The Type I Generalized Half Logistic Distribution

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Abstract. In this paper, we considered the half logistic model and derived a probability density function that generalized it. The cumulative distribution function, the nth moment, the median, the mode and the $100k$-percentage points of the generalized distribution were established. Estimation of the parameters of the distribution through maximum likelihood method was accomplished with the aid of computer program and a theorem that relate the generalized distribution to Pareto distribution was stated and proved. Finally, we obtained the distributions of some order statistics from the generalized half logistic distribution.

1 Introduction

One of the probability distributions which is a member of the family of logistic distribution is the half logistic distribution with probability density function.

$$f_X(x) = \frac{2e^x}{(1+e^x)^2}, \quad 0 < x < \infty, \tag{1.1}$$

and cumulative distribution function

$$F_X(x) = \frac{e^x - 1}{1 + e^x}, \quad 0 < x < \infty. \tag{1.2}$$

Balakrishnan (1985) studied order statistics from the half logistic distribution, Balakrishnan and Puthenpura (1986) obtained best unbiased estimates of the location and scale parameter of the distribution while Olapade (2003) presented some theorems that characterized the distribution. Balakrishnan and Wong (1991) obtained approximate maximum likelihood estimates for the location and scale parameters of the half logistic distribution. Torabi and Bagheri (2010) presented an extended generalized half logistic distribution and studied different methods for estimating its parameters based on complete and censored data.

In the next section, we obtain a generalized form of half logistic distribution through a transformation of an exponential random variable and we name it type I generalized half logistic distribution. In that section, we obtain the cumulative distribution function and the moments and related statistics were tabulated. In section 3, we discuss the estimation of the parameters of the generalized half logistic distribution by the method of maximum likelihood estimation and we applied the new model to analyze a data set related to an athletic event. Section 4 contains the properties of the type I generalized half logistic distribution like median, $100p$-percentage point and mode. In section 5, we stated and proved two theorems that characterized the type I generalized half logistic distribution. In section 6, some functions of order statistics from type I generalized half logistic distribution like largest, smallest and r^{th} order statistics were discussed and their probability density functions and moments were obtained.

2 Type I Generalized Half Logistic Distribution

In this section, we shall derive a form of generalized half logistic distribution which is a special case of the one in the equation (1.1) as Balakrishnan and Leung (1988) derived a type I generalized logistic distribution that generalizes the logistic distribution.

Theorem 1.1. *Let Y be a continuously distributed random variable with density function $f_Y(y)$. The random variable $X = \ln(2e^y - 1)$ is a generalized half logistic random variable if and only if Y follows an exponential distribution with parameter q.*

Proof. If Y is exponentially distributed with parameter q,

$$f_Y(y) = qe^{-qy}, \quad 0 < y < \infty, \quad q > 0. \tag{2.1}$$

Let $x = \ln(2e^y - 1)$, by the method of transformation of random variable $y = \ln(\frac{1+e^x}{2})$. Therefore

$$f_X(x) = |\frac{d}{dx}(g^{-1}(x))|f_Y(x) = |\frac{d}{dx}(y)|f_Y(x) = \frac{2^q qe^x}{(1 + e^x)^{q+1}}. \tag{2.2}$$

The probability density function obtained in equation (2.2) is what we named type I generalized half logistic distribution. It has the cumulative distribution function

$$F_X(x; q) = 1 - 2^q(1 + e^x)^{-q}, \quad 0 \le x < \infty, \quad q > 0. \tag{2.3}$$

It is obvious that when $q = 1$, equations (2.2) and (2.3) reduce to equation (1.1) and (1.2) respectively.

The probability that a type I generalized half logistic random variable X lies in an interval (α_1, α_2) is given as

$$pr(\alpha_1 < X < \alpha_2) = F_X(\alpha_2) - F_X(\alpha_1)$$

$$= 2^q[(1 + e^{\alpha_1})^{-q} - (1 + e^{\alpha_2})^{-q}], \quad \forall \; \alpha_1 < \alpha_2.$$

Hence for any given value of the parameter q and an interval (α_1, α_2), the probability $pr(\alpha_1 < X < \alpha_2)$ can be easily computed for a random variable that has type I generalized half logistic distribution.

2.1 Moments of the Type I Generalized Half Logistic Distribution

Considering the type I generalized half logistic distribution function $f_X(x; q)$ given in equation (2.2). The first moment of X is given as

$$E[X] = \int xf_X(x; q)dx = 2^q q \int_0^\infty \frac{xe^x}{(1 + e^x)^{q+1}}dx. \tag{2.4}$$

Let $u = e^x$, then

$$E[X] = 2^q q \int_1^\infty \frac{\ln u}{(1 + u)^{q+1}}du. \tag{2.5}$$

Similarly, the second moment of X is given as

$$E[X^2] = 2^q q \int_0^\infty \frac{x^2 e^x}{(1 + e^x)^{q+1}}dx = 2^q q \int_1^\infty \frac{\ln^2 u}{(1 + u)^{q+1}}du. \tag{2.6}$$

Equations (2.5) and (2.6) can be evaluated numerically for various values of q. Hence the mean of the type I generalized half logistic distribution can be computed using equation (2.5) and the variance of the distribution can be evaluated for each given value of q using the relation

$$\text{Var}(X;q) = 2^q q \int_1^\infty \frac{\ln^2 u}{(1+u)^{q+1}} du - (2^q q \int_1^\infty \frac{\ln u}{(1+u)^{q+1}} du)^2. \quad (2.7)$$

The nth moment is

$$E[X^n] = \int x^n f_X(x;q) dx = 2^q q \int_0^\infty \frac{x^n e^x}{(1+e^x)^{q+1}} dx = 2^q q \int_1^\infty \frac{\ln^n u}{(1+u)^{q+1}} du. \quad (2.8)$$

The Table 1 below gives the mean μ of the type I generalized half logistic distribution, the variance μ_2, the skewness and the kurtosis for $q = 1, 2, ..., 10$ using the following relations: $\mu_1 = \nu_1$, $\mu_2 = \nu_2 - \nu_1^2$, $\mu_3 = \nu_3 - 3\nu_2\nu_1 + 2\nu_1^3$, $\mu_4 = \nu_4 - 4\nu_3\nu_1 + 6\nu_2\nu_1^2 - 3\nu_1^4$, where ν_i is the ith moment $E[X^i]$ and $\mu_1 = $ the mean, $\mu_2 = $ the variance, the skewness $\beta_1 = \mu_3^2/\mu_2^3$ and the kurtosis $\beta_2 = \mu_4/\mu_2^2$.

Table 1: Mean, Variance, Skewness and Kurtosis of the
Type I Generalized Half Logistic Distribution for some values of q.

q	Mean	Variance	μ_3	Skewness	μ_4	Kurtosis
1	1.3863	1.3680	2.47	0.77	12.32	6.58
2	0.7726	0.4377	0.42	0.73	1.14	5.94
3	0.5452	0.2267	0.16	0.74	0.30	5.90
4	0.4237	0.1415	0.08	0.75	0.12	6.00
5	0.3474	0.0976	0.05	0.77	0.06	6.13
6	0.2948	0.0718	0.03	0.78	0.03	6.27
7	0.2562	0.0551	0.02	0.80	0.02	6.41
8	0.2266	0.0438	0.01	0.81	0.01	6.54
9	0.2033	0.0356	0.01	0.82	0.01	6.65
10	0.1843	0.0296	0.01	0.83	0.01	6.77

3 Estimation of the Parameters of the Type I Generalized Half Logistic Distribution

For the type I generalized half logistic distribution obtained in equation (2.2), when the location parameter μ and scale parameter σ have been

introduced and for a sample of size n, the maximum likelihood function is

$$L(X; \mu, \sigma, q) = \frac{2^{nq} q^n exp \sum_{i=1}^{n} (\frac{x_i - \mu}{\sigma})}{\sigma^n \prod_{i=1}^{n} (1 + exp(\frac{x_i - \mu}{\sigma}))^{q+1}}. \qquad (3.1)$$

Taking the natural logarithm of both sides, we have

$$\ln L(X; \mu, \sigma, q) = nq\ln 2 + n\ln q + \sum_{i=1}^{n} (\frac{x_i - \mu}{\sigma}) - n\ln\sigma - (q+1) \sum_{i=1}^{n} \ln(1 + exp(\frac{x_i - \mu}{\sigma})).$$
$$(3.2)$$

By differentiating the log likelihood function partially with respect to q, μ and σ, we have

$$\frac{\partial \ln L(X; \mu, \sigma, q)}{\partial q} = n\ln 2 + \frac{n}{q} - \sum_{i=1}^{n} \ln(1 + exp(\frac{x_i - \mu}{\sigma})), \qquad (3.2a)$$

$$\frac{\partial \ln L(X; \mu, \sigma, q)}{\partial \mu} = -\frac{n}{\sigma} + \frac{1+q}{\sigma} \sum_{i=1}^{n} (\frac{exp(\frac{x_i - \mu}{\sigma})}{1 + exp(\frac{x_i - \mu}{\sigma})}), \qquad (3.2b)$$

$$\frac{\partial \ln L(X; \mu, \sigma, q)}{\partial \sigma} = -\frac{n}{\sigma} - \frac{\sum_{i=1}^{n} (x_i - \mu)}{\sigma^2} + \frac{1+q}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu) \frac{exp(\frac{x_i - \mu}{\sigma})}{(1 + exp(\frac{x_i - \mu}{\sigma}))}. \qquad (3.2c)$$

From equation (3.2a) we obtain the maximum likelihood estimate of the shape parameter q as

$$\hat{q} = \frac{n}{\sum_{i=1}^{n} \ln(1 + exp(\frac{x_i - \mu}{\sigma})) - n\ln 2}. \qquad (3.3)$$

Since the equations (3.2b) and (3.2c) above are nonlinear in the parameters, we can use numerical iterative method with the aid of computer programme to estimate the parameters from a relevant sample.

3.1 Application of the Type I Generalized Half Logistic Model

In a cross country race of 200 kilometers, a total of 1000 participants started the race and many stopped on the way without completing the race. The distance covered by each participant before stopping was recorded and a frequency distribution table was constructed to reflect the number of participants that are able to cover an interval of the distance. A probability model that will fit the data is needed. The histogram of the data shows that the data can be analyzed using a type I generalized half logistic or the half logistic distribution of Balakrishnan

and Puthenpura (1986). These two models were fitted to the data and we recommend the one that fits better. The distribution of the data and the analysis are shown in Table 2.

The estimates of the parameters of the type I generalized half logistic distribution as obtained from computer programme are ($\hat{\mu} = 0.5$, $\hat{\sigma} = 51.4$, $\hat{q} = 1.188$) while that of the half logistic of Balakrishnan et al are ($\hat{\mu} = 0.5, \hat{\sigma} = 52.74$), the outcomes of the analysis are shown in the Table 2.

The adequacy of the model is tested using the method of chi-square test.

From the table below, $\chi^2_{calculated}$ for the type I generalized half logistic model is 0.8505 while that of half logistic of Balakrishnan et al is 22.3448. From the table of chi-square, $\chi^2_{(7,0.05)} = 14.0671$ and $\chi^2_{(8,0.05)} = 15.5073$. While type I generalized half logistic model is adequate for this data, the half logistic model of Balakrishnan et al is inadequate.

Table 2: Analysis of cross country race data

Class interval of distances	Class mid-point x	Class frequency observed	Type I GHL. estimated frequency	Half logistic estimated frequency	χ^2 Type I GHL	χ^2 Half logistic
1-20	10.5	228	223.9	187.4	0.0751	8.7959
21-40	30.5	203	199.2	174.6	0.0725	4.6195
41-60	50.5	166	164.3	152.5	0.0176	1.1951
61-80	70.5	129	126.9	125.7	0.0348	0.0866
81-100	90.5	95	93.0	98.7	0.0430	0.1387
101-120	110.5	66	65.2	74.7	0.0098	1.0133
121-140	130.5	46	44.4	55.0	0.0577	1.4727
141-160	150.5	32	30.0	39.6	0.1333	1.4586
161-180	170.5	22	19.3	28.0	0.3777	1.2857
181-200	190.5	13	12.4	19.7	0.0290	2.2787
Total		1000			0.8505	22.3448

Note: GHL means generalized half logistic.

4   Properties of the type I Generalized Half Logistic Distribution

4.1   The Median

The median of a probability density function $f(x)$ is a point x_m on the real line which satisfies the equation

$$\int_{-\infty}^{x_m} f(x)dx = 1/2.$$

This implies that

$$1-2^q(1+e^x)^{-q} = 1/2 \Rightarrow 2^q(1+e^x)^{-q} = 1/2 \Rightarrow (1+e^x)^{-q} = 2^{-q-1}, \quad (4.1)$$

solving for x gives $x = \ln(2^{(q+1)/q} - 1)$. Therefore the median of a type I generalized half logistic distribution is

$$x_{median} = \ln(2^{(q+1)/q} - 1) \tag{4.2}$$

which reduces to ln3 when $q = 1$, as the median of an ordinary standard half logistic distribution.

4.2 The $100p$-Percentage Point

The $100p$-percentage point is obtained by setting the cumulative probability distribution function to p. That is

$$F(x_{(p)}) = p \Rightarrow 2^q(1 + e^{x_{(p)}})^{-q} = 1 - p. \tag{4.3}$$

Solving for $x_{(p)}$ gives

$$x_{(p)} = \ln[\sqrt[q]{\frac{2^q}{1-p}} - 1]. \tag{4.4}$$

This gives the value of the point $x_{(p)}$ on the real line that produces a percentage p of the distribution.

4.3 The Mode

The mode of a probability density function is obtained by equating the derivative of the density function to zero and solving for the variable. thus,

$$f_X(x; q) = \frac{2^q q e^x}{(1 + e^x)^{q+1}}, \quad 0 \le x < \infty, \quad q > 0.$$

$$f'_X(x; q) = 2^q q[\frac{(1 + e^x)e^x - (q + 1)e^{2x}}{(1 + e^x)^{q+2}}]. \tag{4.5}$$

Equating the derivative to zero, the solution to this equation gives $x = -\ln q$. This gives the mode of the distribution as confirmed when we put $q = 1$ which gives the mode of standard half logistic distribution.

5 Characterizations of the Type I Generalized Half Logistic Distribution

In this section, we present a theorem that relates the type I generalized half logistic distribution to Pareto distribution and also a differential equation whose solution is the probability density function of the type I generalized half logistic distribution is presented.

Theorem 5.1. *Let X be a continuously distributed random variable with density function $f_X(x)$. Then the random variable $Y = \ln(2x - 1)$ is a generalized half logistic random variable if and only if X has a generalized Pareto distribution with parameter q.*

Proof. If X is a generalized Pareto distributed random variable with parameters q, then

$$f_X(x; q) = \frac{q}{x^{q+1}}, \quad x > 1, \quad q > 0. \tag{5.1}$$

Since the random variable $y = \ln(2x - 1)$ implies that $x = \frac{2}{1+e^y}$ and the Jacobian of the transformation is $|J| = \frac{1}{2}e^y$. Therefore

$$f_Y(y) = \frac{2^q q e^y}{(1 + e^y)^{q+1}}, \quad y > 0 \tag{5.2}$$

which is the probability density function for type I generalized half logistic distribution.

Conversely, if Y has a type I generalized half logistic distribution with distribution function shown in equation (2.2), then the cumulative distribution function of X is given as

$$F_X(x) = Pr[X \le x] = Pr[\frac{1 + e^y}{2} \le x] = F_Y[\ln(2x - 1)] = 1 - x^{-q}. \tag{5.3}$$

Since equation (5.3) is the distribution function for the generalized Pareto distribution given in equation (5.1), which confirms the result.

Theorem 5.2. *The random variable X is type I generalized half logistic with parameter q if and only if the density function f given by equation (2.2) satisfies the homogeneous differential equation*

$$(1 + e^x)f' + (qe^x - 1)f = 0, \tag{5.4}$$

where prime denotes differentiation.

Proof. Suppose X has a type I generalized half logistic with parameter q where

$$f_X(x; q) = \frac{2^q q e^x}{(1 + e^x)^{q+1}},$$

it is easily shown that the function f above satisfies equation (5.4).

Conversely, we assume that f satisfies (5.4). Separating the variables in (5.4) and integrating, we have

$$f = \frac{k e^x}{(1 + e^x)^{q+1}} \tag{5.5}$$

where k is a constant. The value of k that makes f a density function is $k = 2^q q$.

Possible Application of Theorem 5.2: From equation (5.4), we have

$$x = \ln\left(\frac{f - f'}{qf + f'}\right)$$

or equivalently

$$x = \ln\left(\frac{F' - F''}{qF' + F''}\right) \tag{5.6}$$

where F is the corresponding distribution function as shown above. Thus, the importance of Theorem (5.2) lies in the linearizing transformation (5.6). The transformation (5.6) can be regarded as a type I generalized half logistic model alternative to Berkson's logit transform (Berkson (1944)) and Ojo's logit transform when $p = q = 1$ (Ojo (1997)) for the ordinary logistic model.

Thus in the analysis of bioassay and quanta response data, if model (2.2) is used, what Berkson's logit transform does for the ordinary logistic can be done for the model (2.2) by the transformation (5.6).

6   Order Statistics from type I Generalized Half Logistic Distribution

Let $X_1, X_2, ..., X_n$ be a random sample of size n from the type I generalized half logistic distribution and let $X_{1:n} \leq X_{2:n} \leq .... \leq X_{n:n}$ be the corresponding order statistics. Let $F_{X_{r:n}}(x), \ r = 1, 2, ..., n$ be the cumulative distribution function of the rth order statistics $X_{r:n}$ and

$f_{X_{r:n}}(x)$ denotes its probability density function. David (1970) gives the probability density function of $X_{r:n}$ as

$$f_{X_{r:n}}(x) = \frac{1}{B(r, n-r+1)}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x). \qquad (6.1)$$

For the type I generalized half logistic distribution with parameter q which has probability density function and cumulative distribution function given in equations (2.2) and (2.3) respectively. By substitution into equation (6.1), we have

$$f_{X_{r:n}}(x) = \frac{1}{B(r, n-r+1)}[1-2^q(1+e^x)^{-q}]^{r-1}[2^q(1+e^x)^{-q}]^{n-r}\frac{2^q q e^x}{(1+e^x)^{q+1}}$$

$$= \frac{2^{q(n-r+1)}q e^x}{B(r, n-r+1)(1+e^x)^{qn+1}}[(1+e^x)^q - 2^q]^{n-r}. \qquad (6.2)$$

The knowledge of the smallest and largest order statistics are of great importance in statistical studies. They are very useful in the study of extreme values and range.

6.1 The Largest Order Statistics

Consider the probability density of the rth order statistics from the type I generalized half logistic distribution in equation (6.2). Let $r = n$, then the probability density function of the largest order statistic is

$$f_{X_{n:n}}(x) = \frac{2^q n q e^x}{(1+e^x)^{qn+1}}. \qquad (6.3)$$

The pth moment of the largest order statistics from the type I generalized half logistic distribution is

$$E[X_{n:n}^p] = \int x^p f_{X_{n:n}}(x)dx = 2^q n q \int_0^\infty \frac{x^p e^x}{(1+e^x)^{qn+1}}dx. \qquad (6.4)$$

The above integral is not close but can be evaluated numerically, so with n known, we can compute the mean, variance, kurtosis and skewness for a set of largest observations from the type I generalized half logistic distribution.

6.2   The Smallest Order Statistics

Consider the probability density of the rth order statistics from the type I generalized half logistic distribution in equation (6.2). Let $r = 1$, then the probability density function of the smallest order statistic is obtained as

$$f_{X_{1:n}}(x) = \frac{2^{qn} q n e^x}{(1 + e^x)^{qn+1}} [(1 + e^x)^q - 2^q]^{n-1}. \tag{6.5}$$

Using binomial theorem for positive integral exponent,

$$[(1 + e^x)^q - 2^q]^{n-1} = \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} 2^{q(n-1-j)} (1 + e^x)^{qj}$$

and

$$(1 + e^x)^{qj} = \sum_{k=0}^{qj} \binom{qj}{k} e^{x(qj-k)}.$$

Therefore

$$f_{X_{1:n}}(x) = \frac{2^{qn} q n e^x}{(1 + e^x)^{qn+1}} \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} 2^{q(n-1-j)} (1 + e^x)^{qj}$$

$$= 2^{qn} q n e^x \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} 2^{q(n-1-j)} (1 + e^x)^{qj-qn-1}$$

$$= 2^{qn} q n e^x \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} 2^{q(n-1-j)} \sum_{k=0}^{qj-qn-1} \binom{qj - qn - 1}{k} e^{x(qj-qn-1-k)}$$

$$= 2^{qn} q n \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} 2^{q(n-1-j)} \sum_{k=0}^{qj-qn-1} \binom{qj - qn - 1}{k} e^{x(qj-qn-k)}$$

$$= q n \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} 2^{q(2n-1-j)} \sum_{k=0}^{qj-qn-1} \binom{qj - qn - 1}{k} e^{x(qj-qn-k)}$$

$$= q n \sum_{j=0}^{n-1} \sum_{k=0}^{qj-qn-1} (-1)^j \binom{n-1}{j} \binom{qj - qn - 1}{k} 2^{q(2n-1-j)} e^{x(qj-qn-k)}. \tag{6.6}$$

Thus, the pth moment of $X_{1:n}$ is

$$E[X_{1:n}^p] = \theta \int_0^\infty x^p e^{x(qj-qn-k)} dx$$

where $\theta = q n \sum_{j=0}^{n-1} \sum_{k=0}^{qj-qn-1} (-1)^j \binom{n-1}{j} \binom{qj-qn-1}{k} 2^{q(2n-1-j)}$

$$E[X_{1:n}^p] = \theta \int_0^\infty x^p e^{-x(qn+k-qj)} dx.$$

Let $x(qn+k-qj) = t$, $x = t/(qn+k-qj)$ and $dx = dt/(qn+k-qj)$, then

$$E[X_{1:n}^p] = \theta \int_0^\infty \left(\frac{t}{qn+k-qj}\right)^p e^{-t} \frac{dt}{qn+k-qj}$$

$$= \frac{\theta}{(qn+k-qj)^{p+1}} \int_0^\infty t^p e^{-t} dt = \frac{\theta\Gamma(p+1)}{(qn+k-qj)^{p+1}} = \frac{p!\theta}{(qn+k-qj)^{p+1}}.$$
(6.7)

Hence, to obtain the first, second, third and fourth moments of the smallest order statistics from the type I generalized logistic distribution, we use the equation (6.7) for $p = 1, 2, 3$, and 4. The values of these moments can be used to calculate the mean, variance, skewness and kurtosis of the smallest order statistics from the type I generalized half logistic distribution.

6.3 r^{th} Order Statistics from the Type I Generalized Half Logistic Distribution

Consider the probability density function of the rth order statistics from type I generalized half logistic distribution derived earlier in equation (6.2) as

$$f_{X_{r:n}}(x) = \frac{2^{q(n-r+1)}qe^x}{B(r, n-r+1)(1+e^x)^{qn+1}}[(1+e^x)^q - 2^q]^{n-r}.$$

By binomial theorem,

$$[(1+e^x)^q - 2^q]^{n-r} = \sum_{j=0}^{n-1}(-1)^j\binom{n-r}{j}2^{q(n-r-j)}(1+e^x)^{qj}. \quad (6.8)$$

Therefore

$$f_{X_{r:n}}(x) = \frac{2^{q(n-r+1)}qe^x}{B(r, n-r+1)(1+e^x)^{qn+1}}\sum_{j=0}^{n-1}(-1)^j\binom{n-r}{j}2^{q(n-r-j)}(1+e^x)^{qj}$$

$$= \frac{2^{q(n-r+1)}qe^x}{B(r, n-r+1)}\sum_{j=0}^{n-1}(-1)^j\binom{n-r}{j}2^{q(n-r-j)}(1+e^x)^{qj-qn-1}. \quad (6.9)$$

Also by expanding $(1+e^x)^{qj-qn-1}$ binomially,

$$(1+e^x)^{qj-qn-1} = \sum_{k=0}^{qj-qn-1}\binom{qj-qn-1}{k}e^{x(qj-qn-1-k)}. \quad (6.10)$$

Substituting back into $f_{X_{r:n}}(x)$, we have

$$f_{X_{r:n}}(x) = \frac{2^{q(n-r+1)}qe^x}{B(r, n-r+1)} \sum_{j=0}^{n-1} (-1)^j \binom{n-r}{j} 2^{q(n-r-j)}$$

$$\times \sum_{k=0}^{qj-qn-1} \binom{qj-qn-1}{k} e^{x(qj-qn-1-k)}$$

$$= \frac{q}{B(r, n-r+1)} \sum_{j=0}^{n-1} \sum_{k=0}^{qj-qn-1} (-1)^j \binom{n-r}{j}$$

$$\times \binom{qj-qn-1}{k} 2^{q(2n-2r-j+1)} e^{x(qj-qn-k)}. \qquad (6.11)$$

The pth moment of $X_{(r:n)}$ from the type I generalized half logistic distribution is then obtained as

$$E[X_{r:n}^p] = \Lambda \int_0^\infty x^p e^{x(qj-qn-k)} dx = \Lambda \int_0^\infty x^p e^{-x(qn+k-qj)} dx \qquad (6.12)$$

where

$$\Lambda = \frac{q}{B(r, n-r+1)} \sum_{j=0}^{n-1} \sum_{k=0}^{qj-qn-1} (-1)^j \binom{n-r}{j} \binom{qj-qn-1}{k} 2^{q(2n-2r-j+1)}.$$

Therefore,

$$E[X_{r:n}^p] = \frac{\Lambda\Gamma(p+1)}{(qn+k-qj)^{p+1}}. \qquad (6.13)$$

The expression for this moment can be used to calculate the mean, variance, skewness and kurtosis of the rth order statistics from the type I generalized half logistic distribution.

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