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Conditional Maximum Likelihood Estimation of the First-Order Spatial Non-Negative Integer-Valued Autoregressive (SINAR(1,1)) Model

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Abstract. Recently a first-order Spatial Integer-valued Autoregressive SINAR(1,1) model was introduced to model spatial data that comes in counts (Ghodsi et al. , 2012). Some properties of this model have been established and the Yule-Walker estimator has been proposed for this model. In this paper, we introduce the conditional maximum likelihood method for estimating the parameters of the Poisson SINAR(1,1) model. The asymptotic distribution of the estimators are also derived. The properties of the Yule-Walker and conditional maximum likelihood estimators are compared by simulation study. Finally, the Student data (Student , 1906) on the yeast cells count are used to illustrate the fitting of the SINAR(1,1) model.

Keywords. SINAR(1,1) model, Spatial integer-valued autoregressive model, Conditional maximum likelihood estimation, Binomial thinning operator.

MSC: 62M30.

1 Introduction

Time series count data are usually modelled by standard ARMA time series models when the counts are relatively large. However, when the counts are relatively small the approximation by ARMA processes are rather poor. A remedy to this problem would

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be to replace the multiplication operator in ARMA models by '$\circ$' called as *binomial thinning operator* which was introduced by Steutel and Van Harn (1979). The first time series models based on this operator was proposed by McKenzie (1985) and Al-Osh and Alzaid (1987). Al-Osh and Alzaid (1987) proposed the conditional maximum likelihood estimation method to estimate the INAR(1) model parameters. Freeland (1998) derived the score and information of the conditional maximum likelihood function for the INAR(1) model and established the asymptotic properties of the estimators. Freeland and McCabe (2004) extended these results for INAR(p) model.

Spatial data may also come in counts. For instance, the number of observed lip cancer cases in the districts of a country, neuronal cells in a tissue culture well (McShane et al. , 1997), counts of particles throughout a liquid. Count data may also arise in observational studies of ecological phenomena like weed occurrence in a farmer's field (Kruijer et al. , 2007). Kruijer et al. (2007) used hierarchical spatial models to analyze weed counts. In such models it is assumed that the conditional distribution of weed counts is poisson. The number of a special species of plants in a forest is another example of count data. A Poisson or Gaussian distribution is not always suitable for modeling integer-valued spatial process. In view of this fact, Ghodsi et al. (2012) have recently extended the INAR model from time series to the spatial lattice case, thus proposing a First-Order Spatial Integer-Valued Autoregressive SINAR(1,1) model on a regular lattice which defined as follows,

Let $\{X_{ij}, i, j \in \mathbb{Z}\}$ be a spatial non-negative integer-valued process on a two-dimensional regular lattice with mean μ_X and finite variance. The SINAR(1,1) model is given as

$$X_{ij} = \alpha_1 \circ X_{i-1,j} + \alpha_2 \circ X_{i,j-1} + \alpha_3 \circ X_{i-1,j-1} + \varepsilon_{ij}, \tag{1.1}$$

where

$$\alpha_r \circ X_r = \sum_{k_r=1}^{X_r} Y_{k_r}, \quad \{Y_{k_r}\} \overset{iid}{\sim} Bernoulli(\alpha_r),$$

in which X_1, X_2, X_3 are $X_{i-1,j}, X_{i,j-1}, X_{i-1,j-1}$, respectively. $\{Y_{k_r}, r = 1, 2, 3\}$ are mutually independent. $E(Y_{k_r}) = \alpha_r$, $Var(Y_{k_r}) = \alpha_r(1 - \alpha_r)$ and $\alpha_r \in [0, 1)$ for $r = 1, 2, 3$. $\{\varepsilon_{ij}\}$ is a sequence of i.i.d. non-negative integer-valued random variables having finite mean μ_ε and variance σ^2_ε, independent of $\{Y_{k_r}\}$ for $r = 1, 2, 3$. The sequence $\{\varepsilon_{ij}\}$ are also independent of $X_{i-k,j-l}$ for all $k \geq 1$ or $l \geq 1$. Note that, since $\mu_X = \frac{\mu_\varepsilon}{1-\alpha_1-\alpha_2-\alpha_3}$ (see Proposition 1 in Ghodsi et al. (2012)) , the restriction $\alpha_1+\alpha_2+\alpha_3 < 1$ is necessary because the quantity μ_X must be positive and this can only happen if $\alpha_1+\alpha_2+\alpha_3 < 1$.

Noting that the unilateral model is considered for simplifying the asymptotic properties (see Whittle (1954) ; Tjϕstheim (1978) and Tjϕstheim (1983)). Some properties of this model have been studied byGhodsi et al. (2012) and the Yule-Walker estimator has been proposed for this model. The Yule-Walker estimator of the parameters α_1, α_2 and α_3 are given as,

$$\hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{P}}^{-1}\hat{\boldsymbol{\rho}}, \tag{1.2}$$

where $\hat{\boldsymbol{\alpha}}' = (\hat{\alpha_1}, \hat{\alpha_2}, \hat{\alpha_3})$, $\hat{\boldsymbol{\rho}}' = (\hat{\rho}(1,0), \hat{\rho}(0,1), \hat{\rho}(1,1))$ and

$$\hat{\boldsymbol{P}} = \begin{pmatrix} 1 & \hat{\rho}(1,-1) & \hat{\rho}(0,1) \\ \hat{\rho}(1,-1) & 1 & \hat{\rho}(1,0) \\ \hat{\rho}(0,1) & \hat{\rho}(1,0) & 1 \end{pmatrix},$$

noting that $\hat{\rho}(k,l) = \hat{\gamma}(k,l)/\hat{\gamma}(0,0)$, in which $\hat{\gamma}(k,l)$ for a given data set $\left\{X_{ij}; i = 1, ..., n_1 \quad \& \quad j = 1, ..., n_2\right\}$ is given by,

$$\hat{\gamma}(k,l) = \frac{1}{n_1 n_2} \sum_i \sum_j (X_{ij} - \bar{X})(X_{i+k,j+l} - \bar{X}), \tag{1.3}$$

where $\max\{1, 1-k\} < i < \min\{n_1, n_1 - k\}, \max\{1, 1-l\} < j < \min\{n_2, n_2 - l\}$ and $k, l \in \mathbb{Z}$. μ_ε and σ_ε^2 can be estimated as follows,

$$\hat{\mu}_\varepsilon = \bar{X}(1 - \hat{\alpha}_1 - \hat{\alpha}_2 - \hat{\alpha}_3) \tag{1.4}$$

$$\hat{\sigma}_\varepsilon^2 = \hat{\gamma}(0,0)\left\{1 - (\hat{\alpha}_1 + \hat{\alpha}_2\hat{\alpha}_3)\hat{\lambda} - (\hat{\alpha}_2 + \hat{\alpha}_1\hat{\alpha}_3)\hat{\eta} - \hat{\alpha}_3^2\right\} - \bar{X}\sum_{i=1}^{3} \hat{\alpha}_i(1 - \hat{\alpha}_i), \tag{1.5}$$

where $\bar{X} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} X_{ij}/(n_1 n_2)$ and $0 \leq \hat{\lambda} < 1$ and $0 \leq \hat{\eta} < 1$ are obtained using folloing equations,

$$\hat{\lambda} = \frac{(1 + \hat{\alpha}_1^2 - \hat{\alpha}_2^2 - \hat{\alpha}_3^2) - \sqrt{(1 + \hat{\alpha}_1^2 - \hat{\alpha}_2^2 - \hat{\alpha}_3^2)^2 - 4(\hat{\alpha}_1 + \hat{\alpha}_2\hat{\alpha}_3)^2}}{2(\hat{\alpha}_1 + \hat{\alpha}_2\hat{\alpha}_3)} \tag{1.6}$$

and

$$\hat{\eta} = \frac{\hat{\alpha}_2 + \hat{\alpha}_3\hat{\lambda}}{1 - \hat{\alpha}_1\hat{\lambda}}, \tag{1.7}$$

provided $\hat{\alpha}_1 + \hat{\alpha}_2\hat{\alpha}_3 \neq 0$. If $\hat{\alpha}_1 + \hat{\alpha}_2\hat{\alpha}_3 = 0$ then $\hat{\lambda} = 0$ and $\hat{\eta} = \hat{\alpha}_2$.

The SINAR(1,1) model as defined in (1.1) is termed as Poisson SINAR(1,1) model if $\varepsilon_{ij} \sim P(\lambda)$.

The aim of this paper is to extend the results of Freeland and McCabe (2004) in time series to the spatial case. We propose the conditional maximum likelihood (CML) estimation method for estimating the parameters of the Poisson SINAR(1,1) model (1.1) and establish the asymptotic properties of the CML estimators.

2 Conditional Maximum Likelihood Estimation of the SINAR(1,1) Model

The conditional likelihood function of the Poisson SINAR(1,1) model is defined as

$$L(\boldsymbol{\theta}) = \prod_{i=2}^{n_1} \prod_{j=2}^{n_2} P(X_{ij}|\boldsymbol{X}_{ij}^{-}),$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)' = (\alpha_1, \alpha_2, \alpha_3, \lambda)'$, $\boldsymbol{X}_{ij}^{-} = (X_{i-1,j}, X_{i,j-1}, X_{i-1,j-1})$ and $P(X_{ij}|\boldsymbol{X}_{ij}^{-})$ has been derived in Proposition (2.1).

Proposition 2.1. *For the Poisson SINAR(1,1) model the conditional distribution of X_{ij} given $\boldsymbol{X}_{ij}^{-} = (X_{i-1,j}, X_{i,j-1}, X_{i-1,j-1})$ is as follows.*

$$P(X_{ij}|\boldsymbol{X}_{ij}^{-}) = \sum_{s_1=0}^{min\{X_{i-1,j},X_{ij}\}} \sum_{s_2=0}^{min\{X_{i,j-1},X_{ij}-s_1\}} \sum_{s_3=0}^{min\{X_{i-1,j-1},X_{ij}-s_1-s_2\}},$$
$$p_{\alpha_1}(s_1|X_{i-1,j}) p_{\alpha_2}(s_2|X_{i,j-1}) p_{\alpha_3}(s_3|X_{i-1,j-1}) g(X_{ij} - s_1 - s_2 - s_3),$$

where

$$p_{\alpha}(s|X) = \binom{X}{s} \alpha^s (1-\alpha)^{X-s}; \quad s = 0, 1, \ldots, X$$

,

$$g(\varepsilon) = \frac{e^{-\lambda}\lambda^{\varepsilon}}{\varepsilon!}; \qquad \varepsilon = 0, 1, \ldots$$

Proof. Let $Y_{ij} = X_{ij}|\boldsymbol{X}_{ij}^{-}$ and $B_r^{(\alpha_r)} = \alpha_r \circ X_r|\boldsymbol{X}_{ij}^{-}$ where $B_r^{(\alpha_r)} \sim Binomial(X_r, \alpha_r)$ for $r = 1, 2, 3$. As ε_{ij} is independent of $\boldsymbol{X}_{ij}^{-}$, using (1.1) we can write

$$Y_{ij} = B_1^{(\alpha_1)} + B_2^{(\alpha_2)} + B_3^{(\alpha_3)} + \varepsilon_{ij}. \tag{2.1}$$

It is obvious that $B_1^{(\alpha_1)}, B_2^{(\alpha_2)}, B_3^{(\alpha_3)}$ and ε_{ij} are mutually independent. Therefore by letting $S_1 = B_1^{(\alpha_1)}, S_2 = B_2^{(\alpha_2)}$ and $S_3 = B_3^{(\alpha_3)}$, the joint distribution of Y_{ij}, S_1, S_2 and S_3 is given as,

$$f_{Y_{ij},S_1,S_2,S_3}(y_{ij}, s_1, s_2, s_3) = f_{S_1}(s_1)f_{S_2}(s_2)f_{S_3}(s_3)g(y_{ij} - s_1 - s_2 - s_3),$$

where $0 \leq s_1 \leq X_{i-1,j}$, $0 \leq s_2 \leq X_{i,j-1}$, $0 \leq s_3 \leq X_{i-1,j-1}$. Note that $\varepsilon_{ij} \geq 0$ so from (2.1) we have $s_1 + s_2 + s_3 \leq X_{ij}$.
The marginal distribution of Y_{ij} is obtained as follows,

$$\begin{aligned} f_{Y_{ij}}(y_{ij}) &= P(X_{ij}|\boldsymbol{X}_{ij}^{-}) \\ &= \sum_{s_1=0}^{min\{X_{i-1,j},X_{ij}\}} \sum_{s_2=0}^{min\{X_{i,j-1},X_{ij}-s_1\}} \sum_{s_3=0}^{min\{X_{i-1,j-1},X_{ij}-s_1-s_2\}} \\ & \quad f_{Y_{ij},S_1,S_2,S_3}(y_{ij}, s_1, s_2, s_3) \end{aligned}$$

which complets the proof. □

Note that, the conditional log-likelihood of the Poisson SINAR(1,1) model is given by

$$\ell(\boldsymbol{\theta}) = \ln L(\boldsymbol{\theta}) = \sum_{i=2}^{n_1} \sum_{j=2}^{n_2} \ln P(X_{ij}|\boldsymbol{X}_{ij}^{-})$$

The score vector of the $L(\boldsymbol{\theta})$ is defined as

$$\dot{\boldsymbol{\ell}}_{\boldsymbol{\theta}} = (\dot{\ell}_{\theta_1}, \dot{\ell}_{\theta_2}, \dot{\ell}_{\theta_3}, \dot{\ell}_{\theta_4})',$$

where

$$\begin{aligned} \dot{\ell}_{\theta_r} &= \frac{\partial}{\partial \theta_r} \ell(\boldsymbol{\theta}) \\ &= \sum_{i=2}^{n_1} \sum_{j=2}^{n_2} \frac{\partial}{\partial \theta_r} \ln P(X_{ij}|\boldsymbol{X}_{ij}^{-}) \\ &= \sum_{i=2}^{n_1} \sum_{j=2}^{n_2} \frac{\frac{\partial}{\partial \theta_r} P(X_{ij}|\boldsymbol{X}_{ij}^{-})}{P(X_{ij}|\boldsymbol{X}_{ij}^{-})} \end{aligned} \tag{2.2}$$

which is given in Proposition 2.2, for $r = 1, 2, 3, 4$.

Proposition 2.2. *The first derivatives of the conditional log-likelihood function of the Poisson SINAR(1,1) model with respect to the parameters are given as*

$$\begin{aligned}
\dot{\ell}_{\alpha_1} &= \frac{1}{1-\alpha_1}\sum_{i=2}^{n_1}\sum_{j=2}^{n_2} X_{i-1,j}(P_{ij}(1,1,0,0)-1),\\
\dot{\ell}_{\alpha_2} &= \frac{1}{1-\alpha_2}\sum_{i=2}^{n_1}\sum_{j=2}^{n_2} X_{i,j-1}(P_{ij}(1,0,1,0)-1),\\
\dot{\ell}_{\alpha_3} &= \frac{1}{1-\alpha_3}\sum_{i=2}^{n_1}\sum_{j=2}^{n_2} X_{i-1,j-1}(P_{ij}(1,0,0,1)-1),\\
\dot{\ell}_{\lambda} &= \sum_{i=2}^{n_1}\sum_{j=2}^{n_2}(P_{ij}(1,0,0,0)-1),
\end{aligned}$$

where

$$P_{ij}(d_1,d_2,d_3,d_4) = \frac{P(X_{ij}-d_1|X_{i-1,j}-d_2, X_{i,j-1}-d_3, X_{i-1,j-1}-d_4)}{P(X_{ij}|\mathbf{X}_{ij}^{-})}, \tag{2.3}$$

and d_1, d_2, d_3, d_4 *can be* 0 *or* 1.

Proof. (Hint) The first derivatives can be easily obtained by making use of

$$\frac{\partial}{\partial\alpha}\left\{\binom{X}{s}\alpha^s(1-\alpha)^{X-s}\right\} = \left(\frac{s}{\alpha(1-\alpha)} - \frac{X}{1-\alpha}\right)\binom{X}{s}\alpha^s(1-\alpha)^{X-s},$$

$$s\binom{X}{s} = X\binom{X-1}{s-1},$$

$$\frac{\partial}{\partial\lambda}g(\varepsilon;\lambda) = \left(\frac{\varepsilon}{\lambda}-1\right)g(\varepsilon;\lambda)$$

and see the proof of Freeland and McCabe (2004) in one dimension. □

The information matrix also is defined as

$$\ddot{\boldsymbol{\ell}}_{\boldsymbol{\theta}} = [\ddot{\ell}_{\theta_{r_1}\theta_{r_2}}]_{4\times 4},$$

where

$$
\begin{aligned}
\ddot{\ell}_{\theta_{r_1}\theta_{r_2}} &= \frac{\partial^2}{\partial\theta_{r_1}\partial\theta_{r_2}}\ell(\boldsymbol{\theta}) \\
&= \sum_{i=2}^{n_1}\sum_{j=2}^{n_2}\frac{\partial^2}{\partial\theta_{r_1}\partial\theta_{r_2}}\ln P(X_{ij}|\boldsymbol{X}_{ij}^-) \\
&= \sum_{i=2}^{n_1}\sum_{j=2}^{n_2}[\frac{\frac{\partial^2}{\partial\theta_{r_1}\partial\theta_{r_2}}P(X_{ij}|\boldsymbol{X}_{ij}^-)}{P(X_{ij}|\boldsymbol{X}_{ij}^-)} \\
&- \{\frac{\partial}{\partial\theta_{r_1}}\ln P(X_{ij}|\boldsymbol{X}_{ij}^-)\}\{\frac{\partial}{\partial\theta_{r_2}}\ln P(X_{ij}|\boldsymbol{X}_{ij}^-)\}]
\end{aligned} \tag{2.4}
$$

which can be obtained using Proposition 2.3 for $r_1, r_2 = 1, 2, 3, 4$.

Proposition 2.3. *The second derivatives of the conditional log-likelihood function of the Poisson SINAR(1,1) model with respect to the parameters are given as*

$$
\begin{aligned}
\ddot{\ell}_{\alpha_1\alpha_1} &= \frac{1}{(1-\alpha_1)^2}\sum_{i=2}^{n_1}\sum_{j=2}^{n_1}X_{i-1,j}((X_{i-1,j}-1)P_{ij}(2,2,0,0), \\
&\quad -X_{i-1,j}P_{ij}^2(1,1,0,0) + 2P_{ij}(1,1,0,0) - 1), \\
\ddot{\ell}_{\alpha_2\alpha_2} &= \frac{1}{(1-\alpha_2)^2}\sum_{i=2}^{n_1}\sum_{j=2}^{n_1}X_{i,j-1}((X_{i,j-1}-1)P_{ij}(2,0,2,0), \\
&\quad -X_{i,j-1}P_{ij}^2(1,0,1,0) + 2P_{ij}(1,0,1,0) - 1), \\
\ddot{\ell}_{\alpha_3\alpha_3} &= \frac{1}{(1-\alpha_3)^2}\sum_{i=2}^{n_1}\sum_{j=2}^{n_1}X_{i-1,j-1}((X_{i-1,j-1}-1)P_{ij}(2,0,0,2), \\
&\quad -X_{i-1,j-1}P_{ij}^2(1,0,0,1) + 2P_{ij}(1,0,0,1) - 1), \\
\ddot{\ell}_{\lambda\lambda} &= \sum_{i=2}^{n_1}\sum_{j=2}^{n_1}(P_{ij}(2,0,0,0) - P_{ij}^2(1,0,0,0)),
\end{aligned}
$$

$$\ddot{\ell}_{\alpha_1\alpha_2} = \frac{1}{(1-\alpha_1)(1-\alpha_2)}\sum_{i=2}^{n_1}\sum_{j=2}^{n_1} X_{i-1,j}X_{i,j-1}(P_{ij}(2,1,1,0),$$
$$-P_{ij}(1,1,0,0)P_{ij}(1,0,1,0)),$$
$$\ddot{\ell}_{\alpha_1\alpha_3} = \frac{1}{(1-\alpha_1)(1-\alpha_3)}\sum_{i=2}^{n_1}\sum_{j=2}^{n_1} X_{i-1,j}X_{i-1,j-1}(P_{ij}(2,1,0,1),$$
$$-P_{ij}(1,1,0,0)P_{ij}(1,0,0,1))$$

$$\ddot{\ell}_{\alpha_2\alpha_3} = \frac{1}{(1-\alpha_2)(1-\alpha_3)}\sum_{i=2}^{n_1}\sum_{j=2}^{n_1} X_{i,j-1}X_{i-1,j-1}(P_{ij}(2,0,1,1),$$
$$-P_{ij}(1,0,1,0)P_{ij}(1,0,0,1)),$$
$$\ddot{\ell}_{\alpha_1\lambda} = \frac{1}{1-\alpha_1}\sum_{i=2}^{n_1}\sum_{j=2}^{n_1} X_{i-1,j}(P_{ij}(2,1,0,0)-P_{ij}(1,1,0,0)P_{ij}(1,0,0,0)),$$
$$\ddot{\ell}_{\alpha_2\lambda} = \frac{1}{1-\alpha_2}\sum_{i=2}^{n_1}\sum_{j=2}^{n_1} X_{i,j-1}(P_{ij}(2,0,1,0)-P_{ij}(1,0,1,0)P_{ij}(1,0,0,0)),$$
$$\ddot{\ell}_{\alpha_3\lambda} = \frac{1}{1-\alpha_3}\sum_{i=2}^{n_1}\sum_{j=2}^{n_1} X_{i-1,j-1}(P_{ij}(2,0,0,1)-P_{ij}(1,0,0,1)P_{ij}(1,0,0,0)),$$

where $P_{ij}(d_1,d_2,d_3,d_4)$ *is defined as* (2.3) *with* $d_1,d_2,d_3,d_4=0,1$ *or* 2.

(Hint) The second derivatives can be easily obtained by making use of

$$s^2 = s(s-1)+s,$$
$$s(s-1)\binom{X}{s} = X(X-1)\binom{X-2}{s-2},$$

and see the proof of Freeland and McCabe (2004) in one dimension.

2.1 Asymptotic Properties

To establish the asymptotic properties of the conditional maximum likelihood estimators of the SINAR(1,1) model the following lemmas are needed.

Lemma 2.1. *The SINAR(1,1) model* (1.1) *is an ergodic process.*

Let $\{Y(i,j), i = 1, ..., n_1 \quad \& \quad j = 1, ..., n_2\}$ be all Bernoulli sequences $\{Y_{k_1}\}$,$\{Y_{k_2}\}$ and $\{Y_{k_3}\}$ in $\alpha_1 \circ X_{i-1,j} + \alpha_2 \circ X_{i,j-1} + \alpha_3 \circ X_{i-1,j-1}$ where $k_1 = 1, 2, \ldots, X_{i-1,j}$, $k_2 = 1, 2, \ldots, X_{i,j-1}$ and $k_3 = 1, 2, \ldots, X_{i-1,j-1}$.
Since, from (1.1), $X_{i,j}$ is a function of $\varepsilon_{i,j}$ and $Y(i,j)$, we can write

$$\sigma\{X_{i,j}, X_{i-1,j}, X_{i,j-1}, X_{i-1,j-1}, \ldots\} \subset \sigma\{\varepsilon_{i,j}, Y(i,j), \varepsilon_{i-1,j}, Y(i-1,j), \ldots\},$$

where $\sigma\{X_{i,j}, X_{i-1,j}, X_{i,j-1}, X_{i-1,j-1}, \ldots\}$ is the σ-field generated by $\{X_{i,j}, X_{i-1,j}, X_{i,j-1}, X_{i-1,j-1}, \ldots\}$. Hence

$$\begin{aligned} &\bigcap_{i=1}^{\infty}\bigcap_{j=1}^{\infty} \sigma\{X_{i,j}, X_{i-1,j}, X_{i,j-1}, X_{i-1,j-1}, \ldots\} \\ &\subset \bigcap_{i=1}^{\infty}\bigcap_{j=1}^{\infty} \sigma\{\varepsilon_{i,j}, Y(i,j), \varepsilon_{i-1,j}, Y(i-1,j), \ldots\} \end{aligned} \tag{2.5}$$

The right-hand side of (2.5) is a tail σ-field of the process $\{\varepsilon_{i,j}, Y(i,j)\}$. Since $\varepsilon_{i,j}$*'s* are independent and are independent of $Y(i,j)$*'s* and since $Y(i,j)$*'s* are mutually independent for all i, j, then $\{\varepsilon_{i,j}, Y(i,j)\}$ is an independent random process. From Kolmogrov's zero-one law we know that all events in this tail σ-field are trivial (see (See, Breiman , 1992, Chapter 3)). So every set in the tail σ-field of $\{X_{i,j}\}$ (the set on left hand side of (2.5)) has probability 0 or 1, then stationary process $\{X_{i,j}\}$ is ergodic (Breiman , 1992, Chapter 6, Definition 6.30 and Proposition 6.32).

Lemma 2.2. *The scores of the conditional likelihood function of the Poisson SINAR(1,1) model are martingale sequences.*

Proof. Let

$$\begin{aligned} \mathcal{F}_{i,j} &= \sigma\{X_{k,\ell} | (k,\ell) \preceq (i,j)\} \\ &= \sigma(X_{11}, \ldots, X_{1n_2}, X_{21}, \ldots, X_{2n_2}, \ldots, X_{i-1,1}, \ldots, X_{i-1n_2}, X_{i1}, \ldots, X_{ij}), \end{aligned}$$

where $\preceq$ notes the lexicographic order on $\{1, 2, \ldots, n_1\} \times \{1, 2, \ldots, n_2\}$, i.e. $(k,\ell) \preceq (i,j)$ if and only if either $k < i$ or $k = i$ and $\ell \leq j$. By noting

$$U_{ij}^{(\theta)} = \frac{\partial}{\partial\theta} \ln P(X_{ij} | \mathbf{X}_{ij}^{-}), \tag{2.6}$$

where θ can be either $\alpha_1, \alpha_2, \alpha_3$ or λ, for any $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$ we have

$$\begin{aligned} E\{U_{ij}^{(\theta)}|\mathcal{F}_{i,j-1}\} &= E\{\frac{\partial}{\partial\theta}\ln P(X_{ij}|\boldsymbol{X}_{ij}^{-})|\mathcal{F}_{i,j-1}\} \\ &= \sum_{X_{ij}=0}^{\infty}\{\frac{\partial}{\partial\theta}\ln P(X_{ij}|\boldsymbol{X}_{ij}^{-})\}P(X_{ij}|\boldsymbol{X}_{ij}^{-}) \\ &= \sum_{X_{ij}=0}^{\infty}\frac{\partial}{\partial\theta}P(X_{ij}|\boldsymbol{X}_{ij}^{-}) \\ &= \frac{\partial}{\partial\theta}(1) \\ &= 0, \end{aligned} \tag{2.7}$$

which shows that $\{U_{ij}^{(\theta)}\}$ is a martingale difference sequence. The score $\dot{\ell}_\theta$ given in (2.2) can be rewritten as

$$S_{n_1n_2}^{(\theta)} = \sum_{i=2}^{n_1}\sum_{j=2}^{n_1} U_{ij}^{(\theta)}.$$

Using (2.7) we can write

$$\begin{aligned} 0 &= E(U_{n_1,n_2}^{(\theta)}|\mathcal{F}_{n_1,n_2-1}) \\ &= E(S_{n_1n_2}^{(\theta)} - S_{n_1n_2-1}^{(\theta)}|\mathcal{F}_{n_1,n_2-1}) \\ &= E(S_{n_1n_2}^{(\theta)}|\mathcal{F}_{n_1,n_2-1}) - E(S_{n_1n_2-1}^{(\theta)}|\mathcal{F}_{n_1,n_2-1}) \\ &= E(S_{n_1n_2}^{(\theta)}|\mathcal{F}_{n_1,n_2-1}) - S_{n_1n_2-1}, \end{aligned}$$

which implies that

$$E(S_{n_1n_2}^{(\theta)}|\mathcal{F}_{n_1,n_2-1}) = S_{n_1n_2-1}.$$

Therefore $\{S_{n_1n_2}^{(\theta)}\}$ is a martingale sequence. □

Let

$$\boldsymbol{U}_{ij} = \frac{\partial}{\partial\boldsymbol{\theta}}\ln P(X_{ij}|\boldsymbol{X}_{ij}^{-}) = (U_{ij}^{(\alpha_1)}, U_{ij}^{(\alpha_2)}, U_{ij}^{(\alpha_3)}, U_{ij}^{(\lambda)})' \tag{2.8}$$

and

$$\boldsymbol{V}_{ij} = \frac{\partial^2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\ln P(X_{ij}|\boldsymbol{X}_{ij}^{-}) = \left[\frac{\partial^2}{\partial\theta_{r_1}\partial\theta_{r_2}}\ln P(X_{ij}|\boldsymbol{X}_{ij}^{-})\right]_{4\times4}. \tag{2.9}$$

Lemma 2.3. *For the Poisson SINAR(1,1) model if $E(X_{ij}^2) < \infty$, then $Var(\boldsymbol{b}'\boldsymbol{U}_{ij})$ is finite where $\boldsymbol{b} = (b_1, b_2, b_3, b_4)'$ is any four dimensional real valued vector.*

Proof. First we note that

$$U_{ij}^{(\theta)} \leq \frac{X_{ij}}{\theta(1-\theta)}, \qquad \text{for} \quad \theta = \alpha_1, \alpha_2, \alpha_3$$

and

$$U_{ij}^{(\lambda)} \leq \frac{X_{ij}}{\theta}, \qquad \text{for} \quad \theta = \lambda$$

We prove the first expression for $\theta = \alpha_1$. The proof for $\theta = \alpha_2, \alpha_3$ and λ is similar. Using proposition 2.1 and 2.2 we have

$$\begin{aligned}
U_{ij}^{(\alpha_1)} &= \frac{1}{1-\alpha_1} X_{i-1,j}(P_{ij}(1,1,0,0) - 1) \\
&\leq \frac{1}{1-\alpha_1} X_{i-1,j} P_{ij}(1,1,0,0) \\
&= \frac{X_{i-1,j}}{(1-\alpha_1)P(X_{ij}|\boldsymbol{X}_{ij}^-)} \sum_{s_1=0}^{min\{X_{i-1,j}-1, X_{ij}-1\}} \sum_{s_2=0}^{min\{X_{i,j-1}, X_{ij}-1-s_1\}} \\
&\quad \sum_{s_3=0}^{min\{X_{i-1,j-1}, X_{ij}-1-s_1-s_2\}} p(s_1|X_{i-1,j}-1)p(s_2|X_{i,j-1})p(s_3|X_{i-1,j-1}) \\
&\quad g(X_{ij} - 1 - s_1 - s_2 - s_3) \\
&= \frac{X_{i-1,j}}{(1-\alpha_1)P(X_{ij}|\boldsymbol{X}_{ij}^-)} \sum_{s_1=1}^{min\{X_{i-1,j}, X_{ij}\}} \sum_{s_2=0}^{min\{X_{i,j-1}, X_{ij}-s_1\}} \\
&\quad \sum_{s_3=0}^{min\{X_{i-1,j-1}, X_{ij}-s_1-s_2\}} p(s_1 - 1|X_{i-1,j}-1)p(s_2|X_{i,j-1})p(s_3|X_{i-1,j-1}) \\
&\quad g(X_{ij} - s_1 - s_2 - s_3)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{X_{i-1,j}}{(1-\alpha_1)P(X_{ij}|\boldsymbol{X}_{ij}^{-})} \sum_{s_1=1}^{min\{X_{i-1,j},X_{ij}\}} \sum_{s_2=0}^{min\{X_{i,j-1},X_{ij}-s_1\}} \\
&\quad \sum_{s_3=0}^{min\{X_{i-1,j-1},X_{ij}-s_1-s_2\}} \binom{X_{i-1,j}-1}{s_1-1} \alpha_1^{s_1-1}(1-\alpha_1)^{X_{i-1,j}-s_1} \\
&\quad p(s_2|X_{i,j-1})p(s_3|X_{i-1,j-1})g(X_{ij}-s_1-s_2-s_3) \\
&= \frac{1}{\alpha(1-\alpha_1)P(X_{ij}|\boldsymbol{X}_{ij}^{-})} \sum_{s_1=1}^{min\{X_{i-1,j},X_{ij}\}} \sum_{s_2=0}^{min\{X_{i,j-1},X_{ij}-s_1\}} \\
&\quad \sum_{s_3=0}^{min\{X_{i-1,j-1},X_{ij}-s_1-s_2\}} s_1 \binom{X_{i-1,j}}{s_1} \alpha_1^{s_1}(1-\alpha_1)^{X_{i-1,j}-s_1} \\
&\quad p(s_2|X_{i,j-1})p(s_3|X_{i-1,j-1})g(X_{ij}-s_1-s_2-s_3) \\
&\leq \frac{min\{X_{i-1,j},X_{ij}\}}{\alpha(1-\alpha_1)P(X_{ij}|\boldsymbol{X}_{ij}^{-})} \sum_{s_1=0}^{min\{X_{i-1,j},X_{ij}\}} \sum_{s_2=0}^{min\{X_{i,j-1},X_{ij}-s_1\}} \\
&\quad \sum_{s_3=0}^{min\{X_{i-1,j-1},X_{ij}-s_1-s_2\}} \binom{X_{i-1,j}}{s_1} \alpha_1^{s_1}(1-\alpha_1)^{X_{i-1,j}-s_1} \\
&\quad p(s_2|X_{i,j-1})p(s_3|X_{i-1,j-1})g(X_{ij}-s_1-s_2-s_3) \\
&\leq \frac{X_{ij}}{\alpha(1-\alpha_1)P(X_{ij}|\boldsymbol{X}_{ij}^{-})} P(X_{ij}|\boldsymbol{X}_{ij}^{-}) \\
&= \frac{X_{ij}}{\alpha_1(1-\alpha_1)}.
\end{aligned}
$$

Now let $E(X_{ij}^2) < \infty$. Then,

$$\begin{aligned}
E\{\boldsymbol{b}'\boldsymbol{U}_{ij}\}^2 &= E\{\sum_{r=1}^{4} b_r U_{ij}^{(\theta_r)}\}^2 \\
&= E\{\sum_{r=1}^{4}\sum_{s=1}^{4} b_r b_s U_{ij}^{(\theta_r)} U_{ij}^{(\theta_s)}\} \\
&\leq E\{\sum_{r=1}^{4}\sum_{s=1}^{4} |a_r a_s| X_{ij}^2\} \\
&\leq \sum_{r=1}^{4}\sum_{s=1}^{4} |a_r a_s| E(X_{ij}^2) \\
&< \infty,
\end{aligned}$$

which implies that $Var(\boldsymbol{b}'\boldsymbol{U}_{ij}) < \infty$. □

Lemma 2.4. *$Var(\boldsymbol{b}'\boldsymbol{U}_{ij}) = 0$ iff $\boldsymbol{b} = 0$.*

Proof. We know, for non-zero $\boldsymbol{b}$, that $Var(\boldsymbol{b}'\boldsymbol{U}_{ij}) = 0$ iff $\boldsymbol{b}'\boldsymbol{U}_{ij} = c$ for some constant c with probability one. From proposition 2.2 we have

$$\begin{aligned}
\boldsymbol{b}'\boldsymbol{U}_{ij} &= b_1 \frac{1}{1-\alpha_1} X_{i-1,j} P_{ij}(1,1,0,0) + b_2 \frac{1}{1-\alpha_2} X_{i,j-1} P_{ij}(1,0,1,0) \\
&\quad + b_3 \frac{1}{1-\alpha_3} X_{i-1,j-1} P_{ij}(1,0,0,1) + b_4 P_{ij}(1,0,0,0) \\
&= c \qquad (2.10)
\end{aligned}$$

Taking $(X_{i,j}, X_{i-1,j}, X_{i,j-1}, X_{i-1,j-1}) = (2,0,0,0), (1,0,0,0), (1,1,0,0),$ $(1,0,1,0), (1,0,0,1)$ in (2.10) we get,

$$b_4 \frac{P(1|0,0,0)}{P(2|0,0,0)} = \frac{2b_4}{\lambda} = c, \qquad (2.11)$$

$$b_4 \frac{P(0|0,0,0)}{P(1|0,0,0)} = \frac{b_4}{\lambda} = c, \qquad (2.12)$$

$$\frac{b_1}{1-\alpha_1} \frac{P(0|0,0,0)}{P(1|1,0,0)} + b_4 \frac{P(0|1,0,0)}{P(1|1,0,0)} = c, \qquad (2.13)$$

$$\frac{b_2}{1-\alpha_2} \frac{P(0|0,0,0)}{P(1|0,1,0)} + b_4 \frac{P(0|0,1,0)}{P(1|0,1,0)} = c, \qquad (2.14)$$

$$\frac{b_3}{1-\alpha_3} \frac{P(0|0,0,0)}{P(1|0,0,1)} + b_4 \frac{P(0|0,0,1)}{P(1|0,0,1)} = c. \qquad (2.15)$$

Eqs. (2.11) and (2.12) yeild $b_4 = c = 0$. By noting $P(x_1|x_2, x_3, x_4)$ is positive for $x_1, x_2, x_3, x_4 \geq 0$ and replacing $b_4 = c = 0$ in (2.13-2.15) we obtain $b_1 = b_2 = b_3 = 0$ which complets the proof. □

In the next lemma we show that the Fisher information matrix is positive definite and finite which implies the non-singularity of Fisher information matrix.

Lemma 2.5. *Fisher information matrix of the conditional maximum likelihood function of the Poisson SINAR(1,1) model* (1.1) *is positive definite and finite.*

Proof. Using Eqs. (2.2, 2.4, 2.8) and (2.9) we can write $\dot{\boldsymbol{\ell}}_{\boldsymbol{\theta}} = \sum_{i=2}^{n_1}\sum_{j=2}^{n_2} \boldsymbol{U}_{ij}$ and $\ddot{\boldsymbol{\ell}}_{\boldsymbol{\theta}} = \sum_{i=2}^{n_1}\sum_{j=2}^{n_2} \boldsymbol{V}_{ij}$. Since $E(\dot{\ell}_\theta) = EE(\dot{\ell}_\theta|\mathcal{F}_{i,j-1}) = 0$ (see Eq. 2.7), the Fisher information matrix can be written as

$$\begin{aligned} \boldsymbol{I}(\boldsymbol{\theta}) &= Var(\dot{\boldsymbol{\ell}}_{\boldsymbol{\theta}}) \\ &= E(\dot{\boldsymbol{\ell}}_{\boldsymbol{\theta}}\dot{\boldsymbol{\ell}}'_{\boldsymbol{\theta}}) \\ &= -E(\ddot{\boldsymbol{\ell}}_{\boldsymbol{\theta}}) \\ &= -\sum_{i=2}^{n_1}\sum_{j=2}^{n_2} E(\boldsymbol{V}_{ij}) \\ &= \sum_{i=2}^{n_1}\sum_{j=2}^{n_2} E(\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij}) \\ &= \sum_{i=2}^{n_1}\sum_{j=2}^{n_2} Var(\boldsymbol{U}_{ij}). \end{aligned}$$

Therefore $\boldsymbol{b}'\boldsymbol{I}(\boldsymbol{\theta})\boldsymbol{b} = \boldsymbol{b}'\sum_{i=2}^{n_1}\sum_{j=2}^{n_2} Var(\boldsymbol{U}_{ij})\boldsymbol{b} = \sum_{i=2}^{n_1}\sum_{j=2}^{n_2} Var(\boldsymbol{b}'\boldsymbol{U}_{ij})$ which by Lemmas 2.4 and 2.5 is finite and positive for any non-zero four dimensional real valued vector $\boldsymbol{b}$. □

Proposition 2.4. *Let* $\hat{\boldsymbol{\theta}}$ *be the maximum likelihood estimator of the parameter vector* $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)' = (\alpha_1, \alpha_2, \alpha_3, \lambda)'$ *of the Poisson SINAR(1,1) model* (1.1). *Then,*

$$\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} \xrightarrow{d} N(\boldsymbol{0}, \boldsymbol{I}^{-1}(\boldsymbol{\theta})),$$

where

$$\boldsymbol{I}(\boldsymbol{\theta}) = [i_{r_1,r_2}]_{4\times4},$$

$$i_{r_1,r_2} = -E(\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \theta_{r_1} \partial \theta_{r_2}}) = -E(\ddot{\ell}_{\theta_{r_1}\theta_{r_2}}) \qquad r_1, r_2 = 1,2,3,4$$

and $\ddot{\ell}_{\theta_{r_1}\theta_{r_2}}$ *has been derived in Proposition* (2.3).

Proof. For some $\bar{\boldsymbol{\theta}} \in [\hat{\boldsymbol{\theta}}, \boldsymbol{\theta_0}]$ which $\boldsymbol{\theta_0}$ is the true value of $\boldsymbol{\theta}$, using vector form of the Mean Value Theorem we have

$$0 = \dot{\boldsymbol{\ell}}_{\hat{\boldsymbol{\theta}}} = \dot{\boldsymbol{\ell}}_{\boldsymbol{\theta}_0} + \ddot{\boldsymbol{\ell}}_{\bar{\boldsymbol{\theta}}}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta_0})$$

Then, we can get

$$\hat{\boldsymbol{\theta}} - \boldsymbol{\theta_0} = -\ddot{\boldsymbol{\ell}}_{\bar{\boldsymbol{\theta}}}^{-1} \dot{\boldsymbol{\ell}}_{\boldsymbol{\theta}_0},$$

assuming $\ddot{\boldsymbol{\ell}}_{\bar{\boldsymbol{\theta}}}$ is non-singular. Since the scores and informations are functions of the process $\{X_{i,j}\}$ which is stationary and ergodic (see Lemma 2.1) it follows that these processes are also stationary and ergodic. Hence, using ergodic theorem we can conclude that the conditional maximum likelihood estimators which are obtained by solving $\dot{\boldsymbol{\ell}}_{\boldsymbol{\theta}} = \mathbf{0}$, are consistent.

To establish the distribution of $\hat{\boldsymbol{\theta}}$ we can write

$$\hat{\boldsymbol{\theta}} - \boldsymbol{\theta_0} = -\ddot{\boldsymbol{\ell}}_{\bar{\boldsymbol{\theta}}}^{-1} \boldsymbol{I}(\boldsymbol{\theta_0}) \boldsymbol{I}^{-1}(\boldsymbol{\theta_0}) \dot{\boldsymbol{\ell}}_{\boldsymbol{\theta}_0}$$

since $\bar{\boldsymbol{\theta}} \in [\hat{\boldsymbol{\theta}}, \boldsymbol{\theta_0}]$ and by consistency result $\bar{\boldsymbol{\theta}} \to \boldsymbol{\theta_0}$, uniform strong law af large numbers implies that $-\boldsymbol{I}^{-1}(\boldsymbol{\theta_0}) \ddot{\boldsymbol{\ell}}_{\bar{\boldsymbol{\theta}}} \xrightarrow{p} \boldsymbol{I}_4$ the 4×4 unit matrix. Also, using the martingale central limit theorem of Brown (1971) and Scott (1973) (see also Crowder (1976) and Bu and Hadri (2006)) and Cramer and Wold (1936) device $\boldsymbol{I}^{-1}(\boldsymbol{\theta_0}) \dot{\boldsymbol{\ell}}_{\boldsymbol{\theta}_0}$ is asymptotically normal as $N_4(\mathbf{0}, \boldsymbol{I}^{-1}(\boldsymbol{\theta_0}))$ which completes the proof. □

3 Simulation Results and A Real Data Example

In this section we discuss the simulation results of our study. We simulated realisations using (1.1) assuming $\varepsilon_{ij} \sim Poisson(\lambda)$, with three different sets of $\boldsymbol{\theta}' = (\alpha_1, \alpha_2, \alpha_3, \lambda)$ values. The three sets of $\boldsymbol{\theta}$-values used were (i) $\boldsymbol{\theta}' = (0.4, 0.3, 0.1, 1)$, (ii) $\boldsymbol{\theta}' = (0.2, 0.2, 0.5, 1)$ and (iii) $\boldsymbol{\theta}' = (0.1, 0.1, 0.1, 1)$. The size of generated processes are 15×15, 30×30 and 50×50. We investigate the performance of the Yule-Walker (YW) estimators (see Ghodsi et al. (2012)) and the Conditional Maximum Likelihood (CML) estimators through a simulation study. All the simulations were carried out by writing R programs.

Let s represent the number of replications. In this simulation study s is 300. The following computations were carried out from the simulation study,

1. Mean of the point estimates (Mean), $\bar{\hat{\theta}} = \sum_{i=1}^{s} \hat{\theta}_i / s$.

2. Estimated biases (Bias)= $\bar{\hat{\theta}} - \theta$.

3. Estimated stadard deviations (SD) = $\sqrt{\sum_{i=1}^{s} (\hat{\theta}_i - \bar{\hat{\theta}})^2 / (s-1)}$.

4. Estimated root mean squared errors (RMSE) = $\sqrt{\sum_{i=1}^{s} (\hat{\theta}_i - \theta)^2 / s}$.

The results are presented in Table 1. It can be seen that the Bias, SD and RMSE of the estimated parameters using CML are quite small and smaller than those of the YW method. The biases of α's estimators are offen negative but the bias of the λ's estimator is positive in both methods. It can be also seen that, the RMSEs decrease when the value of parameters decrease. The RMSE of λ depends on the values of αs. If the value of at least one of the αs increase, the RMSE of λ also will increase. Also, we can see that the bias and RMSE values of the estimated parameters decrease when the grid size increases.

Figure 1 and 2 show Q-Q normal plots of the Bias of parameters using CML method when $\boldsymbol{\theta}' = (0.4, 0.3, 0.1, 1)$ and grid sample sizes are 15×15 and 50×50, respectively. From these figures it can be seen that the estimators are normally distributed not only for large grid sizes of 50×50 but also for small grid sizes of 15×15.

To illustrate the fitting of the SINAR(1,1) model, we consider the data set from a regular grid presented by Student (1906) on the yeast cell counts (see Hand et al. (1994)). Yeast cell counts were made on each of 400 small regions on a microscope slide. The 400 squares were arranged as a 20× 20 grid and each small square was of side 1/20 mm. Mean and variance of the data are 4.675 and 4.464, respectively.

The fitted models to the Student's data using YW and CML estimation methods are respectively as follows

$$X_{ij} = 0.077 \circ X_{i-1,j} + 0.044 \circ X_{i,j-1} + \varepsilon_{ij}, \qquad \varepsilon_{ij} \sim P(4.114)$$

$$X_{ij} = 0.040 \circ X_{i-1,j} + 0.013 \circ X_{i,j-1} + \varepsilon_{ij}, \qquad \varepsilon_{ij} \sim P(4.427)$$

Note that, since the estimated values for α_3 using both methods were negative, the terms containing $X_{i-1,j-1}$ were omitted from the model. The fitted model can be used for prediction.

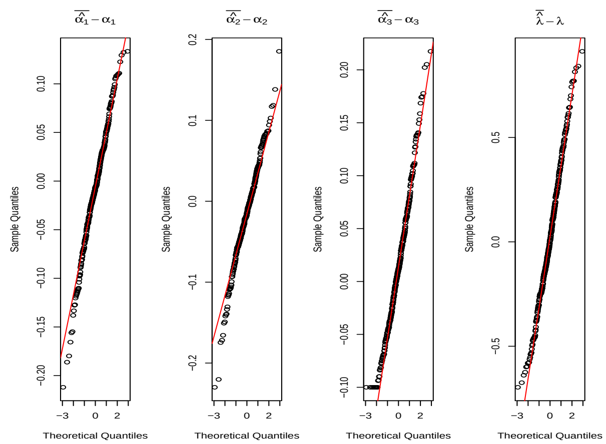

Figure 1: Q-Q normal plots of the Bias of the parameters for set (i) and grid sample sizes of 15×15 using CML method.

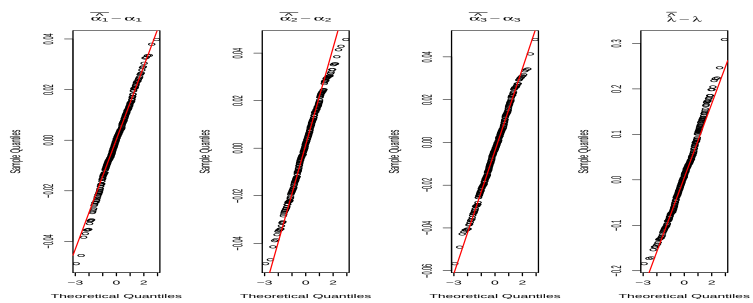

Figure 2: Q-Q normal plots of the Bias of parameters for set (i) and grid sample sizes of 50×50 using CML method.

4 Conclusion

The models described in this paper can be useful in the analysis of count data on a regular grid. In this paper, the conditional maximum likelihood (CML) estimation method to estimate the parameters of the Poisson SINAR(1,1) mode was derived. The asymptotic distribution of the estimators also established. The simulation study to compare the RMSE of CML estimators with YW estimators was performed and it was found that the Bias, SD and RMSE of the CML method are quite small and less than of the YW method. It is hoped that more researchers would consider using this model for various applications.

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Table 1: Mean, Bias, SD and RMSE of the estimates.

			$\hat{\alpha}_1$				$\hat{\alpha}_2$			
Set	Size	Method	Mean	Bias	SD	RMSE	Mean	Bias	SD	RMSE
(i)	15× 15	CML	0.3920	-0.0080	1.6e-17	0.0080	0.2846	-0.0154	1.2e-17	0.0154
		YW	0.3678	-0.0323	2.0e-17	0.0322	0.2751	-0.0246	8.3e-19	0.0249
	30×30	CML	0.3975	-0.0025	4.4e-18	0.0025	0.2989	-0.0011	1.5e-17	0.0011
		YW	0.3906	-0.0093	8.3e-18	0.0093	0.2993	-0.0007	2.4e-17	0.0007
	50× 50	CML	0.3997	-0.0003	2.7e-17	0.0003	0.3012	0.0012	2.7e-17	0.0012
		YW	0.3958	-0.0042	2.2e-17	0.0042	0.3021	0.0021	1.7e-17	0.0021
(ii)	15× 15	CML	0.1944	-0.0056	9.4e-17	0.0059	0.1890	-0.0110	1.0e-17	0.0110
		YW	0.2231	0.0231	1.3e-17	0.0231	0.2158	0.0158	1.1e–17	0.0158
	30×30	CML	0.1953	-0.0047	1.2e-17	0.0047	0.1965	-0.0035	7.7e-18	0.0035
		YW	0.2117	0.0117	3.3e-18	0.0117	0.2129	0.0129	1.3e-17	0.0129
	50× 50	CML	0.1984	-0.0016	3.98e-18	0.0016	0.1988	-0.0012	8.0e-18	0.0012
		YW	0.2088	0.0088	1.4e-18	0.0088	0.2095	0.0095	3.1e-18	0.0095
(iii)	15× 15	CML	0.1030	0.0030	6.9e-20	0.0030	0.1096	0.0096	1.8e-18	0.0096
		YW	0.0970	-0.0031	5.0e-19	0.0040	0.1000	0.00001	4.9e-18	0.00001
	30×30	CML	0.0974	-0.0026	4.2e-19	0.0026	0.1042	0.0042	6.5e-19	0.0042
		YW	0.095	-0.0053	5.8e-18	0.0053	0.1017	0.0017	3.7e-18	0.0017
	50× 50	CML	0.0989	-0.0011	5.1e-18	0.0011	0.0998	-0.0002	6.3e-18	0.0002
		YW	0.0971	-0.0029	5.6e-18	0.0029	0.0984	-0.0016	5.3e-18	0.0016

Table 1: (Continued) Mean, Bias and RMSE of the estimates.

			$\hat{\alpha}_3$				$\hat{\lambda}$			
Set	Size	Method	Mean	Bias	SD	RMSE	Mean	Bias	SD	RMSE
(i)	15× 15	CML	0.1164	-0.0164	6.7e-18	0.0164	1.0319	0.0319	8.5e-17	0.0319
		YW	0.0873	-0.0249	8.3e-18	0.0127	1.3401	0.3401	7.9e-17	0.3401
	30×30	CML	0.1009	0.0009	3.3e-18	0.0009	1.0156	0.0156	2.8e-17	0.0156
		YW	0.0846	-0.0154	3.1e-18	0.0154	1.1246	0.1246	3.5e-17	0.1246
	50× 50	CML	0.0966	-0.0034	6.0e-18	0.0034	1.0119	0.0119	1.0e-16	0.0119
		YW	0.0868	-0.0132	4.6e-20	0.0132	1.0753	0.0753	8.9e-18	0.0753
(ii)	15× 15	CML	0.4940	-0.0060	18e-17	0.0060	1.1929	0.1929	8.6e-17	0.1929
		YW	0.3817	-0.1183	2.4e-17	0.1183	1.7546	0.7546	9.8e-17	0.7546
	30×30	CML	0.4998	-0.0002	1.8e-17	0.0002	1.0816	0.0816	4.1e-17	0.0816
		YW	0.4406	-0.0594	2.6e-17	0.0594	1.3330	0.3330	4.0e-17	0.3330
	50× 50	CML	0.5003	0.0003	5.0e-17	0.0003	1.0213	0.0214	5.1e-17	0.0214
		YW	0.4646	-0.0354	1.1e-17	0.0354	1.1635	0.1635	7.0e-17	0.1635
(iii)	15× 15	CML	0.1100	0.0100	4.1e-18	0.0100	0.9694	-0.0306	1.9e–17	0.0306
		YW	0.0950	-0.0050	2.5e-18	0.0050	1.0149	0.0149	2.6e-17	0.0149
	30×30	CML	0.0979	-0.0021	5.4e-18	0.0021	1.0031	0.0031	3.7e-17	0.0031
		YW	0.0910	-0.0090	3.4e-18	0.0090	1.0196	0.0196	2.7e-17	0.0196
	50× 50	CML	0.0997	-0.0003	2.3e-18	0.0003	1.0022	0.0022	4.2e-17	0.0022
		YW	0.0957	-0.0043	1.9e-18	0.0043	1.0128	0.0128	6.1e-17	0.0128