(2^{j_0} Z_m - k)}{1 - G(Z_m)}\}^2 - n^{-1} a_{j_0k}^2 \\ &= (-1)^{2d} 2^{j_0(2d+1)} n^{-1} \\ &\quad \int \varphi^{2(d)}(y) \frac{f_1((y+k)/2^{j_0})}{1 - G((y+k)/2^{j_0})} dy - n^{-1} a_{j_0k}^2, \end{aligned}$$

Then for $k \in K_{j_0}$ we obtain

$$\begin{aligned} \sum_k E(\bar{a}_{j_0k} - a_{j_0k})^2 &= 2^{j_0(2d+1)} n^{-1} \\ &\quad \times \int \varphi^{2(d)}(y) \sum_k 2^{-j_0} \frac{f_1((y+k)/2^{j_0})}{1 - G((y+k)/2^{j_0})} dy \\ &\quad - n^{-1} \sum_k a_{j_0k}^2 \end{aligned}$$

since $\sum_k 2^{-j_0} f_1((y+k)/2^{j_0})/(1 - G((y+k)/2^{j_0})) \rightarrow \int f_1/(1 - G)$ and $\sum_k a_{j_0k}^2 = O(\int f_1^2) \int \varphi_{j_0k}^{2(d)}$, then $E \sum_k (\bar{a}_{j_0k} - a_{j_0k})^2 = 2^{j_0(2d+1)} n^{-1} \int f_1/(1 - G) \int \varphi^{2(d)} + o(2^{j_0(2d+1)} n^{-1})$. Now, using the relation $2^{j_0} = n^{1/(1+2s)}$ we obtain $T_{21} = O(2^{j_0(2d+1)} n^{-1}) = O(n^{-2(s-d)/(1+2s)})$.

By (4.4),

$$T_{22} \leq n^{-1} \sum_k EW_{j_0k}^2(Z_1) \leq 2n^{-1} \sum_k (EU_{j_0k}^2(Z_1) + EV_{j_0k}^2(Z_1)).$$

in view of (4.5) and (4.6), applying Cauchy-Schwarz inequality and using the compact support of ϕ , we finally can obtain

$$EU_{j_0k}^2(Z_1) \leq \frac{1}{[(1-H(T))][(1-G(T))]} 2^{j_0(2d+1)} \times \int \varphi^{2(d)}(y) f_1^2((y+k)/2^{j_0}) dy.$$

Hence,

$$\begin{aligned} n^{-1} \sum_k EU_{j_0k}^2(Z_1) &= O(2^{j_0(2d+1)} n^{-1}) \\ &\quad \times \int \varphi^{2(d)}(y) \sum_k 2^{-j_0} f_1^2((y+k)/2^{j_0}) dy \\ &= O(2^{j_0(2d+1)} n^{-1}) \\ &= O(n^{-2(s-d)/(1+2s)}). \end{aligned}$$

Similarly, we obtain

$$EV_{j_0k}^2(Z_1) \leq \frac{1}{[(1-H(T))]^2[(1-G(T))]^2} 2^{j_0(2d+1)} \times \int \varphi^{2(d)}(y) f_1^2((y+k)/2^{j_0}) dy.$$

Thus, $n^{-1} \sum_k EV_{j_0k}^2(Z_1) = O(2^{j_0(2d+1)} n^{-1})$. Hence

$$T_{22} = o(2^{j_0(2d+1)} n^{-1}) = O(n^{-2(s-d)/(1+2s)}).$$

By (4.3),

$$T_{23} = O(n^{-2}) \sum_k \int \varphi_{j_0k}^{2(d)} dF = O(2^{j_0(2d+1)} n^{-2}).$$

Hence $T_2 = O(2^{j_0(2d+1)} n^{-1})$

Now, using the bounds obtained for T_1 and T_2 and, choosing j_0 such that $2^{j_0} = n^{\frac{1}{1+2s}}$ in Eq.(4.7) the proof is completed. □

Proof of Theorem 3.2. Observing that

$$\mathbf{E} \|\hat{f}_{1j_0}^{(d)}(x) - f_1^{(d)}(x)\|_2^2 \leq 2(\|f_1^{(d)} - \hat{f}_{1j_0}^{(d)}\|_2^2 + \mathbf{E} \|\hat{f}_{1j_0}^{(d)} - f_{1j_0}^{(d)}\|_2^2) = 2(T_1 + T_2) \tag{4.8}$$

By using Lemma 4.2 and (2.6) for $1 < p \leq 2$, we have

$$\|\delta_j\|_2 \leq \|\delta_j\|_p \leq M2^{-j(s-1/p+1/2)}.$$

Thus, we have $\sum_k \delta_{jk}^2 \leq M^2 2^{-2j(s-1/p+1/2)}$. Since $sp > 1$, we have

$$\begin{aligned} T_1 &\leq \sum_{j \geq j_0} M^2 2^{-2j(s-1/p+1/2)} = M^2 2^{-2j_0(s-1/p+1/2)} 2^{-2(s-1/p+1/2)} \\ &\quad \times (1 - 2^{-2(s-1/p+1/2)})^{-1} \\ &\leq M^2 2^{-2j_0(s-1/p+1/2)}, \end{aligned}$$

Thus, with nothing to ($d > 1/p - 1/2$), We have $T_1 = O(n^{-2(s-d)/(1+2s)})$, And by use the same argument as in Theorem 3.1, we have $T_2 = O(2^{j_0(2d+1)} n^{-1})$.

Now, using the bounds obtained for T_1 and T_2 and, choosing j_0 such that $2^{j_0} = n^{\frac{1}{1+2s}}$ in Eq.(4.8) the proof is completed . \square

References

- Antoniadis, A., Gregoire, G., and Nason, G. (1999), Density and hazard rate estimation for right-censored data by using wavelet methods. *J. Roy. Statist. Soc*, **61**, 63-84.
- Cai, T. (1999), Adaptive wavelet estimation: A block thresholding and oracle inequality approach. *The Annals of Statistics*, **27**, 898-924.
- Chaubey, Y. P., Doosti, H., and Prakasa Rao, B. L. S. (2008), Wavelet based estimation of derivative of a density for negatively associated process. *Journal of Statistical Theory and Practice*, **2**(3) 453-463.
- Dabrowska, D. M. (1995), Nonparametric Regression with Censored Covariates. *Journal of Multivariate Analysis*, **54**, 253-283.
- Daubechies, I. (1992), Ten Lectures on Wavelets. CBMS-NSF regional conferences series in applied mathematics. SIAM, Philadelphia.
- Daubechies, I. (1988), Orthogonal bases of compactly supported wavelets. *Statistica Sinica*, **41**, 909-996.

- Donoho, D. L., Johnstone, I. M., Kerkyacharian, G., and Picard, D. (1996), Density estimation by wavelet thresholding. *The Annals of Statistics*, **2**, 508-539.
- Hall, P., Kerkyacharian, G., and Picard, D. (1999), On the minimax optimality of block thresholded wavelet estimators. *The Annals of Statistics*, **9**, 33-50.
- Heuchenne, C. and Keilegom, I. V. (2007), Location estimation in nonparametric regression with censored data. *Multivariate Analysis*, **98**, 1558-1582.
- Kerkyacharian, G. and Picard, D. (1992), Density estimation in Besov spaces. *Statistics and Probability Letters*, **13**, 15-24.
- Leblanc, F. (1996), Wavelet linear density estimator for a discrete-time stochastic process: L_p -losses. *Statistics and Probability Letters*, **27**, 71-84.
- Li, L. (2003), Non-linear wavelet-based density estimators under random censorship. *Journal of Statistical Planning and Inference*, **117**, 35-58.
- Li, L. (2007), On the minimax optimality of wavelet estimators with censored data. *Journal of Statistical Planning and Inference*, **137**, 1138-1150.
- Lopez, O. and Patilea, V. (2009), Nonparametric lack-of-fit tests for parametric mean-regression models with censored data. *Journal of Multivariate Analysis*, **100**, 210-230.
- Prakasa Rao, B. L. S. (1996), Nonparametric estimation of the derivatives of a density by the method of wavelets. *Bull. Inform. Cyb.*, **28**, 91-100.
- Stute, W. (1995), The central limit theorem under random censorship. *Ann. Statist.*, **23**, 422-439.
- Triebel, H. (1992), *Theory of Function Spaces II*. Berlin: Birkhäuser Verlag.
- Tribouley, K. (1995), Density estimation by cross-validation with wavelet method. *Statistica Neerlandica*, **45**, 41-62.