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A New Class of Zero-Inflated Logarithmic Series Distribution

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Abstract. Through this paper we suggest an alternative form of the modified zero-inflated logarithmic series distribution of Kumar and Riyaz (Statistica, 2013) and study some of its important aspects. The method of maximum likelihood is employed for estimating the parameters of the distribution and certain test procedures are considered for testing the significance of the additional parameter of the model. Further, all the procedures are illustrated with the help of two real life data sets.

Keywords. Generalized likelihood ratio test, logarithmic series distribution, maximum likelihood estimation, probability generating function, Rao's score test.

MSC: 60E05, 60E10.

1 Introduction

The logarithmic series distribution (LSD) and its generalized versions have received much attention in the literature. For example see Khang and Ong (2007), Kumar and Riyaz (2013a) or Johnson et.al. (2005). A limitation of the LSD in certain practical situations is that it excludes zero observation from its support. Khatri (1961) considered a distribution namely "the logarithmic-with-zeros distribution (LZD)" for modelling such experimental phenomena with excess zero observations.

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Khatri (1961) defined the LZD through the following probability mass function (pmf), in which $0 < \omega, \theta < 1$.

$$f_1(x) = \begin{cases} \omega, & for \quad x = 0 \\ \frac{(1-\omega)\theta^x}{-x ln(1-\theta)}, & x = 1, 2, ... \end{cases} \tag{1.1}$$

Some aspects of the LZD is also available in Johnson et al. (2005, pp.355). Kumar and Riyaz (2013b) considered another zero-inflated logarithmic series distribution, namely the "zero-inflated logarithmic series distribution (ZILSD)", through the following pmf, for $x = 0, 1, 2, \cdots$

$$f_2(x) = \frac{A\theta^x}{x+1}, \tag{1.2}$$

where $A = \theta[-ln(1-\theta)]^{-1}$ and $0 < \theta < 1$. The probability generating function of the ZILSD with pmf (1.2) is

$$Q_1(z) = -Az^{-1} ln(1-\theta z). \tag{1.3}$$

Note that ZILSD with pmf (1.2) belongs to the generalized power series family of distributions as well as the Kemp family of distributions, studied by Kumar (2009).

Kumar and Riyaz (2013b) considered a modified form of the ZILSD, namely the "the modified zero-inflated logarithmic series distribution (MZILSD)", through the following pgf.

$$Q_2(z) = -\Lambda(\theta_1 z + \theta_2 z^2)^{-1} ln(1 - \theta_1 z - \theta_2 z^2). \tag{1.4}$$

where $\Lambda = [-ln(1-\theta_1-\theta_2)]^{-1}(\theta_1+\theta_2), \theta_1 > 0, \theta_2 \geq 0$ such that $\theta_1 + \theta_2 < 1$. Note that the MZILSD with pgf (1.4) reduces to the pgf (1.3) of the ZILSD when $\theta_2 = 0$.

Through this paper we consider the alternative form of the MZILSD and named it as "the alternative modified zero-inflated logarithmic series distribution (AMZILSD)", and derive some of its important properties. In section 2 we present the definition of its pgf and derive expression for its pmf, mean and variance. Certain recurrence relations for its probabilities, raw moments and factorial moments are also obtained here. In section 3 we discuss the estimation of the parameters of the AMZILSD by the method of maximum likelihood. In section 4 we describe certain test procedures for testing the significance of the additional parameter of the AMZILSD by using generalized likelihood ratio test and Rao's efficient score test. Both the estimation and testing procedures are illustrated with the help of two real life data sets.

2 Definition and Properties

In this section we present the definition and some important properties of the AMZILSD.

Definition 2.1. A non-negative integer valued random variable Y is said to follow "the alternate modified zero-inflated logarithmic series distribution (AMZILSD)" if its pgf is of the following, in which $\Delta = [-ln(1-\theta_1-\theta_2-\alpha)]^{-1}(\theta_1+\theta_2+\alpha), \theta_1 > 0, \theta_2 \geq 0, \alpha \geq -1$ and $|\theta_1+\theta_2+\alpha| < 1$ such that $\theta_1+\theta_2 \neq -\alpha$.

$$G(z) = \Delta(\theta_1 z + \theta_2 z^2 + \alpha)^{-1}[-ln(1-\theta_1 z - \theta_2 z^2 - \alpha)] \tag{2.1}$$

A distribution with pgf (2.1) we call "alternate modified zero-inflated logarithmic series distribution" or in short "AMZILSD". Clearly, when $\alpha = 0$ the pgf (2.1) reduces to the pgf of the MZILSD as given in (1.4). The pgf of AMZILSD given in (2.1) can also be written as

$$G(z) = \Delta {}_2F_1(1,1;2;\theta_1 z + \theta_2 z^2 + \alpha), \tag{2.2}$$

with

$$\Delta^{-1} = {}_2F_1(1,1;2;\theta_1 + \theta + 2 + \alpha), \text{where}$$

$${}_2F_1(a,b;c;z) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} \frac{z^r}{r!} \tag{2.3}$$

is the Gauss hypergeometric function, in which $(d)_r = d(d+1)\cdots(d+r-1)$ for $r \geq 1$ and $(d)_0 = 1$ for any $d \in R = (-\infty, \infty)$. For details regarding Gauss hypergeometric function see Mathai and Haubold (2008) or Slater (1966).

Now, we obtain the following results in the light of the series representation:

$$\sum_{x=0}^{\infty}\sum_{r=0}^{\infty} A(r,x) = \sum_{x=0}^{\infty}\sum_{r=0}^{x} A(r,x-r) \tag{2.4}$$

and

$$\sum_{x=0}^{\infty}\sum_{r=0}^{\infty} B(r,x) = \sum_{x=0}^{\infty}\sum_{r=0}^{[\frac{x}{2}]} B(r,x-2r), \tag{2.5}$$

where $[a]$ denote the integer part of a, for any $a \geq 0$.

Result 2.1. *For $x = 0, 1, 2, \ldots$, the pmf $g(x) = P(Y = x)$ of the AMZILSD with pgf (2.1) is the following, in which, $\beta_\nu(\alpha) = {}_2F_1(1+\nu, 1+\nu; 2+\nu; \alpha)$, for any $\nu \geq 0$.*

$$g(x) = \Delta \sum_{r=0}^{[\frac{x}{2}]} \beta_{x-r}(\alpha) \frac{(x-r)!}{(x-r+1)} \frac{\theta_1^{x-2r}}{(x-2r)!} \frac{\theta_2^r}{r!} \tag{2.6}$$

Proof. By the definition of pgf given in (2.1), we have

$$\begin{aligned} G(z) &= \sum_{x=0}^{\infty} g(x) z^x & (2.7)\\ &= \Delta \sum_{x=0}^{\infty} (\theta_1 z + \theta_2 z^2 + \alpha)^{-1} [-ln(1 - \theta_1 z - \theta_2 z^2 - \alpha)] z^x. & (2.8) \end{aligned}$$

On expanding the logarithmic function in (2.8), to obtain

$$\begin{aligned} G(z) &= \Delta \sum_{x=1}^{\infty} \frac{(\theta_1 z + \theta_2 z^2 + \alpha)^{x-1}}{x} \\ &= \Delta \sum_{x=0}^{\infty} \frac{(\theta_1 z + \theta_2 z^2 + \alpha)^{x}}{x+1} \\ &= \Delta \sum_{x=0}^{\infty} \sum_{p=0}^{x} \binom{x}{p} \frac{(\theta_1 z + \theta_2 z^2)^{x-p}}{(x+1)} \alpha^p \\ &= \Delta \sum_{x=0}^{\alpha} \sum_{p=0}^{x} \sum_{r=0}^{x-p} \binom{x}{p} \frac{\binom{x-p}{r}}{(x+1)} \theta_1^{x-p-r} \theta_2^r \alpha^p z^{x+r-p} & (2.9) \end{aligned}$$

by binomial theorem. Now by applying (2.4) in (2.9), we get

$$\begin{aligned} G(z) &= \Delta \sum_{x=0}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{x} \frac{\binom{x+p}{p}\binom{x}{r}}{(x+p+1)} \theta_1^{x-r} \theta_2^r \alpha^p z^{x+r} & (2.10)\\ &= \Delta \sum_{x=0}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \binom{x+p+r}{p} \frac{\binom{x+r}{r}}{(x+p+r+1)} \theta_1^x \theta_2^r \alpha^p z^{x+2r}, & (2.11) \end{aligned}$$

by (2.4). By applying (2.5) in (2.11) we obtain

$$G(z) = \Delta \sum_{x=0}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{[\frac{x}{2}]} \binom{x+p-r}{p} \frac{\binom{x-r}{r}}{(x+p-r+1)} \theta_1^{x-2r} \theta_2^r \alpha^p z^x \quad (2.12)$$

$$= \Delta \sum_{x=0}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{[\frac{x}{2}]} \frac{(x+p-r)!(x-r)!}{(x-r)!(x+p-r+1)} \frac{\theta_1^{x-2r}}{(x-2r)!} \frac{\theta_2^r}{r!} \frac{\alpha^p}{p!} z^x, \quad (2.13)$$

which implies the following, in the light of the relation $(1+\nu)_p = \frac{(\nu+p)!}{\nu!}$ and the definition of the generalized hypergeometric function.

$$G(z) = \Delta \sum_{x=0}^{\infty} \sum_{r=0}^{[\frac{x}{2}]} \beta_{x-r}(\alpha) \frac{(x-r)!}{(x-r+1)} \frac{\theta_1^{x-2r}}{(x-2r)!} \frac{\theta_2^r}{r!} z^x \quad (2.14)$$

Now on equating the coefficients of z^x on the right hand side expression of (2.7) and (2.14) we get (2.6).

Remark 2.1. When $\alpha = 0$, the pmf of (2.6) of the AMZILSD reduces to the following pmf of the MZILSD, given in (1.4) of Kumar and Riyaz (2013b).

$$g_1(x) = \Lambda \sum_{r=0}^{[\frac{x}{2}]} \frac{(x-r)!}{(x-r+1)} \frac{\theta_1^{x-2r}}{(x-2r)!} \frac{\theta_2^r}{r!}, \quad (2.15)$$

in which Λ is as given in (1.4).

In order to establish certain recurrence relations for probabilities, raw moments and factorial moments of the AMZILSD, we need the following notations. For $i = 0, 1, 2, \cdots$

$$\underline{u} + i = (1+i, 1+i, 2+i), \quad (2.16)$$

$$\delta_i = \frac{(1+i)(1+i)}{(2+i)}, \quad (2.17)$$

and

$$\Delta_i = {}_2F_1(1+i, 1+i;\ 2+i;\ \theta_1 + \theta_2 + \alpha) \quad (2.18)$$

with $\Delta_0 = \Delta$, as defined in (2.2). Here after, we denote the pmf $g(x)$ of the AMZILSD by $g(x;\underline{u})$. In the light of notations (2.16) to (2.18), we can write $g(x;\underline{u}+i)$ as

$$g(x;\underline{u}+i) = e^{\alpha}\Delta_i{}^{-1}\delta_i \sum_{r=0}^{[\frac{x}{2}]} \frac{\theta_1^{x-2r}}{(x-2r)!}\frac{\theta_2^r}{r!}. \tag{2.19}$$

proof of (2.19) is given in Appendix A.

Result 2.2. *For $x \geq 0$, a simple recurrence relation for probabilities $g(x;\underline{u})$ of the AMZILSD is the following, in which for any positive integer r, $g(-r;u) = 0$.*

$$(x+1)g(x+1;\underline{u}) = \delta_0\Delta_0^{-1}\Delta_1[\theta_1 g(x;\underline{u}+1) + 2\theta_2 g(x-1;\underline{u}+1)] \tag{2.20}$$

Proof. From (2.2) we have

$$\begin{aligned} G(z) &= \sum_{x=0}^{\infty} g(x;\underline{u})z^x & (2.21)\\ &= \Delta_0^{-1}{}_2F_1(1,1;\ 2;\ \theta_1 z + \theta_2 z^2 + \alpha). & (2.22)\end{aligned}$$

Differentiating the right hand side expression of (2.21) and (2.22) with respect to z, we get

$$\sum_{x=0}^{\infty} xg(x;\underline{u})z^{x-1} = \delta_0\Delta_0^{-1}(\theta_1 + 2\theta_2 z)\,{}_2F_1(2,2;\ 3;\ \theta_1 z + \theta_2 z^2 + \alpha) \tag{2.23}$$

From (2.21) and (2.22) we obtain the following.

$$\Delta_1 \sum_{x=0}^{\infty} g(x;\underline{u}+1)z^x = {}_2F_1(2,2;\ 3;\ \theta_1 z + \theta_2 z^2 + \alpha) \tag{2.24}$$

By applying (2.24) in (2.23) we have

$$\sum_{x=0}^{\infty} xg(x;\underline{u})z^{x-1} = \delta_0\Delta_0^{-1}\Delta_1 \sum_{x=0}^{\infty}(\theta_1 + 2\theta_2 z)g(x;\underline{u}+1)z^x \tag{2.25}$$

Equating the coefficients of z^x on both sides of (2.25) we obtain (2.20).

Result 2.3. *For $r \geq 0$, a simple recurrence relation for raw moments $\mu_r(\underline{u})$ of AMZILSD is the following.*

$$\mu_{r+1}(\underline{u}) = \delta_0\Delta_0^{-1}\Delta_1 \sum_{j=0}^{r}\binom{r}{j}(\theta_1 + 2^{j+1}\theta_2)\mu_{r-j}(\underline{u}+1) \tag{2.26}$$

Proof. The characteristic function of the AMZILSD with pgf given in (2.2) has the following representation. For $z \in R$ and $i = \sqrt{-1}$,

$$\begin{aligned} \phi_Y(z) &= \sum_{r=0}^{\infty} \mu_r(\underline{u}) \frac{(iz)^r}{r!} && (2.27) \\ &= \Delta_0^{-1} {}_2F_1(1,1;2;\theta_1 e^{iz} + \theta_2 e^{2iz} + \alpha). && (2.28) \end{aligned}$$

On differentiating the right side expressions of (2.27) and (2.28) with respect to z, we get

$$\begin{aligned} &\sum_{r=0}^{\infty} \mu_r(\underline{u}) \frac{(iz)^{r-1}}{(r-1)!} \\ &\quad = \delta_0 \Delta_0^{-1} (\theta_1 e^{iz} + 2\theta_2 e^{2iz}) {}_2F_1(2,2;3;\theta_1 e^{iz} + \theta_2 e^{2iz} + \alpha). \quad (2.29) \end{aligned}$$

From (2.27) and (2.28) we also obtain

$$\Delta_1 \sum_{r=0}^{\infty} \mu_r(\underline{u}+1) \frac{(iz)^r}{r!} = {}_2F_1(2,2;3;\theta_1 e^{iz} + \theta_2 e^{2iz} + \alpha). \quad (2.30)$$

Equations (2.29) and (2.30) together implies

$$\begin{aligned} \sum_{r=0}^{\infty} \mu_r(\underline{u}) \frac{(iz)^{r-1}}{(r-1)!} &= \delta_0 \Delta_0^{-1} \Delta_1 \sum_{r=0}^{\infty} (\theta_1 e^{iz} + 2\theta_2 e^{2iz}) \mu_r(\underline{u}+1) \frac{(iz)^r}{r!} \\ &= \delta_0 \Delta_0^{-1} \Delta_1 [\theta_1 \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \mu_r(\underline{u}+1) \frac{(iz)^{r+j}}{r!j!} \\ &\quad + 2\theta_2 \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \mu_r(\underline{u}+1) \frac{(2iz)^{r+j}}{r!j!}] \\ &= \delta_0 \Delta_0^{-1} \Delta_1 [\theta_1 \sum_{r=0}^{\infty} \sum_{j=0}^{r} \mu_{r-j}(\underline{u}+1) \frac{(iz)^r}{(r-j)!j!} \\ &\quad + 2\theta_2 \sum_{r=0}^{\infty} \sum_{j=0}^{r} \mu_{r-j}(\underline{u}+1) \frac{(2iz)^r}{(r-j)!j!}] \quad (2.31) \end{aligned}$$

by (2.4). Now, on equating the coefficients of $(r!)^{-1}(iz)^r$ on both sides of (2.31) we get (2.26).

Result 2.4. *For $r \geq 1$, a simple recurrence relation for factorial moments $\mu_{[r]}(\underline{u})$ of the AMZILSD is the following, in which $\mu_{[0]}(\underline{u}) = 1$.*

$$\mu_{[r+1]}(\underline{u}) = \delta_0 \Delta_0^{-1} \Delta_1 [(\theta_1 + 2\theta_2) \mu_{[r]}(\underline{u}+1) + 2\theta_2 r \mu_{[r-1]}(\underline{u}+1)] \quad (2.32)$$

Proof. From (2.2) we have the following factorial moment generating function $F(z)$ of the AMZILSD

$$\begin{aligned} F(z) &= \sum_{r=0}^{\infty} \mu_{[r]}(\underline{u}) \frac{z^r}{r!} && (2.33) \\ &= \Delta_0^{-1} {}_2F_1[1,1;2;\theta_1(1+z)+\theta_2(1+z)^2+\alpha]. && (2.34) \end{aligned}$$

On differentiating the right hand side expressions of (2.33) and (2.34) with respect to z, we get

$$\sum_{r=0}^{\infty} \mu_{[r]}(\underline{u}) \frac{z^{r-1}}{(r-1)!}$$
$$= \delta_0 \Delta_0^{-1} [\theta_1 + \theta_2(1+z)] {}_2F_1[2,2;3;\theta_1(1+z)+\theta_2(1+z)^2+\alpha]. \quad (2.35)$$

From (2.33) and (2.34) we have

$$\Delta_1 \sum_{r=0}^{\infty} \mu_{[r]}(\underline{u}+1) \frac{z^r}{r!} = {}_2F_1[2,2;3;\theta_1(1+z)+\theta_2(1+z)^2+\alpha]. \quad (2.36)$$

Equations (2.33) and (2.34) together implies the following.

$$\sum_{r=0}^{\infty} \mu_{[r]}(\underline{u}) \frac{z^{r-1}}{(r-1)!} = \delta_0 \Delta_0^{-1} \Delta_1 \sum_{r=0}^{\infty} [\theta_1 + 2\theta_2(1+z)] \mu_{[r]}(\underline{u}+1) \frac{z^r}{r!} \quad (2.37)$$

On equating the coefficients of $(r!)^{-1} z^r$ on both sides of (2.37) we get (2.32).

By using the Result 2.4, we obtain the mean and variance of the AMZILSD, as given in the following result.

Result 2.5. *The mean and variance of the AMZILSD are given below.*

$$\begin{aligned} E(Y) &= \delta_0 \Delta_0^{-1} \Delta_1 (\theta_1 + 2\theta_2) \\ Var(Y) &= \delta_0 \Delta_0^{-2} (\delta_1 \Delta_0 \Delta_2 - \delta_0 \Delta_1^2)(\theta_1 + 2\theta_2) + \delta_0 \Delta_0^{-1} \Delta_1 (\theta_1 + 2\theta_2). \end{aligned}$$

Proof is simple and hence omitted.

3 Estimation

In this section we discuss the estimation of the parameters θ_1, θ_2 and α of the AMZILSD by the method of maximum likelihood and illustrate the procedures using certain real life data sets.

Let $a(x)$ be the observed frequency of x events and let y be the highest value of x. Then the likelihood function of the sample is

$$L = \prod_{x=0}^{y} [g(x)]^{a(x)}, \tag{3.1}$$

where $g(x)$ is the pmf of the AMZILSD as given in (2.6). Now taking logarithm on both sides of (3.1), we have

$$\begin{aligned} logL &= \sum_{x=0}^{y} a(x) \log[g(x)] \\ &= \sum_{x=0}^{y} a(x)[\log \Delta + \log \Psi(x;\theta_1,\theta_2,\alpha)], \end{aligned} \tag{3.2}$$

where Δ is as given in (2.1) and

$$\Psi(x;\theta_1,\theta_2,\alpha) = \sum_{r=0}^{[\frac{x}{2}]} \beta_{x-r}(\alpha) \frac{(x-r)!}{(x-r+1)} \frac{\theta_1^{x-2r}}{(x-2r)!} \frac{\theta_2^r}{r!}.$$

Let $\hat{\theta_1}$, $\hat{\theta_2}$ and $\hat{\alpha}$ denote the maximum likelihood estimators of the AMZILSD respectively. On differentiating(3.2) partially with respect to the parameters θ_1, θ_2 and α and equating to zero, we get the following likelihood equations, in which

$$D = (\theta_1+\theta_2+\alpha)^{-1} - [-ln(1-\theta_1-\theta_2-\alpha)]^{-1}(1-\theta_1-\theta_2-\alpha)^{-1}.$$

$$\sum_{x=0}^{y} a(x) \left[D + \frac{\sum_{r=0}^{[\frac{x}{2}]} \beta_{x-r}(\alpha) \frac{(x-r)!}{(x-r+1)} \frac{\theta_1^{x-2r-1}}{(x-2r-1)!} \frac{\theta_2^r}{r!}}{\Psi(x;\theta_1,\theta_2,\alpha)} \right] = 0 \tag{3.3}$$

$$\sum_{x=0}^{y} a(x) \left[D + \frac{\sum_{r=0}^{[\frac{x}{2}]} \beta_{x-r}(\alpha) \frac{(x-r)!}{(x-r+1)} \frac{\theta_1^{x-2r}}{(x-2r)!} \frac{\theta_2^{r-1}}{(r-1)!}}{\Psi(x;\theta_1,\theta_2,\alpha)} \right] = 0 \tag{3.4}$$

and

$$\sum_{x=0}^{y} a(x)\left[D+\frac{\sum_{r=0}^{[\frac{x}{2}]}\beta_{x-r+1}(\alpha)\frac{(x-r+1)!}{(x-r+2)}\frac{\theta_1^{x-2r}}{(x-2r)!}\frac{\theta_2^{r}}{r!}}{\Psi(x;\theta_1,\theta_2,\alpha)}\right]=0. \tag{3.5}$$

When likelihood equations do not have a solution, the maximum of the likelihood function attained at the border of the domain of the parameters. So we obtained the second order partial derivatives of log[g(x)] with respect to the parameters θ_1,θ_2 and α by using *MATHCAD* software observed that these eqations give negative values for all $\theta_1 > 0$, $\theta_1 \geq 0$, $\alpha \geq -1$ and $|\theta_1 + \theta_2 + \alpha| < 1$ such that$\theta_1 + \theta_2 \neq \alpha$. Thus the density of the AMZILSD is log-concave and hence the maximum likelihood estimators of the parameters θ_1, θ_2 and α are unique (cf. Puig, 2007). Now, on solving these likelihood equations (3.3), (3.4) and (3.5) by using some mathematical software such as *MATHLAB*, *MATHCAD*, *MATHEMATICA* etc., one can obtain the maximum likelihood estimators of the parameters θ_1,θ_2 and α of the AMZILSD.

For numerical illustration, we have considered two real life data sets of which the first data is on *Ribes, Kaniksu, Idaho* taken from Fracker and Brischle (1944) and the second data set on *Juncuss effuses seedings* taken from Evans (1953). We have fitted the ZILSD, the LZD, the MZILSD and the AMZILSD to these two data sets and the results obtained along with the corresponding values of the expected frequencies, Chi-square statistic, degrees of freedom (d.f), P, Akaike information criterion (AIC), Bayesian information criterion (BIC) and the second order Akaike information criterion (AICc) in respect of each of the models are presented in Tables 1 and 2. Based on the computed values of Chi-square statistic, P, AIC, BIC and AICc, it can be observed that the AMZILSD give the best fit to both the data sets where the existing models - the ZILSD, the LZD and the MZILSD fails.

Table 1. *Observed frequencies and computed values of expected frequencies of the ZLSD, the LZD, the MZILSD and the AMZILAD by the method of moments and the method of maximum likelihood for the first data set.*

Ribes	*Observed*	*ZILSD*	*LZD*	*MZILSD*	*AMZILSD*
0	43	33.203	37.040	34.960	43.920
1	15	14.610	22.878	12.320	13.440
2	8	8.571	8.964	8.640	8.320
3	6	5.657	4.405	5.600	5.550
4	3	3.982	2.511	4.160	3.040
5	4	2.920	1.527	3.040	2.000
6	0	2.203	0.967	2.320	1.360
7	1	8.854	1.978	9.920	2.370
Total	80	80	80	80	80
Estimates of the Parameters		$\hat{\theta} = 0.88$	$\hat{\omega} = 0.463$ $\hat{\theta}_2 = 0.76$	$\hat{\theta}_1 = 0.65$ $\hat{\theta}_2 = 0.21$	$\hat{\theta}_1 = 0.61$ $\hat{\theta}_2 = 0.13$ $\hat{\alpha} = 0.001$
Chi-square Values		10.414	4.325	11.779	0.317
d.f		4	1	3	1
P- values		0.034	0.038	0.003	0.573
AIC		108.994	106.554	112.152	104.304
BIC		109.352	106.260	111.858	104.013
AICc		109.012	106.710	112.598	104.62

Table 2. *Observed frequencies and computed values of expected frequencies of the ZLSD, the LZD, the MZILSD and the AMZILAD by the method of maximum likelihood for the second data set.*

No.of individuals	Observed	ZILSD	LZD	MZILSD	AMZILSD
0	150	123.350	77.976	136.116	148.428
1	34	46.256	91.782	37.392	38.076
2	28	23.128	30.288	22.572	25.136
3	8	13.010	13.327	12.312	9.576
4	4	7.806	6.579	7.296	5.244
5	2	4.789	3.483	4.332	1.010
6	2	9.571	4.547	7.980	0.530
Total	228	228	228	228	228
Estimates of the Parameters		$\hat{\theta} = 0.75$	$\hat{\omega} = 0.342$ $\hat{\theta}_2 = 0.66$	$\hat{\theta}_1 = 0.55$ $\hat{\theta}_2 = 0.13$	$\hat{\theta}_1 = 0.51$ $\hat{\theta}_2 = 0.09$ $\hat{\alpha} = 0.01$
Chi-square Values		21.374	108.239	11.64	1.069
d.f		4	3	3	1
P- values		< 0.0001	< 0.0001	0.009	0.301
AIC		290.648	267.380	227.696	224.632
BIC		291.006	268.096	228.412	225.346
AICc		290.666	267.433	227.751	224.739

4 Testing of the Hypothesis

In this section we discuss the testing of hypothesis $H_0 : \alpha = 0$ against the alternative hypothesis $H_1 : \alpha \neq 0$ by using generalized likelihood ratio test and Rao's efficient score test.

In case of generalized likelihood ratio test, the test statistic is

$$-2\log\lambda = 2[\log L(\hat{\underline{\Omega}}; x) - \log L(\hat{\underline{\Omega}}^*; x)], \tag{4.1}$$

where $\hat{\underline{\Omega}}$ is the maximum likelihood estimator of $\underline{\Omega} = (\theta_1, \theta_2, \alpha)$ with no restrictions, and $\hat{\underline{\Omega}}^*$ is the maximum likelihood estimator of $\underline{\Omega}$ when $\alpha = 0$. The test statistic $-2\log\lambda$ given in (4.1) is asymptotically distributed as Chi-square with one degree of freedom. For details see Rao, (1973). We have computed the values of $\log L(\hat{\underline{\Omega}}; x)$, $\log L(\hat{\underline{\Omega}}^*; x)$ and the test statistic for the AMZILSD for these two data sets and presented in Table 3.

Table 3 *: The computed the values of* $\log L(\underline{\hat{\Omega}}; x)$, $\log L(\underline{\hat{\Omega}}^*; x)$ *and the generalized likelihood ratio test under* H_0.

	$\log L(\underline{\hat{\Omega}}^*; x)$	$\log L(\underline{\hat{\Omega}}; x)$	Test statistic
Data set 1	-54.020	-50.628	6.784
Data set 2	-111.846	109.609	4.654

In case of Rao's Score test, the statistic is,

$$S = T'\varphi^{-1}T, \tag{4.2}$$

where $T' = \left(\frac{1}{\sqrt{n}}\frac{\partial \log L}{\partial \theta_1}, \frac{1}{\sqrt{n}}\frac{\partial \log L}{\partial \theta_2}, \frac{1}{\sqrt{n}}\frac{\partial \log L}{\partial \alpha}\right)$ and φ is the Fisher information matrix. The test statistic given in (4.2) follows Chi-square distribution with one degree of freedom, (see Rao, 1973). We have computed the values of S for the AMZILSD in the case of first data set as S_1 and for the the AMZILSD in the case second data set as S_2 as given below.

$$S_1 = \begin{pmatrix} 2.902 & 7.399 & -1.522 \end{pmatrix} \begin{bmatrix} 0.101 & -0.106 & 0.013 \\ -0.106 & 0.240 & -0.157 \\ 0.013 & -0.157 & 0.169 \end{bmatrix} \begin{pmatrix} 2.902 \\ 7.399 \\ -1.522 \end{pmatrix}$$
$$= 13.231$$

$$S_2 = \begin{pmatrix} 8.75 & 16.94 & -0.959 \end{pmatrix} \begin{bmatrix} 0.026 & -0.0002 & -0.033 \\ -0.0002 & 0.0059 & -0.0066 \\ -0.033 & -0.0066 & 0.057 \end{bmatrix} \begin{pmatrix} 8.75 \\ 16.94 \\ -0.959 \end{pmatrix}$$
$$= 4.447$$

Since the critical value for the test at 5% level of significance and one degree of freedom is 3.84, the null hypothesis is rejected in all the above cases in respect of generalized likelihood ratio test and Rao's efficient score test.

5 Appendix - A

Proof of Equation(2.19). From (2.7) we have the following, in the light of (2.16) and (2.18).

$$\begin{aligned} G(z) &= \sum_{x=0}^{\infty} g(x; \underline{u} + i) z^x & (5.1) \\ &= \Delta_i^{-1} {}_2F_1(1+i, 1+i; 2+i; \theta_1 z + \theta_2 z^2 + \alpha). & (5.2) \end{aligned}$$

On expanding the Gauss hypergeometric function in (5.2), we obtain

$$G(z) = \Delta_i^{-1}\sum_{x=0}^{\infty}\frac{(1+i)_x(1+i)_x}{(2+i)_x}\frac{(\theta_1 z+\theta_2 z^2+\alpha)}{x!} \tag{5.3}$$

$$= \Delta_i^{-1}\sum_{x=0}^{\infty}\frac{(1+i)_x(1+i)_x}{(2+i)_x}\sum_{p=0}^{x}\binom{x}{p}\frac{(\theta_1 z+\theta_2 z^2)^{x-p}}{x!}\alpha^p$$

$$= \Delta_i^{-1}\sum_{x=0}^{\infty}\frac{(1+i)_x(1+i)_x}{(2+i)_x} \times \sum_{p=0}^{x}\sum_{r=0}^{x-p}\binom{x}{p}\frac{\binom{x-p}{r}}{x!}\theta_1^{x-p-r}\theta_2^r\alpha^p z^{x+r-p} \tag{5.4}$$

by binomial theorem. Apply (2.4) in (5.4) to get

$$G(z) = \Delta_i^{-1}\sum_{x=0}^{\infty}\frac{(1+i)_x(1+i)_x}{(2+i)_x}\sum_{p=0}^{\infty}\sum_{r=0}^{x}\frac{\binom{x+p}{p}\binom{x}{r}}{(x+p)!}\theta_1^{x-r}\theta_2^r\alpha^p z^{x+r} \tag{5.5}$$

$$= \Delta_i^{-1}\sum_{x=0}^{\infty}\frac{(1+i)_x(1+i)_x}{(2+i)_x} \times \sum_{p=0}^{\infty}\sum_{r=0}^{x}\binom{x+p+r}{p}\frac{\binom{x+r}{r}}{(x+p+r)!}\theta_1^{x}\theta_2^r\alpha^p z^{x+2r}, \tag{5.6}$$

by applying (2.4) again. Now, by using (2.5), we have the following from (5.6).

$$G(z) = \Delta_i^{-1}\sum_{x=0}^{\infty}\frac{(1+i)_x(1+i)_x}{(2+i)_x} \times \sum_{p=0}^{\infty}\sum_{r=0}^{[\frac{x}{2}]}\binom{x+p-r}{p}\frac{\binom{x-r}{r}}{(x+p-r)!}\theta_1^{x-2r}\theta_2^r\alpha^p z^{x} \tag{5.7}$$

$$= e^{\alpha}\Delta_i^{-1}\sum_{x=0}^{\infty}\frac{(1+i)_x(1+i)_x}{(2+i)_x}\sum_{r=0}^{[\frac{x}{2}]}\frac{\theta_1^{x-2r}}{(x-2r)!}\frac{\theta_2^r}{r!}z^x. \tag{5.8}$$

On equating the coefficients of z^x on the right hand side expressions of (5.1) and (5.8), we get (2.19).

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