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Mixed Estimators of Ordered Scale Parameters of Two Gamma Distributions with Arbitrary Known Shape Parameters

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Abstract. When an ordering among parameters is known in advance, the problem of estimating the smallest or the largest parameters arises in various practical problems. Suppose independent random samples of size n_i drawn from two gamma distributions with known arbitrary shape parameter $\nu_i > 0$ and unknown scale parameter $\beta_i > 0, i = 1, 2$. We consider the class of mixed estimators of β_1 and β_2 under the restriction $0 < \beta_1 \leq \beta_2$. It has been shown that a subclass of mixed estimators of β_i, beats the usual estimators $\overline{X}_i/\nu_i, i = 1, 2$, and a class of admissible estimators in the class of mixed estimators are derived under scale-invariant squared error loss function. Also it has been shown that the mixed estimator of (β_1, β_2), $0 < \beta_1 \leq \beta_2$, beats the usual estimator $\left(\overline{X}_1/\nu_1, \overline{X}_2/\nu_2\right)$ simultaneously, and a class of admissible estimators in the class of mixed estimators of (β_1, β_2) are derived. Finally the results are extended to some subclass of exponential family.

Key words and phrases: Admissibility, mixed estimators, ordered parameters, scale-invariant squared error loss function, simultaneous estimation.

1 Introduction

The problem of estimating order restricted parameters has its origins in the study of isotonic regression and has found applications in areas such as bio-assays, reliability and life testing and various agricultural and industrial experiments. For example, suppose it is desired to estimate the average yields, say, β_1 and β_2, under treatments τ_1 and τ_2 respectively, where the treatment τ_2 is using certain fertilizer for the crop, while treatment τ_1 is not using any fertilizer. In this situation, it is reasonable to assume that $\beta_1 \leq \beta_2$.

Estimation of ordered parameters has received attention of several researchers during the past 50 years. Most of the earlier work on this problem deals with methods for finding maximum likelihood estimators (MLEs) when the underlying distributions are normal, gamma, binomial, etc. Barlow et al. (1972) and Roberstson et al. (1988) discuss these results in details. Katz (1963) introduced mixed estimators for simultaneous estimation of two ordered binomial parameters and showed that they are better than the unrestricted MLEs. Kumar and Sharma (1988) consider mixed estimators for two ordered normal means and discuss the minimaxity and inadmissibility of them. Mixed estimators for ordered parameters of two exponential population have been studied by Vijayasree and Singh (1991, 1993), Kaur and Singh (1991), Kumar and Kumar (1993, 1995) and Misra and Singh (1994). Some estimation problem in connection with ordered scale parameters of two (or $k \geq 2$) gamma populations can be found in Vijayasree et al. (1995), Chang and Shinozaki (2002) and Misra et al. (2002). For estimation of ordered parameters of uniform, normal and von mises distributions, see Misra and Dhariyal (1995), Misra and van der Meulen (1997), Misra et al. (2004), Kumar et al. (2005), and Singh et al. (2005). For a classified and extensively reviewed work in this area, see van Eeden (2006).

In estimation of ordered scale parameters of two exponential populations, Kaur and Singh (1991) showed that the unrestricted MLEs of two exponential means are inadmissible and are dominated by their respective restricted MLEs. Vijayasree and Singh (1991, 1993) discussed componentwise and simultaneous estimation of ordered mean of two exponential distributions and considered mixed estimators based on sample mean, and obtained classes of estimators that are minimal complete in the class(es) of mixed estimators. However the work of these authors are in the framework of ordered scale parameters of gamma populations with known shape parameters which are integers. In their inadmissibility and admissibility results they employed to use the relationship between: binomial and negative binomial, binomial and incomplete beta function, and gamma and poisson distribution. This method only can be used for the shape parameters that are integers.

In estimation of the ordered scale parameters of two gamma distribution, Misra et al. (2002) derived smooth estimator that improve upon the best scale equivariant estimators, and compared it to non-smooth improved estimators of Vijayasree et al. (1995). Also Chang and Shinozaki (2002) considered estimation of linear functions of ordered scale parameters, and showed that the inadmissibility results of Kaur and Singh (1991) are special cases of their results.

In this paper we extend the results of Vijayasree and Singh (1991, 1993) in estimation of ordered scale parameters of two gamma populations with arbitrary known shape parameters under scale-invariant squared error loss function and then extend it to some subclass of exponential family of distributions.

Suppose $X_{ij}, j = 1, 2, \cdots, n_i,\ i = 1, 2$ be two independent random samples from gamma distribution with known shape parameter $\nu_i > 0$ and unknown scale parameter $\beta_i > 0, i = 1, 2$, with density

$$f_{X_{ij}}(x) = \frac{1}{\beta_i^{\nu_i}\Gamma(\nu_i)} x^{\nu_i - 1} e^{-x/\beta_i},$$
$$x > 0,\ \nu_i > 0,\ \beta_i > 0,\quad j = 1, \ldots, n_i,\quad i = 1, 2. \tag{1.1}$$

We assume $0 < \beta_1 \leq \beta_2$, and want to estimate β_1 and β_2 componentwise under the scale-invariant squared error loss function

$$L(\beta_i, \delta_i) = \left(\frac{\delta_i}{\beta_i} - 1\right)^2, \quad i = 1, 2, \tag{1.2}$$

and simultaneously estimate $\boldsymbol{\beta} = (\beta_1, \beta_2)$ under the following loss

$$L(\boldsymbol{\beta}, \boldsymbol{\delta}) = \sum_{i=1}^{2} \left(\frac{\delta_i}{\beta_i} - 1\right)^2, \tag{1.3}$$

where $\boldsymbol{\delta} = (\delta_1, \delta_2)$.

In Section 2, a subclass of mixed estimators of β_i that beats the usual estimators $\overline{X}_i/\nu_i,\ \ i = 1, 2$, is obtained. In Section 3 the class of admissible estimators in the class of mixed estimators are derived under the loss (1.2). In Section 4, for simultaneous estimation of $(\beta_1, \beta_2),\ 0 < \beta_1 \leq \beta_2$, a subclass of mixed estimators of (β_1, β_2)

that beats the usual estimator $\left(\overline{X}_1/\nu_1, \overline{X}_2/\nu_2\right)$ is obtained and the class of admissible estimators in the class of mixed estimators are derived under the loss(1.3). Finally, an extension to some subclass of exponential families of distributions is considered in Section 5 and a discussion is given in Section 6.

2 Inadmissibility of Usual Estimators

Let $X_{ij}, j = 1, 2, \ldots, n_i,\ i = 1, 2$ be two independent random samples from Gamma(ν_i, β_i)- distribution, $i = 1, 2$, with density(1.1) where $0 < \beta_1 \leq \beta_2$ and ν_1, ν_2 are known positive real valued shape parameters. Let $m_i = n_i\nu_i$ and $\delta_i = \sum_{j=1}^{n_i} X_{ij}/m_i = \overline{X}_i/\nu_i, i = 1, 2$. Then $\frac{m_i\delta_i}{\beta_i} \sim Gamma(m_i, 1),\ i = 1, 2$, and δ_1 and δ_2 are MLEs of β_1 and β_2, respectively, when β_1 and β_2 are unrestricted. Define the component-wise mixed estimator of β_1 and β_2, as

$$\begin{aligned}\delta_{1\alpha} &= \min\left(\delta_1, \alpha\delta_1 + (1-\alpha)\delta_2\right) \\ &= \alpha\delta_1 + (1-\alpha)\min\left(\delta_1, \delta_2\right), \quad 0 \leq \alpha < 1 \qquad (2.1)\end{aligned}$$

and

$$\begin{aligned}\delta_{2\alpha} &= \max\left(\delta_2, \alpha\delta_2 + (1-\alpha)\delta_1\right) \\ &= \alpha\delta_2 + (1-\alpha)\max\left(\delta_1, \delta_2\right), \quad 0 \leq \alpha < 1 \qquad (2.2)\end{aligned}$$

respectively. When $\alpha = \frac{m_1}{m_1+m_2}$, then $\delta_{1\alpha}$ is the MLE of β_1 and if $\alpha = \frac{m_2}{m_1+m_2}$, then $\delta_{2\alpha}$ is the MLE of β_2 when $0 < \beta_1 \leq \beta_2$. Note that in general m_1 and m_2 are not integers.

In Vijayasree and Singh (1993), $\delta_{2\alpha}$ is defined by $\delta_{2\alpha} = \max(\delta_2, \alpha\delta_1 + (1-\alpha)\delta_2),\ 0 < \alpha \leq 1$. Since we want to construct the admissibility results for simultaneous mixed estimators $\boldsymbol{\delta}_\alpha = (\delta_{1\alpha}, \delta_{2\alpha})$, we use $\delta_{2\alpha}$ in the form (2.2) which is used in Vijayasree and Singh (1991).

In this section, we show that $\delta_{i\alpha}$ dominates the usual estimator δ_i of $\beta_i, i = 1, 2$, for some $0 \leq \alpha < 1$. The risk functions of $\delta_{i\alpha}$ and δ_i with respect to the loss(1.2) are given by

$$R(\delta_{i\alpha}, \boldsymbol{\beta}) = E\left[\left(\frac{\delta_{i\alpha}}{\beta_i} - 1\right)^2\right], \quad i = 1, 2$$

and

$$R(\delta_i, \boldsymbol{\beta}) = E\left[\left(\frac{\delta_i}{\beta_i} - 1\right)^2\right] = \frac{1}{m_i}, \quad i = 1, 2$$

respectively. Let $y_1 = \beta_2/\beta_1, y_2 = \beta_1/\beta_2$ and $z = m_1 y_1/(m_1 y_1 + m_2)$. Since $0 < \beta_1 \leq \beta_2,$ we have $y_1 \geq 1, \quad 0 < y_2 \leq 1$ and $0 < z < 1$.

Theorem 2.1. *(a) For $\alpha \in [0, 1)$ and $m_2 \geq m_1 - 1$,*

$$R(\delta_{1\alpha}, \boldsymbol{\beta}) < R(\delta_1, \boldsymbol{\beta}).$$

(b) Let $\alpha_1 = m_1/(m_1 + m_2 + 1)$, then for $\alpha \in (\alpha_1, 1)$ and $0 < \beta_1 \leq \beta_2$,

$$R(\delta_{1\alpha_1}, \boldsymbol{\beta}) < R(\delta_{1\alpha}, \boldsymbol{\beta}) < R(\delta_1, \boldsymbol{\beta}) \tag{2.3}$$

Proof. (a) Let $T_1 = \frac{m_2 \delta_2}{m_1 y_1 \delta_1 + m_2 \delta_2}$ and $T_2 = \frac{m_1 \delta_1}{\beta_1} + \frac{m_2 \delta_2}{\beta_2}$. Then $\delta_1 = \frac{\beta_1 T_2 (1 - T_1)}{m_1}, \delta_2 = \frac{\beta_2 T_1 T_2}{m_2}$ and T_1 and T_2 are statistically independent with $T_1 \sim Beta(m_2, m_1)$ and $T_2 \sim Gamma(m_1 + m_2, 1)$. If $\Delta_1 = R(\delta_1, \boldsymbol{\beta}) - R(\delta_{1\alpha}, \boldsymbol{\beta})$, then

$$\begin{aligned}
\Delta_1 &= (1-\alpha) \times \\
&\quad E\left[\frac{\delta_1 - \delta_2}{\beta_1}\left\{\frac{(1+\alpha)\delta_1 + (1-\alpha)\delta_2}{\beta_1} - 2\right\} I_{[0,\infty)}(\delta_1 - \delta_2)\right] \\
&= \frac{1-\alpha}{m_1^2 m_2^2} E[\{m_2 - (m_1 y_1 + m_2) T_1\}\{[m_2(1+\alpha) + \\
&\quad (m_1 y_1 (1-\alpha) - m_2(1+\alpha)) T_1)] T_2^2 - 2 m_1 m_2 T_2\} I_{[0,1-z]}(T_1)] \\
&= \frac{(1-\alpha)(m_1 + m_2)}{m_1^2 m_2^2} E\left[g_{1,y_1,\alpha}(T_1) I_{[0,1-z]}(T_1)\right] \qquad (2.4)
\end{aligned}$$

where

$$\begin{aligned}
g_{1,y_1,\alpha}(x) &= [m_2 - (m_1 y_1 + m_2)x] \times \\
&\quad [(m_2(1+\alpha)(1-x) + m_1 y_1 (1-\alpha) x)(m_1 + m_2 + 1) \\
&\quad -2 m_1 m_2] \\
&= A_1(y_1, \alpha) x^2 + B_1(y_1, \alpha) x + C_1(y_1, \alpha) \qquad (2.5)
\end{aligned}$$

and

$$\begin{aligned} A_1(y_1,\alpha) &= (m_1+m_2+1)(m_1y_1+m_2)\times \\ &\quad [\alpha(m_1y_1+m_2)+m_2-m_1y_1], \\ B_1(y_1,\alpha) &= 2m_2[(m_1y_1+m_2)(m_1-\alpha(m_1+m_2+1))- \\ &\quad m_2(m_1+m_2+1)], \\ C_1(y_1,\alpha) &= m_2^2[\alpha(m_1+m_2+1)+m_2+1-m_1] \end{aligned} \tag{2.6}$$

Note that $C_1(y_1,\alpha)>0$ for all $y_1\geq 1$ when $m_2\geq m_1-1$ or $\alpha>\frac{m_1}{m_1+m_2+1}=\alpha_1$. The risk difference Δ_1 in (2.4) and the coefficients in (2.5) are the same as the risk difference and coefficients in (2.1) of Vijayasree and Singh (1993) with replacing n_i by $m_i, i=1,2$, where m_i is not an integer. Hence the rest of the proof is similar to proof of Theorem 2.1 of Vijayasree and Singh (1993), and is omitted. So, $g_{1,y_1,\alpha}(x)>0$ for $x\in[0,1-z]$. and hence $\Delta_1>0$ for all $0<\beta_1\leq\beta_2$, when $\alpha\in(\alpha_1,1)$ or $\alpha\in[0,1)$ and $m_2\geq m_1-1$.

(b) The right inequality in (2.3) follows from the proof of part (a). For a proof of the left inequality in (2.3), from (2.4) we have

$$\frac{\partial R(\partial_{1\alpha},\boldsymbol{\beta})}{\partial\alpha}=-\frac{\partial\Delta_1}{\partial\alpha}=\frac{2(m_1+m_2)}{m_1^2m_2^2}E[h_{1,y_1,\alpha}(T_1)I_{[0,1-z]}(T_1)] \tag{2.7}$$

where

$$\begin{aligned} h_{1,y_1,\alpha}(x) &= [m_2-(m_1y_1+m_2)x]\{-m_1m_2+(m_1+m_2+1)\times \\ &\quad [\alpha m_2(1-x)+(1-\alpha)m_1y_1x]\} \\ &= A_1^*(y_1,\alpha)x^2+B_1^*(y_1,\alpha)x+C_1^*(y_1,\alpha) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} A_1^*(y_1,\alpha) &= (m_1+m_2+1)(m_1y_1+m_2)(\alpha m_2-(1-\alpha)m_1y_1) \\ B_1^*(y_1,\alpha) &= m_2[(m_1y_1+m_2)(m_1-2\alpha(m_1+m_2+1)) \\ &\quad +m_1y_1(m_1+m_2+1)] \\ C_1^*(y_1,\alpha) &= m_2^2[\alpha(m_1+m_2+1)-m_1]. \end{aligned} \tag{2.9}$$

Note that $C_1^*(y_1,\alpha)>0$ for all $y_1\geq 1$ and $\alpha>\alpha_1$. When $A_1(y_1,\alpha)\neq 0$, the quadratic function (2.8) has the roots

$$x_1=1-z,\quad x_2=1-z+\frac{m_1m_2[m_2(y_1-1)+y_1]}{A_1^*(y_1,\alpha)}.$$

If $A_1^*(y_1,\alpha)> 0$ then $x_1 = 1 - z$ is the smaller positive root, and if $A_1^*(y_1,\alpha) < 0$ then $x_1 = 1 - z$ is the only positive root when $\alpha \in (\alpha_1, 1)$. For the case $A_1^*(y_1,\alpha) = 0$, $x_1 = 1 - z$ is the only root. Thus from (2.8), $h_{1,y_1,\alpha}(x) > 0$ for $x \in [0, 1 - z]$ and hence $\frac{\partial R(\delta_{1\alpha,\beta})}{\partial\alpha} > 0$ for all $0 < \beta_1 \le \beta_2$ when $\alpha \in (\alpha_1, 1), i.e.,$ $R(\delta_{1\alpha_1},\boldsymbol{\beta}) < R(\delta_{1\alpha},\boldsymbol{\beta})$ for $\alpha \in (\alpha_1, 1)$, which completes the proof.

Remark 2.1. For driving Δ_1 and $g_{1,y_1,\alpha}(x)$ in (2.4), Vijayasree and Singh (1993) used the method of Kaur and Singh(1991) in which they used the relationships between: binomial and negative binomial, binomial and incomplete beta function, and gamma and poisson distribution. Their method only applied for the shape parameters when they are integers. However we use the relationship between gamma and beta random variables to drive Δ_1 and $g_{1,y_1,\alpha}(x)$, which is cover positive real valued shape parameters and hence their results are special cases of our results.

Remark 2.2. Since $\frac{m_1}{m_1+m_2} > \frac{m_1}{m_1+m_2+1} = \alpha_1$, by Theorem 2.1 (b) the ML estimator of β_1, i.e., $\delta_{1\alpha}$ with $\alpha = m_1/(m_1 + m_2)$, is beaten by the mixed estimator $\delta_{1\alpha_1}$.

Theorem 2.2. *Let* $\alpha_2 = \frac{m_1(2m_1+m_2)}{(m_1+m_2)(m_1+m_2+1)}$ *and* $\alpha_2^* = 1 - \alpha_2$,

(a) If $m_1 = 1$ *and* $\beta_1 = \beta_2$, *then* $R(\delta_{2\alpha_2^*},\boldsymbol{\beta}) = R(\delta_2,\boldsymbol{\beta})$.

(b) If $m_1 \ge 1$ *and* $0 < \beta_1 \le \beta_2$, *then for* $\alpha \in [\alpha_2^*, 1)$, $R(\delta_{2\alpha},\boldsymbol{\beta}) < R(\delta_2,\boldsymbol{\beta})$.

Proof. Let $\Delta_2 = R(\delta_2,\boldsymbol{\beta}) - R(\delta_{2\alpha},\boldsymbol{\beta})$, then similar to the proof of Theorem 2.1 we have

$$\begin{aligned}
\Delta_2 &= (1-\alpha)\times \\
&\quad E\left[\frac{\delta_1-\delta_2}{\beta_2}\left\{2-\frac{(1-\alpha)\delta_1+(1+\alpha)\delta_2}{\beta_2}\right\}I_{[0,\infty)}(\delta_1-\delta_2)\right] \\
&= \frac{(1-\alpha)(m_1+m_2)}{m_1^2m_2^2}E\left[g_{2,y_2,\alpha}(T_1)I_{[0,1-z]}(T_1)\right] \qquad (2.10)
\end{aligned}$$

where

$$\begin{aligned}
g_{2,y_2,\alpha}(x) &= [m_2y_2-(m_1+m_2y_2)x][2m_1m_2-(m_1+m_2+1)\times \\
&\quad (m_2y_2(1-\alpha)(1-x)+m_1(1+\alpha)x)] \\
&= A_2(y_2,\alpha)x^2+B_2(y_2,\alpha)x+C_2(y_2,\alpha), \qquad (2.11)
\end{aligned}$$

and

$$\begin{aligned} A_2(y_2,\alpha) &= (m_1+m_2+1)(m_1+m_2y_2)\times \\ &\quad [2m_1-(1-\alpha)(m_1+m_2y_2)], \\ B_2(y_2,\alpha) &= -2m_2\{(m_1+m_2+1)y_2[m_1-(1-\alpha)(m_1+m_2y_2)] \\ &\quad +m_1(m_1+m_2y_2)\}, \\ C_2(y_2,\alpha) &= m_2^2y_2\{2m_1-(1-\alpha)y_2(m_1+m_2+1)\}. \qquad (2.12) \end{aligned}$$

The risk difference Δ_2 in (2.10) and the coefficients in (2.11) are the same as the risk difference and coefficients in (2.2) of Vijayasree and Singh (1993), with replacing α by $1-\alpha$ and n_i by $m_i, i=1,2$, where m_i is not an integer. Hence the rest of the proof is similar to the proof of Theorem 2.2 of Vijayasree and Singh (1993), and is omitted. Note that in Theorem 2.2 of Vijayasree and Singh (1993) we replace α by $1-\alpha$, hence $1-\alpha \le \alpha_2$, i.e., $\alpha \ge 1-\alpha_2 = \alpha_2^*$.

3 Class of Admissible Mixed Estimators

In this section we find the class of admissible estimators in the class of mixed estimators (2.1) and (2.2) of β_1 and β_2 respectively. The admissible mixed estimators of β_1 are given in the next theorem.

Theorem 3.1. *Let $\rho = \frac{m_2}{m_1+m_2}$ and*

$$\alpha^* = 1-\rho-\frac{\rho}{m_2B_\rho(m_2,m_1)[\rho^{-m_2}(1-\rho)^{-m_1}]-\frac{1-2\rho}{1-\rho}}\ , \qquad (3.1)$$

where $B(.,.)$ is the beta function and $B_\rho(.,.)$ is the incomplete beta function given by $B_\rho(a,b)=\int_0^\rho x^{a-1}(1-x)^{b-1}dx$. Then for $\alpha \in [0,\alpha^*]$ and $m_2(m_2+1) > 2m_1$, the estimator $\delta_{1\alpha}$ is admissible.

Proof. see the Appendix.

Remark 3.1. If $m_1 = m_2 = m$, *i.e.*, $n_1\nu_1 = n_2\nu_2$, then the condition $m_2(m_2+1) > 2m_1$ reduces to $m > 1$ and using the fact $B_{\frac{1}{2}}(m,m) = \frac{1}{2}B(m,m)$, α^* simplifies to $\alpha^* = \frac{1}{2} - \frac{1}{m2^{2m}B(m,m)}$, which is the value given in Theorem 3.1 of Vijayasree and Singh (1993) with replacing m by integer n. So, Theorem 3.1.a of Vijayasree and Singh (1993) is a special case of Theorem 3.1. Note that this is the case when $n_1 = n_2$ and $\nu_1 = \nu_2$.

Remark 3.2. In Theorem 2.1.b and Theorem 3.1, it is shown that the estimator $\delta_{1\alpha}$ in the class of mixed estimators (2.1) is inadmissible when $\alpha \in (\alpha_1, 1)$ and is admissible when $\alpha \in [0, \alpha^*]$ where α^* is given in (3.1) and $\alpha_1 = \frac{m_1}{m_1+m_2+1}$. It can be shown that $\frac{1-\rho}{2} < \alpha^* < \alpha_1 < 1 - \rho$ and hence the admissibility of $\delta_{1\alpha}$ for $\alpha \in (\alpha^*, \alpha_1]$ remained unsolved.

The admissible and inadmissible class of mixed estimators of β_2 are given in the next theorem.

Theorem 3.2. *Let $\alpha^{**} = 1 - \alpha^*$, then*
(a) For $\alpha \in (\alpha^{**}, 1)$, and for all $\beta_1 \le \beta_2$, $R(\delta_{2\alpha^{**}}, \boldsymbol{\beta}) < R(\delta_{2\alpha}, \boldsymbol{\beta})$.
(b) For $\alpha \in [0, \alpha^{**}]$ and $m_2 > 1$, the estimator $\delta_{2\alpha}$ is admissible.

Proof. see the Appendix.

Remark 3.3. Theorem 3.2 shows that the estimator $\delta_{2\alpha}$ in the class of mixed estimators (2.2) is admissible, if and only if $\alpha \in [0, \alpha^{**}]$ and $m_2 > 1$. Since $\alpha^{**} > \rho = \frac{m_2}{m_1+m_2}$, therefore the ML estimator of β_2, i.e., δ_α with $\alpha = \frac{m_2}{m_1+m_2}$, is admissible in the class of mixed estimators of β_2.

Remark 3.4. If $m_1 = m_2 = m$ then $\alpha^{**} = 1 - \alpha^* = 1 - \left(\frac{1}{2} - \frac{1}{m2^{2m}B(m,m)}\right)$, which is the same as $1 - \alpha^*$ given in Theorem 3.1 of Vijayasree and Singh (1993) with replacing m by integer n and α by $1 - \alpha$, respectively. So, Theorem 3.1.b and 3.1.c of Vijayasree and Singh (1993) are special cases of Theorem 3.2. Note that $0 \le \alpha \le \alpha^{**} = 1 - \alpha^*$ is equivalent to $\alpha^* \le 1 - \alpha \le 1$, and hence the results of Vijayasree and Singh (1993) follows with replacing α by $1 - \alpha$.

4 Simultaneous Estimation

In this section we consider simultaneous estimation of (β_1, β_2) when $\beta_1 \le \beta_2$. We compare the mixed estimator $\boldsymbol{\delta}_\alpha = (\delta_{1\alpha}, \delta_{2\alpha})$ and the usual estimator $\boldsymbol{\delta} = (\delta_1, \delta_2) = \left(\overline{X}_1/\nu_1, \overline{X}_2/\nu_2\right)$ under the loss function (1.3).

Theorem 4.1. *(a)* $R(\boldsymbol{\delta}_\alpha, \boldsymbol{\beta}) < R(\boldsymbol{\delta}, \boldsymbol{\beta})$, *for all* $0 < \alpha < 1$, $0 < \beta_1 \leq \beta_2$ *and* $m_2 \geq m_1 - 1$.

(b) $R(\boldsymbol{\delta}_0, \boldsymbol{\beta}) = R(\boldsymbol{\delta}, \boldsymbol{\beta})$, *for* $\beta_1 = \beta_2$.

(c) $R(\boldsymbol{\delta}_0, \boldsymbol{\beta}) < R(\boldsymbol{\delta}, \boldsymbol{\beta})$, *for* $0 < \beta_1 < \beta_2$ *and* $m_2 \geq m_1 - 1$.

Proof. Using (2.4) and (2.10), we have

$$\begin{aligned} \Delta &= \Delta_1 + \Delta_2 = R(\boldsymbol{\delta}, \boldsymbol{\beta}) - R(\boldsymbol{\delta}_\alpha, \boldsymbol{\beta}) \\ &= \frac{(1-\alpha)(m_1+m_2)}{m_1^2 m_2^2 y_1^2} E\left[H_{y_1,\alpha}(T_1) I_{[0,1-z]}(T_1)\right] \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} H_{y_1,\alpha}(x) &= y_1^2 g_{1,y_1,\alpha}(x) + y_1^2 g_{2,y_2,\alpha}(x) \\ &= [m_2 - (m_1 y_1 + m_2)x]\{2m_1 m_2 y_1 (1-y_1) \\ &\quad + (m_1+m_2+1)[\alpha(y_1^2+1)\big(m_2(1-x) - m_1 y_1 x\big) \\ &\quad + (y_1^2-1)(m_2(1-x) + m_1 y_1 x)]\} \\ &= A_3(y_1,\alpha)x^2 + B_3(y_1,\alpha)x + C_3(y_1,\alpha) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} A_3(y_1,\alpha) &= (m_1+m_2+1)(m_1y_1+m_2)[\alpha(m_1y_1+m_2)(y_1^2+1) \\ &\quad +(m_2-m_1y_1)(y_1^2-1)], \\ B_3(y_1,\alpha) &= 2m_2\{(m_1y_1+m_2)[y_1^2(m_1-\alpha(m_1+m_2+1)) - \\ &\quad \alpha(m_1+m_2+1) - m_1y_1] + (m_1+m_2+1)m_2(1-y_1^2)\}, \\ C_3(y_1,\alpha) &= m_2^2[\alpha(m_1+m_2+1)(y_1^2+1) + (m_2-m_1+1)(y_1^2-1) \\ &\quad +2m_1(y_1-1)]. \end{aligned} \tag{4.3}$$

When $\alpha > 0$ and $m_2 \geq m_1 - 1$, $C_3(y_1, \alpha) > 0$ for all $y_1 \geq 1$. When $A_3(y_1, \alpha) \neq 0$, the quadratic form (4.2) has the roots

$$x_1 = 1 - z, \quad x_2 = 1 - z + \frac{2m_1 m_2 y_1 (y_1 - 1)[(m_2+1)y_1 + (m_1+1)]}{A_3(y_1,\alpha)}.$$

Now, if $A_3(y_1, \alpha) < 0$ then $x_1 x_2 = \frac{C_3(y_1,\alpha)}{A_3(y_1,\alpha)} < 0$ and $x_1 = 1 - z$ is the only positive root, and if $A_3(y_1, \alpha) > 0$ then $x_1 = 1 - z$ is the smaller positive root. For the case $A_3(y_1, \alpha) = 0$, $x_1 = 1 - z$ is the only root. Thus from (4.2) $H_{y_1,\alpha}(x) > 0$ for $x \in [0, 1 - z]$, which yields part (a) of the Theorem.

For $\alpha = 0$ and $y_1 = 1$, (4.2) and (4.3) implies that $H_{y_1,\alpha}(x) = 0$, i.e., $R(\boldsymbol{\delta}_0, \boldsymbol{\beta}) = R(\boldsymbol{\delta}, \boldsymbol{\beta})$. For $\alpha = 0$ and $y_1 > 1$, a similar argument as in the proof of part (a) implies that $H_{y_1,0}(x) > 0$ for $x \in [0, 1 - z]$ and $m_2 \geq m_1 - 1$, which completes the proof.

Remark 4.1. Vijayasree and Singh (1991) used the sum of squared error loss

$$L(\boldsymbol{\beta}, \boldsymbol{\delta}) = \sum_{i=1}^{2} (\delta_i - \beta_i)^2$$

which does not have the same behavior and risk difference as the one we used. Also their method only applied for integer shape parameters. However our method covers positive real valued shape parameters under the loss (1.3), which is more appropriate in estimation of scale parameters (β_1, β_2) than the loss they used.

In the next theorem we show that $\boldsymbol{\delta}_\alpha$ is inadmissible for $\alpha \in (1/2, 1)$.

Theorem 4.2. *For* $\alpha \in (\frac{1}{2}, 1)$, $R(\boldsymbol{\delta}_{\frac{1}{2}}, \boldsymbol{\beta}) < R(\boldsymbol{\delta}_\alpha, \boldsymbol{\beta})$ *for all* $0 < \beta_1 \leq \beta_2$ *and* $m_2 \geq m_1 - 1$.

Proof. From (4.1) and (4.2) we have

$$\begin{aligned}
\Delta^* &= R(\boldsymbol{\delta}_\alpha, \boldsymbol{\beta}) - R(\boldsymbol{\delta}_{\frac{1}{2}}, \boldsymbol{\beta}) \\
&= [R(\boldsymbol{\delta}, \boldsymbol{\beta}) - R(\boldsymbol{\delta}_{\frac{1}{2}}, \boldsymbol{\beta})] - [R(\boldsymbol{\delta}, \boldsymbol{\beta}) - R(\boldsymbol{\delta}_\alpha, \boldsymbol{\beta})] \\
&= \frac{(m_1 + m_2)}{m_1^2 m_2^2 y_1^2} E\left[(\frac{1}{2} H_{y_1, \frac{1}{2}}(T_1) - (1 - \alpha) H_{y_1,\alpha}(T_1)) I_{[0,1-z]}(T_1)\right] \\
&= \frac{(\alpha - \frac{1}{2})(m_1 + m_2)}{m_1^2 m_2^2 y_1^2} E\left[H_{y_1, \alpha - \frac{1}{2}}(T_1) I_{[0,1-z]}(T_1)\right]
\end{aligned}$$

So, the quadratic form in Δ^* is the same as Δ in (4.2) with replacing $\alpha > 0$ by $\alpha - 1/2 > 0$. Therefore $H_{y_1, \alpha - \frac{1}{2}}(x) > 0$ for $x \in [0, 1 - z]$, $m_2 \geq m_1 - 1$ and $\alpha > 1/2$, which completes the proof.

Now we find the class of admissible estimators in the class of mixed estimators.

Theorem 4.3. *For $0 \le \alpha \le \frac{1}{2}$ and $2m_1 < m_2(m_2+1)$, the estimators $\boldsymbol{\delta}_\alpha$ are admissible in the class of mixed estimators.*

Proof. From (4.1) and (4.2) we have

$$\begin{aligned}\frac{\partial R(\boldsymbol{\delta}_\alpha, \boldsymbol{\beta})}{\partial \alpha} &= -\frac{\partial \Delta}{\partial \alpha} = -\frac{m_1+m_2}{m_1^2 m_2^2 y_1^2} E[\{[m_2 - (m_1 y_1 + m_2)T_1]^2 \\ &\quad \times (m_1+m_2+1)(1-2\alpha)(y_1^2+1) \\ &\quad -[m_2 - (m_1 y_1 + m_2)T_1][(m_1+m_2+1)(y_1^2-1) \\ &\quad \times (m_2 + (m_1 y_1 - m_2)T_1) \\ &\quad +2m_1 m_2 y_1 (1-y_1)]\} I_{[0,1-z]}(T_1)]. \end{aligned} \tag{4.4}$$

So, $\frac{\partial R(\boldsymbol{\delta}_\alpha, \boldsymbol{\beta})}{\partial \alpha}$ is a strictly increasing function of α, i.e., $R(\boldsymbol{\delta}_\alpha, \boldsymbol{\beta})$ for fixed $\boldsymbol{\beta}$, is a strictly convex function of α. Therefore for fixed $\boldsymbol{\beta}$, $R(\boldsymbol{\delta}_\alpha, \boldsymbol{\beta})$ will be minimized at the point α given by $\frac{\partial R(\boldsymbol{\delta}_\alpha, \boldsymbol{\beta})}{\partial \alpha} = 0$ which gives

$$\begin{aligned}\alpha(y_1) &= \frac{1}{2} - \frac{A(y_1)}{B(y_1)} \\ &= \frac{1}{2} - \frac{y_1^2-1}{2(y_1^2+1)} - \frac{m_1 y_1 (1-y_1)}{(m_1+m_2+1)(1+y_1^2)} \times \gamma(y_1) \end{aligned} \tag{4.5}$$

where

$$\begin{aligned}A(y_1) &= E[\{m_2 - (m_1 y_1 + m_2)T_1\}\{(m_1+m_2+1)(y_1^2-1) \\ &\quad \times (m_2 + (m_1 y_1 - m_2)T_1) + 2m_1 m_2 y_1 (1-y_1)\} I_{[0,1-z]}(T_1)], \\ B(y_1) &= 2E[\{m_2 - (m_1 y_1 + m_2)T_1\}^2 (m_1+m_2+1) \\ &\quad \times (y_1^2+1) I_{[0,1-z]}(T_1)], \end{aligned}$$

$$\gamma(y_1) = \frac{E[\{m_2 - (m_1 y_1 + m_2)T_1\}[m_2 - (m_1+m_2+1)(1+y_1)T_1] I_{[0,1-z]}(T_1)]}{E[\{m_2 - (m_1 y_1 + m_2)T_1\}^2 I_{[0,1-z]}(T_1)]}.$$

Note that $y_1 \to \infty$ if and only if $1 - z \to 0$, so by L'Hospital's rule it can be easily shown that $\lim_{y_1 \to \infty} \gamma(y_1) = \frac{m_1+m_2+1}{m_1} - \frac{(m_2+1)(m_2+2)}{2m_1}$. Therefore for $2m_1 < m_2(m_2+1)$ we have

$$\begin{aligned}\lim_{y_1 \to \infty} \alpha(y_1) &= \frac{m_1}{m_1+m_2+1}\{\frac{m_1+m_2+1}{m_1} - \frac{(m_2+1)(m_2+2)}{2m_1}\} \\ &= \frac{2m_1 - m_2(m_2+1)}{2(m_1+m_2+1)} < 0. \end{aligned}$$

From (4.5) $\alpha(1) = 1/2$ and for all $y_1 > 1$, $\alpha(y_1)$ is continuous in y_1. Thus for $\alpha \in [0, \frac{1}{2}]$ there is a y_1 for which $R(\boldsymbol{\delta}_\alpha, \boldsymbol{\beta})$ is minimum, which implies that for $0 \leq \alpha \leq 1/2$ and $2m_1 < m_2(m_2+1)$, $\boldsymbol{\delta}_\alpha$ is admissible in the class of mixed estimators.

Remark 4.2. If $m_1 = m_2 = m$, then the condition of Theorem 4.2 on m_1 and m_2 always satisfy and condition of Theorem 4.3 on m_1 and m_2 reduces to $m > 1$. Also if the conditions of Theorems 4.2 and 4.3 holds, i.e., $2m_1 < m_2(m_2+1)$ and $m_2 > m_1 - 1$, then $\boldsymbol{\delta}_\alpha$ is admissible in the class of mixed estimators if and only if $\alpha \in [0, \frac{1}{2}]$.

5 Extension to a Subclass of Exponential Family

Let $\mathbf{X}_i = (X_{i1}, X_{i2}, \cdots, X_{in_i})$, $i = 1, 2$ has the joint probability density function

$$f(\mathbf{x}_i, \theta_i) = C(\mathbf{x}_i, n_i)\theta_i^{-\gamma_i} e^{-T_i(\mathbf{x}_i)/\theta_i}, \qquad i = 1, 2, \tag{5.1}$$

where $\mathbf{x}_i = (x_{i1}, \cdots, x_{in_i})$, $C(\mathbf{x}_i, n_i)$ is a function of $\mathbf{x}_i$ and n_i, $\theta_i = \tau_i^r$ for some $r > 0, \gamma_i$ is a function of n_i and $T_i(\mathbf{x}_i)$ is a complete sufficient statistic for θ_i with $Gamma(\gamma_i, \theta_i)$- distribution. For example $Exponential(\beta_i)$ with $\theta_i = \beta_i$, $Gamma(\nu_i, \beta_i)$ with $\theta_i = \beta_i$ and known v_i, $Inverse\ Gaussian(\infty, \lambda_i)$ with $\theta_i = \frac{1}{\lambda_i}$, $Normal(0, \sigma_i^2)$ with $\theta_i = \sigma_i^2$, $Weibull(\eta_i, \beta_i)$ with $\theta_i = \eta_i^{\beta_i}$ and known β_i, $Rayleigh(\beta_i)$ with $\theta_i = \beta_i^2$, $Generalized\ \ Gamma(\alpha_i, \lambda_i, p_i)$ with $\theta_i = \lambda_i^{p_i}$ and known p_i and α_i, $Generalized\ \ laplace(\lambda_i, k_i)$ with $\theta_i = \lambda_i^{k_i}$ and known k_i belong to the family of distributions (5.1). An admissible linear estimator of $\theta_i = \tau_i^r$ in this family under the entropy loss function can be found in Parsian and Nematollahi (1996).

Since $T_i = T_i(\mathbf{X}_i), i = 1, 2$, has a $Gamma(\gamma_i, \theta_i)-$ distribution, therefore we can extend the results of sections 2-4 to the subclass of exponential family (5.1) by replacing $m_i = n_i\nu_i$, β_i and $\sum\limits_{j=1}^{n_i} X_{ij} = m_i\delta_i$ by γ_i, θ_i and $T_i(\mathbf{X}_i)$, respectively.

The results of sections 2-4 can be extended to some other families of distributions which do not necessarily belong to a scale families, such as pareto or beta distributions. Considered the one parameter exponential family

$$f(\mathbf{x}_i, \eta_i) = e^{a_i(\mathbf{x}_i)b(\eta_i) + c(\eta_i) + h(\mathbf{x}_i)}, \quad i = 1, 2. \tag{5.2}$$

Rahman and Gupta (1993) showed that $-2a_i(\mathbf{X}_i)b(\eta_i)$ has a $Gamma\left(\frac{k_i}{2}, 2\right)$-distribution if and only if

$$\frac{2c'(\eta_i)b(\eta_i)}{b'(\eta_i)} = k_i. \tag{5.3}$$

When k_i is an integer, $-2a_i(\mathbf{X}_i)b(\eta_i)$ follow a chi-square distribution with k_i degrees of freedom. They called the one parameter exponential family (5.2) which satisfies (5.3), the family of transformed chi-square distributions. For example, beta, pareto, exponential, log-normal and some other distributions belong to this family of distributions (see Table 1 of Rahman and Gupta,1993).

Now it is easy to show that if condition (5.3) holds then the one parameter exponential family (5.2) is in the form of the scale parameter exponential family (5.1) with $\gamma_i = \frac{k_i}{2}$, $T_i(\mathbf{X}_i) = a_i(\mathbf{X}_i)$ and $\theta_i = -1/b(\eta_i)$ (see Jafari Jozani et al., 2002). Hence with these substitutions, we can extend the results of sections 2-4 to the family of transformed chi-square distributions.

6 Discussion

In previous sections under the order restriction $0 < \beta_1 \leq \beta_2$ of gamma scale parameters, we derive the conditions that the mixed estimators based on usual ML estimators $\delta_i = \sum_{j=1}^{n_i} X_{ij}/m_i, \quad i = 1, 2$, dominates δ_1 and δ_2 respectively, and characterize the admissible estimators in the class of mixed estimators under the loss function (1.2). Also, similar results was obtained for simultaneous estimation of (β_1, β_2) when $\beta_1 \leq \beta_2$ under the loss function (1.3).

Under the scale-invariant squared error loss function (1.2), the best scale-invariant estimators of β_1 and β_2 in gamma-distribution (1.1) is given by $\delta_i^* = \sum_{j=1}^{n_i} X_{ij}/(n_i\nu_i + 1) = \sum_{j=1}^{n_i} X_{ij}/(m_i + 1)$, $i = 1, 2$, respectively. By replacing m_i by $m_i^* = m_i + 1$ and δ_i by δ_i^*, the results of sections 2-4 followed for mixed estimators based on the best scale-invariant estimators δ_1^* and δ_2^* of β_1 and β_2, respectively. Note that in the new framework we have

$$R(\delta_i^*, \boldsymbol{\beta}) = E\left(\frac{\delta_i^*}{\beta_i} - 1\right)^2 = \frac{n_i v_i}{(n_i v_i + 1)^2} = \frac{m_i^* - 1}{m_i^{*2}}$$

7 Appendix

Proof of Theorem 3.1. From (2.7) we have

$$\begin{aligned}\frac{\partial R(\delta_{1\alpha},\boldsymbol{\beta})}{\partial\alpha} &= \frac{2(m_1+m_2)}{m_1^2m_2^2}E[\{m_2-(m_1y_1+m_2)T_1\}\\ &\quad\times\{\alpha[m_2-(m_1y_1+m_2)T_1](m_1+m_2+1)\\ &\quad+(m_1+m_2+1)m_1y_1T_1-m_1m_2\}I_{[0,1-z]}(T_1)].\end{aligned}$$

So, $\frac{\partial R(\delta_{1\alpha},\boldsymbol{\beta})}{\partial\alpha}$ is a strictly increasing function of α, i.e., $R(\delta_{1\alpha},\boldsymbol{\beta})$ for fixed $\boldsymbol{\beta}$ is a strictly convex function of α. Therefore for fixed $\boldsymbol{\beta}$, $R(\delta_{1\alpha},\boldsymbol{\beta})$ will be minimized at the point α given by $\frac{\partial R(\delta_{1\alpha},\boldsymbol{\beta})}{\partial\alpha}=0$ which reduces to

$$\begin{aligned}&\alpha_1(y_1,m_1,m_2)\\ &= \frac{E[\{m_1m_2-m_1(m_1+m_2+1)y_1T_1\}\{m_2-(m_1y_1+m_2)T_1)I_{[0,1-z]}(T_1)]}{(m_1+m_2+1)E[\{m_2-(m_1y_1+m_2)T_1)\}^2I_{[0,1-z]}(T_1)]}\\ &= 1-\frac{(1-z)E[\{T_1-(1-z)\}\{(m_1+m_2+1)T_1-(m_2+1)\}I_{[0,1-z]}(T_1)]}{(m_1+m_2+1)E[\{T_1-(1-z)\}^2I_{[0,1-z]}(T_1)]}.\end{aligned}$$

Note that $y_1\to\infty$ if and only if $1-z\to 0$, so by L'Hospital's rule it can be easily shown that

$$\lim_{y_1\to\infty}\alpha_1(y_1,m_1,m_2)=1-\frac{(m_2+2)(m_2+1)}{2(m_1+m_2+1)}=\frac{2m_1-m_2(m_2+1)}{2(m_1+m_2+1)}.$$

Therefore $\lim_{y_1\to\infty}\alpha_1(y_1,m_1,m_2)<0$ when $m_2(m_2+1)>2m_1$. Using the fact $B_\rho(a+1,b)=\frac{a}{a+b}B_\rho(a,b)-\frac{\rho^a(1-\rho)^b}{a+b}$, we have

$$\begin{aligned}&\alpha_1(1,m_1,m_2)\\ &= 1-\rho-\frac{\rho E[(T_1-\rho)\{(m_1+m_2+1)\rho-(m_2+1)\}I_{[0,\rho]}(T_1)]}{(m_1+m_2+1)E[(T_1-\rho)^2I_{[0,\rho]}(T_1)]}\\ &= 1-\rho-\frac{[\rho(1-\rho)][\rho^{m_2}(1-\rho)^{m_1}]}{(m_1+m_2)\rho B_\rho(m_2,m_1)(1-\rho)-(1-2\rho)\rho^{m_2}(1-\rho)^{m_1}}\\ &= 1-\rho-\frac{\rho}{m_2B_\rho(m_2,m_1)[\rho^{-m_2}(1-\rho)^{-m_1}]-\frac{1-2\rho}{1-\rho}}=\alpha^*.\end{aligned}$$

Since $\alpha_1(y_1,m_1,m_2)$ is continuous in y_1, therefore for each $\alpha\in[0,\alpha^*]$ there is a y_1 for which $R(\delta_{1\alpha},\boldsymbol{\beta})$ is minimum, which implies that for

$\alpha \in [0, \alpha^*]$ and $m_2(m_2+1) > 2m_1$, $\delta_{1\alpha}$ is admissible in the class of mixed estimators.

Proof of Theorem 3.2. (**a**) Let

$$\begin{aligned}\Delta_2^* &= R(\delta_{2\alpha}, \boldsymbol{\beta}) - R(\delta_{2\alpha^{**}}, \boldsymbol{\beta})\\ &= [R(\delta_2, \boldsymbol{\beta}) - R(\delta_{2\alpha^{**}}, \boldsymbol{\beta})] - [R(\delta_2, \boldsymbol{\beta}) - R(\delta_{2\alpha}, \boldsymbol{\beta})].\end{aligned}$$

Then from (2.10) we have

$$\Delta_2^* = \frac{m_1+m_2}{m_1^2 m_2^2} E[G_{y_2,\alpha,\alpha^{**}}(T_1) I_{[0,1-z]}(T_1)] \tag{7.1}$$

where

$$\begin{aligned}G_{y_2,\alpha,\alpha^{**}}(x) &= (1-\alpha^{**}) g_{2,y_2,\alpha^{**}}(x) - (1-\alpha) g_{2,y_2,\alpha}(x)\\ &= (\alpha-\alpha^{**})(m_2 y_2 - (m_1+m_2 y_2)x) \times\\ &\quad \{2m_1 m_2 - (m_1+m_2+1)[(2-\alpha-\alpha^{**})m_2 y_2(1-x)\\ &\quad + m_1(\alpha+\alpha^{**})x]\}\\ &= (\alpha-\alpha^{**})\{A_4(y_2,\alpha,\alpha^{**})x^2 + B_4(y_2,\alpha,\alpha^{**})x\\ &\quad + C_4(y_2,\alpha,\alpha^{**})\}\end{aligned} \tag{7.2}$$

and

$$\begin{aligned}A_4(y_2,\alpha,\alpha^{**}) &= (m_1+m_2+1)(m_1+m_2 y_2) \times\\ &\quad [2m_1 - (2-\alpha-\alpha^{**})(m_1+m_2 y_2)],\\ B_4(y_2,\alpha,\alpha^{**}) &= 2m_2\{(m_1+m_2+1)y_2[(2-\alpha-\alpha^{**}) \times\\ &\quad (m_1+m_2 y_2) - m_1] - m_1(m_1+m_2 y_2)\},\\ C_4(y_2,\alpha_2,\alpha^{**}) &= m_2^2 y_2[2m_1 - (m_1+m_2+1)(2-\alpha-\alpha^{**})y_2].\end{aligned}$$

Note that $C_4(y_2,\alpha,\alpha^{**}) > 0$ for all $\alpha > \alpha^{**} > \rho = \frac{m_2}{m_1+m_2} > \frac{m_2}{m_1+m_2+1}$. When $A_4(y_2,\alpha,\alpha^{**}) \neq 0$, the quadratic form in (7.2) has the following roots

$$x_1 = 1-z, \quad x_2 = 1-z+\frac{2m_1 m_2(m_1 - y_2(m_1+1)}{A_4(y_2,\alpha,\alpha^{**})}.$$

For $0 < y_2 \leq \frac{m_1}{m_1+1}$, if $A_4(y_2, \alpha, \alpha^{**}) > 0$ then $x_1 = 1 - z$ is the smaller positive root, and if $A_4(y_2, \alpha, \alpha^{**}) < 0$ then $x_1 = 1 - z$ is the only positive root when $\alpha > \alpha^{**}$. For the case $A_4(y_2, \alpha, \alpha^{**}) = 0$, $x_1 = 1 - z$ is the only root. Thus from (7.1), $G_{y_2,\alpha,\alpha^{**}}(x) > 0$ for $x \in [0, 1 - z]$ and $0 < y_2 \leq \frac{m_1}{m_1+1}$, and hence $\Delta_2^* > 0$.

For $\frac{m_1}{m_1+1} < y_2 \leq 1$, it can be shown that $\frac{\partial^2 \Delta_2^*}{\partial y_2^2} < 0$, so Δ_2^* is a concave function of y_2 for $\frac{m_1}{m_1+1} < y_2 \leq 1$. Also by similar argument as in the proof of Theorem 3.1 we have

$$\Delta_2^*\Big|_{y_2=1} = \left\{\frac{(\alpha - \alpha^{**})(m_1 + m_2)(m_1 + m_2 + 1)}{m_1^2\rho^2} E[(T_1 - \rho)^2 I_{[0,\rho]}(T_1)]\right\} h(\rho)$$

where

$$\begin{aligned} h(\rho) &= \alpha + \alpha^{**} \\ &\quad -2\left\{\frac{\rho E[(T_1 - \rho)\{(m_1 + m_2 + 1)T_1 - (m_2 + 1)\}I_{[0,\rho]}(T_1)]}{(m_1 + m_2 + 1)E[(T_1 - \rho)^2 I_{[0,\rho]}(T_1)]}\right\} \\ &= \alpha + \alpha^{**} - 2\alpha^{**} = \alpha - \alpha^{**}. \end{aligned}$$

So, $h(\rho) > 0$ for $\alpha > \alpha^{**}$ and hence $\Delta_2^*\Big|_{y_2=1} > 0$. Thus $\Delta_2^* > 0$ for all $0 < y_2 \leq 1$, which completes the proof of part a.

(**b**) By similar argument as in the proof of Theorem 3.1, it can be shown from (2.10) that $R(\delta_{2\alpha}, \boldsymbol{\beta})$ is a strictly convex function of α and minimized at the point $\alpha = \alpha_2(y_2, m_1, m_2)$ where

$$\alpha_2(y_2, m_1, m_2) = 1 - \frac{zE[(T_1 - (1 - z))\{(m_1 + m_2 + 1)T_1 - m_2\}I_{[0,1-z]}(T_1)]}{(m_1 + m_2 + 1)E[(T_1 - (1 - z))^2 I_{[0,1-z]}(T_1)]}$$

and $\alpha_2(y_2, m_1, m_2) \to -\infty$ as $y_2 \to 0$ when $m_2 > 1$. Also

$$\begin{aligned} \alpha_2(1, m_1, m_2) &= \rho - \frac{(1 - \rho)E[(T_1 - \rho)\{\rho(m_1 + m_2 + 1) - m_2\}I_{[0,\rho]}(T_1)]}{(m_1 + m_2 + 1)E[(T_1 - \rho)^2 I_{[0,\rho]}(T_1)]} \\ &= \rho + \frac{\rho}{m_2 B_\rho(m_2, m_1)[\rho^{-m_2}(1 - \rho)^{-m_1}] - \frac{1-2\rho}{1-\rho}} \\ &= 1 - \alpha^* = \alpha^{**}. \end{aligned}$$

Now the result follows using similar arguments to the one in the proof of Theorem 3.1.

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