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Karlin's Basic Composition Theorems and Stochastic Orderings

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Abstract. Suppose λ, x, ζ traverse the ordered sets Λ, X and Z, respectively and consider the functions $f(\lambda, x, \zeta)$ and $g(\lambda, \zeta)$ satisfying the following conditions,

(a) $f(\lambda, x, \zeta) > 0$ and f is TP_2 in each pairs of variables when the third variable is held fixed; and

(b) $g(\lambda, \zeta)$ is TP_2.

Then the function

$$h(\lambda, x) = \int_Z f(\lambda, x, \zeta) g(\lambda, \zeta) d\mu(\zeta),$$

defined on $\Lambda \times X$ is TP_2 in (λ, x). The aim of this note is to use a new *stochastic ordering* argument to prove the above result and simplify it's proof given by Karlin (1968). We also prove some other new versions of this result.

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1 Introduction

Karlin (1968) introduced the concept of Sign-Regular of order 2, which is of great importance in various fields of Mathematics and Statistics with many applications.

Definition 1.1. (Karlin (1968)). We say that a function $h(x, y)$ is Sign-Regular of order 2 (SR_2) if $\varepsilon_1 h(x, y) \geq 0$ and

$$\varepsilon_2 \begin{vmatrix} h(x_1, y_1) & h(x_1, y_2) \\ h(x_2, y_1) & h(x_2, y_2) \end{vmatrix} \geq 0, \tag{1.1}$$

whenever $x_1 < x_2$, $y_1 < y_2$ for ε_1 and ε_2 equal to +1 or -1.

If the above relations hold with $\varepsilon_1 = +1$ and $\varepsilon_2 = +1$, then h is said to be *Totally Positive of order 2* (TP_2); and if they hold with $\varepsilon_1 = +1$ and $\varepsilon_2 = -1$ then h is said to be *Reverse Regular of order* 2 (RR_2).

Karlin (1968) proved the following theorem which is well known as *Basic Composition Theorem.* In the following μ represents a σ-finite measure.

Theorem 1.1. *Let A, B and C be subsets of the real line and let $L(x, z)$ be SR_2 for $x \in A$, $z \in B$ and $M(z, y)$ be SR_2 for $z \in B$, $y \in C$. Then*

$$K(x, y) = \int L(x, z)M(z, y)\, d\mu(z) \tag{1.2}$$

is SR_2 for $x \in A$, $y \in C$ and $\varepsilon_i(K) = \varepsilon_i(L) \times \varepsilon_i(M)$ $\forall\ i = 1, 2$.

That is, if the functions L and M both are either TP_2 or RR_2, then the function h is TP_2 and if one of the functions L or M is TP_2 and the other is RR_2, then the function h is RR_2.

The following theorem was also proved by Karlin (1968)

Theorem 1.2. *Suppose λ, x, ζ traverse the ordered sets Λ, X and Z, respectively and consider the functions $f(\lambda, x, \zeta)$ and $g(\lambda, \zeta)$ satisfying the following conditions,*

(a) *$f(\lambda, x, \zeta) > 0$ and f is TP_2 in each pairs of variables when the third variable is held fixed; and*

(b) *$g(\lambda, \zeta)$ is TP_2.*

Then the function

$$h(\lambda, x) = \int_Z f(\lambda, x, \zeta)g(\lambda, \zeta)d\mu(\zeta),$$

defined on $\Lambda \times X$ is TP_2 in (λ, x).

The aim of this note is to use a new *stochastic ordering* argument to prove Theorem 1.2 and simplify the proof of this theorem given by Karlin (1968). We also prove some other new versions of these theorems.

Now, we need to recall some definitions of stochastic orderings that we use later in this note.

Let X and Y be two random variables with distribution functions F and G and density functions f and g, respectively.

Definition 1.2. X is said to be *stochastically* smaller than Y (denoted by $X \leq_{st} Y$) if for all x, $\overline{F}(x) \leq \overline{G}(x)$. It is well known that $X \leq_{st} Y$ is equivalent to that

$$E[\phi(X)] \leq (\geq) \, E[\phi(Y)] \tag{1.3}$$

for all increasing (decreasing) functions $\phi : \mathcal{R} \to \mathcal{R}$, for which the expectations exist.

A stronger notion of stochastic dominance is that of *likelihood ratio* ordering.

Definition 1.3. X is said to be smaller than Y in the *likelihood ratio* ordering (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in x.

For a more comprehensive review and details on the above stochastic orderings, see Chapter 1 of Shaked and Shanthikumar (2007).

2 Main Results

We first give the new proof of Theorem 1.2.

Theorem 2.1. *Suppose λ, x, ζ traverse the ordered sets Λ, X and Z, respectively and consider the functions $f(\lambda, x, \zeta)$ and $g(\lambda, \zeta)$ satisfying the following conditions,*

(a) $f(\lambda, x, \zeta) > 0$ *and f is TP_2 in each pairs of variables when the third variable is held fixed; and*

(b) $g(\lambda, \zeta)$ *is TP_2.*

Then the function

$$h(\lambda, x) = \int_Z f(\lambda, x, \zeta) g(\lambda, \zeta) d\mu(\zeta),$$

defined on $\Lambda \times X$ is TP_2 in (λ, x).

Proof. Let $x_1 \leq x_2$. Then

$$
\begin{aligned}
\frac{h(\lambda_2, x_2)}{h(\lambda_2, x_1)} &= \frac{\int_Z f(\lambda_2, x_2, \zeta) g(\lambda_2, \zeta) d\mu(\zeta)}{\int_Z f(\lambda_2, x_1, \zeta) g(\lambda_2, \zeta) d\mu(\zeta)} \\
&= \int_Z \frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)} \frac{f(\lambda_2, x_1, \zeta) g(\lambda_2, \zeta)}{\int_Z f(\lambda_2, x_1, u) g(\lambda_2, u) d\mu(u)} d\mu(\zeta) \\
&\geq \int_Z \frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)} \frac{f(\lambda_1, x_1, \zeta) g(\lambda_1, \zeta)}{\int_Z f(\lambda_1, x_1, u) g(\lambda_1, u) d\mu(u)} d\mu(\zeta) \quad (2.4) \\
&\geq \int_Z \frac{f(\lambda_1, x_2, \zeta)}{f(\lambda_1, x_1, \zeta)} \frac{f(\lambda_1, x_1, \zeta) g(\lambda_1, \zeta)}{\int_Z f(\lambda_1, x_1, u) g(\lambda_1, u) d\mu(u)} d\mu(\zeta) \quad (2.5) \\
&= \frac{h(\lambda_1, x_2)}{h(\lambda_1, x_1)}.
\end{aligned}
$$

Let $\Theta^*(\lambda)$ be a random variable having density function given by

$$\frac{f(\lambda, x_1, \zeta) g(\lambda, \zeta)}{\int_Z f(\lambda, x_1, u) g(\lambda, u) d\mu(u)}$$

with respect to μ. Then the assumptions that f and g are TP_2 in (λ, ζ) implies the fact that for $\lambda_1 \leq \lambda_2$, $\Theta^*(\lambda_1) \leq_{lr} \Theta^*(\lambda_2)$, which in turn implies that $\Theta^*(\lambda_1) \leq_{st} \Theta^*(\lambda_2)$. On the other hand the assumption that f is TP_2 in (x, ζ) is equivalent to that $\frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)}$ is increasing in ζ. Combining these observations, the inequality (2.4) follows from (1.3). The second inequality follows from the assumption that f is TP_2 in (λ, x). This completes the proof of the required result. ■

Next, we prove a new version of Theorem 2.1 which covers Lemma A.1 of Khaledi and Kochar (2000) as special case.

Theorem 2.2. *Suppose λ, x, ζ traverse the ordered sets Λ, X and Z, respectively and consider the functions $f(\lambda, x, \zeta)$ and $g(\lambda, \zeta)$ satisfying the following conditions,*

(a) $f(\lambda, x, \zeta) > 0$, *f is TP_2 in (λ, x), RR_2 in (x, ζ) and (λ, ζ).*

(b) *g is RR_2 in (λ, ζ)*

Then the function

$$h(\lambda, x) = \int_Z f(\lambda, x, \zeta) g(\lambda, \zeta) d\mu(\zeta),$$

defined on $\Lambda \times X$ is TP_2 in (λ, x).

Proof. Let $x_1 \leq x_2$. Then

$$\begin{aligned}
\frac{h(\lambda_2,x_2)}{h(\lambda_2,x_1)} &= \frac{\int_Z f(\lambda_2,x_2,\zeta)g(\lambda_2,\zeta)d\mu(\zeta)}{\int_Z f(\lambda_2,x_1,\zeta)g(\lambda_2,\zeta)d\mu(\zeta)} \\
&= \int_Z \frac{f(\lambda_2,x_2,\zeta)}{f(\lambda_2,x_1,\zeta)} \frac{f(\lambda_2,x_1,\zeta)g(\lambda_2,\zeta)}{\int_Z f(\lambda_2,x_1,u)g(\lambda_2,u)d\mu(u)} d\mu(\zeta) \\
&\geq \int_Z \frac{f(\lambda_2,x_2,\zeta)}{f(\lambda_2,x_1,\zeta)} \frac{f(\lambda_1,x_1,\zeta)g(\lambda_1,\zeta)}{\int_Z f(\lambda_1,x_1,u)g(\lambda_1,u)d\mu(u)} d\mu(\zeta) \quad (2.6) \\
&\geq \int_Z \frac{f(\lambda_1,x_2,\zeta)}{f(\lambda_1,x_1,\zeta)} \frac{f(\lambda_1,x_1,\zeta)g(\lambda_1,\zeta)}{\int_Z f(\lambda_1,x_1,u)g(\lambda_1,u)d\mu(u)} d\mu(\zeta) \quad (2.7) \\
&= \frac{h(\lambda_1,x_2)}{h(\lambda_1,x_1)}.
\end{aligned}$$

Let $\Theta^*(\lambda)$ be a random variable having density function given by

$$\frac{f(\lambda,x_1,\zeta)g(\lambda,\zeta)}{\int_Z f(\lambda,x_1,u)g(\lambda,u)d\mu(u)}$$

with respect to μ. Then the assumptions that f and g are RR_2 in (λ,ζ) implies that for $\lambda_1 \leq \lambda_2$, $\Theta^*(\lambda_2) \leq_{lr} \Theta^*(\lambda_1)$, which in turn implies that $\Theta^*(\lambda_2) \leq_{st} \Theta^*(\lambda_1)$. On the other hand the assumption that f is RR_2 in (x,ζ) is equivalent to that $\frac{f(\lambda_2,x_2,\zeta)}{f(\lambda_2,x_1,\zeta)}$ is decreasing in ζ. Combining these observations, the inequality (2.6) follows from (1.3). The second inequality follows from the assumption that f is TP_2 in (λ,x). This completes the proof of the required result. ■

Next two theorems deal with conditions under which the function $h(\lambda,x)$ is RR_2 in (λ,x).

Theorem 2.3. *Suppose λ,x,ζ traverse the ordered sets Λ, X and Z, respectively and consider the functions $f(\lambda,x,\zeta)$ and $g(\lambda,\zeta)$ satisfying the following conditions,*

(a) $f(\lambda,x,\zeta) > 0$, *f and g are RR_2 in (λ,ζ).*

(b) *$f(\lambda,x,\zeta)$ is RR_2 in (λ,x) and is TP_2 in (x,ζ).*

Then the function

$$h(\lambda,x) = \int_Z f(\lambda,x,\zeta)g(\lambda,\zeta)d\mu(\zeta),$$

defined on $\Lambda \times X$ is RR_2 in (λ,x).

Proof. Let $x_1 \le x_2$. Then

$$
\begin{aligned}
\frac{h(\lambda_2, x_2)}{h(\lambda_2, x_1)} &= \frac{\int_Z f(\lambda_2, x_2, \zeta) g(\lambda_2, \zeta) d\mu(\zeta)}{\int_Z f(\lambda_2, x_1, \zeta) g(\lambda_2, \zeta) d\mu(\zeta)} \\
&= \int_Z \frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)} \frac{f(\lambda_2, x_1, \zeta) g(\lambda_2, \zeta)}{\int_Z f(\lambda_2, x_1, u) g(\lambda_2, u) d\mu(u)} d\mu(\zeta) \\
&\le \int_Z \frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)} \frac{f(\lambda_1, x_1, \zeta) g(\lambda_1, \zeta)}{\int_Z f(\lambda_1, x_1, u) g(\lambda_1, u) d\mu(u)} d\mu(\zeta) \quad (2.8) \\
&\le \int_Z \frac{f(\lambda_1, x_2, \zeta)}{f(\lambda_1, x_1, \zeta)} \frac{f(\lambda_1, x_1, \zeta) g(\lambda_1, \zeta)}{\int_Z f(\lambda_1, x_1, u) g(\lambda_1, u) d\mu(u)} d\mu(\zeta) \quad (2.9) \\
&= \frac{h(\lambda_1, x_2)}{h(\lambda_1, x_1)}.
\end{aligned}
$$

Let $\Theta^*(\lambda)$ be a random variable having density function given by

$$
\frac{f(\lambda, x_1, \zeta) g(\lambda, \zeta)}{\int_Z f(\lambda, x_1, u) g(\lambda, u) d\mu(u)}
$$

with respect to μ. Then the assumptions that f and g are RR_2 in (λ, ζ) implies that for $\lambda_1 \le \lambda_2$, $\Theta^*(\lambda_2) \le_{lr} \Theta^*(\lambda_1)$, which in turn implies that $\Theta^*(\lambda_2) \le_{st} \Theta^*(\lambda_1)$. On the other hand the assumption that f is TP_2 in (x, ζ) is equivalent to that $\frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)}$ is increasing in ζ. Combining these observations, the inequality (2.8) follows from (1.3). The second inequality follows from the assumption that f is RR_2 in (λ, x). This completes the proof of the required result. ■

Theorem 2.3 was also proved in Khaledi and Kochar (2001) in a different complicated way.

Next, we prove another new version of Theorem 2.1.

Theorem 2.4. *Suppose λ, x, ζ traverse the ordered sets Λ, X and Z, respectively and consider the functions $f(\lambda, x, \zeta)$ and $g(\lambda, \zeta)$ satisfying the following conditions,*

(a) $f(\lambda, x, \zeta) > 0$, *f and g are TP_2 in (λ, ζ).*

(b) $f(\lambda, x, \zeta)$ *is RR_2 in (λ, x) and (x, ζ).*

Then the function

$$
h(\lambda, x) = \int_Z f(\lambda, x, \zeta) g(\lambda, \zeta) d\mu(\zeta),
$$

defined on $\Lambda \times X$ is RR_2 in (λ, x).

Proof. Let $x_1 \le x_2$. Then

$$\begin{aligned}
\frac{h(\lambda_2, x_2)}{h(\lambda_2, x_1)} &= \frac{\int_Z f(\lambda_2, x_2, \zeta) g(\lambda_2, \zeta) d\mu(\zeta)}{\int_Z f(\lambda_2, x_1, \zeta) g(\lambda_2, \zeta) d\mu(\zeta)} \\
&= \int_Z \frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)} \frac{f(\lambda_2, x_1, \zeta) g(\lambda_2, \zeta)}{\int_Z f(\lambda_2, x_1, u) g(\lambda_2, u) d\mu(u)} d\mu(\zeta) \\
&\le \int_Z \frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)} \frac{f(\lambda_1, x_1, \zeta) g(\lambda_1, \zeta)}{\int_Z f(\lambda_1, x_1, u) g(\lambda_1, u) d\mu(u)} d\mu(\zeta) \quad (2.10) \\
&\le \int_Z \frac{f(\lambda_1, x_2, \zeta)}{f(\lambda_1, x_1, \zeta)} \frac{f(\lambda_1, x_1, \zeta) g(\lambda_1, \zeta)}{\int_Z f(\lambda_1, x_1, u) g(\lambda_1, u) d\mu(u)} d\mu(\zeta) \quad (2.11) \\
&= \frac{h(\lambda_1, x_2)}{h(\lambda_1, x_1)}.
\end{aligned}$$

Let $\Theta^*(\lambda)$ be a random variable having density function given by

$$\frac{f(\lambda, x_1, \zeta) g(\lambda, \zeta)}{\int_Z f(\lambda, x_1, u) g(\lambda, u) d\mu(u)}$$

with respect to μ. Then the assumptions that f and g are TP_2 in (λ, ζ) implies that for $\lambda_1 \le \lambda_2$, $\Theta^*(\lambda_1) \le_{lr} \Theta^*(\lambda_2)$, which in turn implies that $\Theta^*(\lambda_1) \le_{st} \Theta^*(\lambda_2)$. On the other hand the assumption that f is RR_2 in (x, ζ) is equivalent to that $\frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)}$ is decreasing in ζ. Combining these observations, the inequality (2.10) follows from (1.3). The second inequality follows from the assumption that f is RR_2 in (λ, x). This completes the proof of the required result. ■

Remark 2.1. The results of Theorem 1.1 are special cases of Theorem 2.1 - 2.4.

We end this note by proving a result given in Kirmani and Kochar (1995), using a similar argument used to prove Theorem 2.1.

Theorem 2.5. *Let $\psi_1(x, u)$ and $\psi_2(x, u)$ be two positive real-valued functions such that*

(i) for each $u_1 \le u_2$, $\frac{\psi_2(x, u_2)}{\psi_2(x, u_1)}$ is increasing in x,

(ii) for each x, $\frac{\psi_1(x, u)}{\psi_2(x, u)}$ is increasing in u and

(iii) for each fixed u, $\frac{\psi_1(x, u)}{\psi_2(x, u)}$ is increasing in x.

Then, $X \leq_{lr} Y$ implies that the ratio

$$\frac{E[\psi_1(x,Y)]}{E[\psi_2(x,X)]}$$

is increasing in x, provided that the expectations exist.

Proof. Let $x_1 \leq x_2$ and denote the density functions of X and Y by F and G, respectively. Then

$$\begin{aligned}
\frac{E[\psi_1(x_2,Y)]}{E[\psi_2(x_2,X)]} &= \frac{\int \psi_1(x_2,u)g(u)du}{\int \psi_2(x_2,u)f(u)du} \\
&= \int \left(\frac{\psi_1(x_2,u)}{\psi_2(x_2,u)}\right)\left(\frac{g(u)}{f(u)}\right)\frac{\psi_2(x_2,u)f(u)}{\int \psi_2(x_2,u)f(u)du}du \\
&\geq \int \left(\frac{\psi_1(x_2,u)}{\psi_2(x_2,u)}\right)\left(\frac{g(u)}{f(u)}\right)\frac{\psi_2(x_1,u)f(u)}{\int \psi_2(x_1,u)f(u)du}du \\
&\geq \int \left(\frac{\psi_1(x_1,u)}{\psi_2(x_1,u)}\right)\left(\frac{g(u)}{f(u)}\right)\frac{\psi_2(x_1,u)f(u)}{\int \psi_2(x_1,u)f(u)du}du \\
&= \frac{\int \psi_1(x_1,u)g(u)du}{\int \psi_2(x_1,u)f(u)du} \\
&= \frac{E[\psi_1(x_1,Y)]}{E[\psi_2(x_1,X)]}.
\end{aligned}$$

The proof of the above inequalities follows from the similar kind of arguments used to prove inequalities (2.4) and (2.5). ■

Remark 2.2.

(a) If in Theorem 2.5, $X \geq_{lr} Y$,

(i) for each $u_1 \leq u_2$, $\frac{\psi_2(x,u_2)}{\psi_2(x,u_1)}$ is decreasing in x,

(ii) for each x, $\frac{\psi_1(x,u)}{\psi_2(x,u)}$ is decreasing in u and

(iii) for each fixed u, $\frac{\psi_1(x,u)}{\psi_2(x,u)}$ is decreasing in x,

then
$$\frac{E[\psi_1(x,Y)]}{E[\psi_2(x,X)]}$$
is increasing in x.

(b) If in Theorem 2.5, $X \leq_{lr} Y$,

(i) for each $u_1 \leq u_2$, $\frac{\psi_2(x,u_2)}{\psi_2(x,u_1)}$ is decreasing in x,

(ii) for each x, $\frac{\psi_1(x,u)}{\psi_2(x,u)}$ is increasing in u and

(iii) for each fixed u, $\frac{\psi_1(x,u)}{\psi_2(x,u)}$ is increasing in x,

then

$$\frac{E[\psi_1(x,Y)]}{E[\psi_2(x,X)]}$$

is decreasing in x.

(c) If in Theorem 2.5, $X \geq_{lr} Y$,

(i) for each $u_1 \leq u_2$, $\frac{\psi_2(x,u_2)}{\psi_2(x,u_1)}$ is increasing in x,

(ii) for each x, $\frac{\psi_1(x,u)}{\psi_2(x,u)}$ is decreasing in u and

(iii) for each fixed u, $\frac{\psi_1(x,u)}{\psi_2(x,u)}$ is decreasing in x,

then

$$\frac{E[\psi_1(x,Y)]}{E[\psi_2(x,X)]}$$

is decreasing in x.

Concluding Remark

In this paper, using a new stochastic ordering arguments, we could prove two main theorems (Theorems 2.1 and 2.5) in a simpler way that have been used by several authors to establish some stochastic ordering results. We also prove some new versions of these theorems that can effectively be used and applied to establish more stochastic inequalities among mixture distributions and among some useful statistics such as order statistics and spacings.

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