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# Optimal Simple Step-Stress Plan for Type-I Censored Data from Geometric Distribution

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**Abstract.** A simple step-stress accelerated life testing plan is considered when the failure times in each level of stress are geometrically distributed under Type-I censoring. The problem of choosing the optimal plan is investigated using the asymptotic variance-optimality as well as determinant-optimality and probability-optimality criteria. To illustrate the results of the paper, an example is presented and a sensitivity analysis is performed. A simulation study is also done to investigate the robustness of the criteria with respect to estimation error of the parameters. Eventually, some conclusions are presented.

**Keywords.** Asymptotic variance-optimality, determinant-optimality, Fisher information, order statistics, probability-optimality.

**MSC:** 62N05, 62B10, 62F10, 62F12, 62N01.

## 1 Introduction

The accelerated life testing (ALT) experiments are commonly used in product life testing and analysis to shorten the time of the experiments which may not terminate on an adequate time under the normal conditions. Under the ALT plans, units are tested at higher than operating levels of stress such as temperature, vibration, voltage, pressure and humidity. Nelson (1990), Meeker and Escobar (1998) and Bagdonavicius and Nikulin (2002) are some key references in this field. The step-stress

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accelerated life testing (SSALT) is a special type of the ALT in which the stress levels of the experiment are increased during the test period at some pre-specified times. Determination of the change times of stress levels is one of the most important design problems in the SSALT. Miller and Nelson (1983) investigated the optimum SSALT with two levels of stress (i.e., the case of a simple SSALT) for the exponentially distributed life times. Bai *et al.* (1989) extended the results of Miller and Nelson (1983) to the case of time-censored data. The case of three stress levels was dealt by Khamis and Higgins (1996). For the general case of $k$-level SSALT, some numerical investigations was taken by Khamis (1997). Khamis and Higgins (1998) also considered the problem under a Weibull distribution for the lifetimes of units subjected to stress. Gouno *et al.* (2004) studied the problem of determining the optimal change times of stress levels for a general $k$-level SSALT plan under the large-sample case, when the available data are progressively Type-I censored. Fard and Li (2009) studied optimal simple SSALT for the Weibull distributed failure times under Type-I censored data. Balakrishnan and Han (2009) considered the optimality problem in a $k$-level SSALT under progressively Type-I censored data from exponential distribution. Wang and Yu (2009) considered a simple SSALT with progressively Type-II censored exponential data and determined the optimum change time on the basis of the variance of the uniformly minimum variance unbiased estimators. Ling *et al.* (2011) found the optimal simple SSALT under Type-I hybrid censored data. Kateri *et al.* (2011) investigated this problem under Type-II censoring.

There are situations in reliability and survival analysis for which the life-testing experiment must be investigated in a discrete set-up. For example, suppose that the lifetimes of the units in an experiment depend on the number of times the units are switched on and off or the number of shocks they receive. The number of rotations of a machine or the number of pages a printer prints may also be included. Let $w$ be the number of times a unit can withstand the operating stress until it fails, so, $w$ is the associated failure time. For more details about the results on order statistics of a random sample taken from a discrete population see, for example, Nagaraja (1992). Censored samples in discrete set-up are of great interest, as revealed from the works of Rezaei and Arghami (2002), Davarzani and Parsian (2011) and Balakrishnan *et al.* (2011). In a simple SSALT, assuming the failure times at each level of stress are geometrically distributed, frequentist and Bayesian analysis have been investigated by Arefi and Razmkhah (2013) and Arefi *et al.* (2011), respectively. See, also, Xu *et al.* (2010) and Wang *et al.* (2012).

The main goal of this paper is to investigate the optimal simple SSALT in a discrete set-up. Toward this end, we use three different optimality criteria, namely, the AV-, D- and P-optimality criteria. The AV-optimality criterion in this paper focuses on the minimizing the asymptotic variance of the reliability estimate under normal operating conditions. The D-optimal plan is obtained by minimizing the determinant of the covariance matrix or equivalently by maximizing the determinant of the Fisher information matrix. The P-optimal plan is derived by minimizing the probability of non-existence of the parameter estimates.

The rest of the paper is as follows: In Section 2, some preliminaries are presented. In Section 3, the main results are given. In Section 4, the optimization criteria are investigated. To illustrate the results of the paper, an example is presented and a sensitivity analysis is performed in Section 5. A simulation study is done in Section 6. Eventually, some conclusions are presented in Section 7.

## 2 Preliminaries

In a simple SSALT, $n$ identical units are simultaneously placed on a test under an initial level of stress $S_1$ and the level of stress is increased to $S_2$ at the pre-fixed time $\tau$. Assume that the failure times at level $S_i$ $(i = 1, 2)$ are geometrically distributed with successive probability $\theta_i$ $(i = 1, 2)$, for which $\theta_1 < \theta_2$. That is, the probability mass function (pmf) and cumulative distribution function (cdf) at level $S_i$ are given by

$$f_i(x; \theta_i) = \theta_i(1 - \theta_i)^{x-1},\ x = 1, 2, \ldots$$

and

$$F_i(x; \theta_i) = 1 - (1 - \theta_i)^x,\ x = 1, 2, \ldots, \tag{1}$$

respectively, where $0 < \theta_i < 1$. Using the memoryless property of the geometric random variable, it is easy to show that when the stress level jumps from $S_1$ to $S_2$ at point $\tau$, the pmf of the model is as follows

$$g(x) = \begin{cases} g_1(x) = \theta_1(1 - \theta_1)^{x-1},\ x = 1, 2, \cdots, \tau, \\ g_2(x) = \theta_2(1 - \theta_1)^{\tau}(1 - \theta_2)^{x-(\tau+1)},\ x = \tau + 1, \tau + 2, \cdots \end{cases} \tag{2}$$

(see, also, Arefi *et al.*, 2011). Using (2), the corresponding cdf is

$$G(x) = \begin{cases} G_1(x) = 1 - (1 - \theta_1)^x,\ x = 1, 2, \cdots, \tau, \\ G_2(x) = 1 - (1 - \theta_1)^{\tau}(1 - \theta_2)^{x-\tau},\ x = \tau + 1, \tau + 2, \cdots . \end{cases} \tag{3}$$

Suppose that in an experiment, the lifetimes of the units are counted in number of cycles, number of rotations, number of switches on and off, etc. Moreover, suppose that the test terminates when all surviving units have operated $\eta$ times, where $\eta$ is a pre-fixed integer number, then a Type-I censoring scheme has been performed. Under this scheme, the failure times greater than $\eta$ are not observed. So, we will observe the following data set under Type-I censoring scheme:

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{N_1:n} \leq \tau < X_{N_1+1:n} \leq \cdots \leq X_{N_1+N_2:n} \leq \eta,$$

where $X_{i:n}$ denotes the $i$th smallest order statistic in a sample of size $n$ from the pmf in (2), $N_1$ is the number of observations that are less than or equal to $\tau$ and $N_2$ stands for the number of data points that are less than or equal to $\eta$ and greater than $\tau$, for which $N_1 + N_2 \leq n$. Summing up, the situation studied in the present paper is data with the given structure and with an assumed geometric distribution for the number of times the experimental units can withstand the operating stress until they fail.

## 3 Main Results

As seen in (2), there exist two parameters $\theta_1$ and $\theta_2$ in a simple SSALT. Assuming $\hat{\theta}_1$ and $\hat{\theta}_2$ are the maximum likelihood estimators (MLEs) of these parameters on the basis of the data set $\mathbf{X}^\star = (X_{1:n}, \cdots, X_{N_1+N_2:n}, N_1, N_2)$, we will find the optimal simple SSALT by considering the properties of $\hat{\theta}_1$ and $\hat{\theta}_2$ individually or a function of both of them such as $\varphi(\hat{\theta}_1, \hat{\theta}_2)$. Using the tie-run technique in the case of discrete order statistics which is defined by Gan and Bain (1995) regarding the number and lengths of runs of tied observations, Arefi *et al.* (2011) showed that the likelihood function of $\theta_1$ and $\theta_2$ is

$$L(\theta_1, \theta_2) = \frac{n!}{(n - r_1 - r_2)!} \left( \prod_{j=1}^{k} z_j! \right)^{-1} \theta_1^{r_1} (1-\theta_1)^{d_1} \theta_2^{r_2} (1-\theta_2)^{d_2}, \quad (4)$$

where $r_1$ and $r_2$ are the observed values of $N_1$ and $N_2$, respectively, $k$ stands for the number of tie-runs with length $z_j$ for the $j$th one, and $d_1$ and $d_2$ are the observed values of

$$D_1 = \sum_{i=1}^{N_1} X_{i:n} - N_1 + \tau(n - N_1) \quad (5)$$

and

$$D_2 = \sum_{i=N_1+1}^{N_1+N_2} X_{i:n} - (\tau+1)N_2 + (\eta-\tau)(n-N_1-N_2), \tag{6}$$

respectively. Moreover, they showed that

$$\hat{\theta}_1 = \frac{N_1}{N_1+D_1} \quad \text{and} \quad \hat{\theta}_2 = \frac{N_2}{N_2+D_2}. \tag{7}$$

Notice that the MLEs of positive parameters $\theta_1$ and $\theta_2$ in (7) exist, if $N_1 \neq 0$ and $N_2 \neq 0$, respectively. Therefore, we introduce the event $B = \{1 \le N_1 \le n-1,\ 1 \le N_2 \le n - N_1\}$ on which the estimates are positive. We emphasize that all the results and computations presented hereafter in the paper are conditionally obtained when the event $B$ occurs, otherwise, the results are meaningless. Moreover, note that for simplicity in notation, the condition is omitted in the next results. That is, $P(N_1 = r_1, N_2 = r_2) \equiv P(N_1 = r_1, N_2 = r_2|B)$, also for $i = 1, 2$, we have $E(N_i) \equiv E(N_i|B)$, $E(D_i) \equiv E(D_i|B)$, etc. The following results are applied to determine the optimal SSALT plan.

• By some algebraic calculations, it can be shown that the probability of occurrence of the event $B$ is given by

$$P(B) = 1 - (1-p_1)^n - (1-p_2)^n + p_3^n, \tag{8}$$

where

$$\begin{aligned} p_1 &= G_1(\tau) = 1 - (1-\theta_1)^\tau, \\ p_2 &= G_2(\eta) - G_1(\tau) = (1-\theta_1)^\tau (1-(1-\theta_2)^{\eta-\tau}), \\ p_3 &= 1 - p_1 - p_2 = (1-\theta_1)^\tau (1-\theta_2)^{\eta-\tau}. \end{aligned}$$

• Notice that given the event $B$, the random variables $N_1$ and $N_2$ have a truncated trinomial distribution. Using (8), the corresponding conditional pmf, for $1 \le r_1 \le n-1$ and $1 \le r_2 \le n - r_1$, is

$$P(N_1 = r_1, N_2 = r_2) = \frac{c_{r_1,r_2}\, p_1^{r_1} p_2^{r_2} p_3^{n-r_1-r_2}}{1-(1-p_1)^n - (1-p_2)^n + p_3^n}, \tag{9}$$

where

$$c_{r_1,r_2} = \frac{n!}{r_1! r_2! (n-r_1-r_2)!}.$$

• Using (9) and by performing some algebraic calculations, it is deduced that given $B$, the conditional expected values of $N_1$ and $N_2$ are as follows

$$\begin{aligned} E(N_1) &= \sum_{r_1=1}^{n-1} \sum_{r_2=1}^{n-r_1} \frac{r_1 c_{r_1,r_2}\, p_1^{r_1} p_2^{r_2} p_3^{n-r_1-r_2}}{1-(1-p_1)^n-(1-p_2)^n+p_3^n} \\ &= \frac{np_1\big(1-(1-p_2)^{n-1}\big)}{1-(1-p_1)^n-(1-p_2)^n+p_3^n} \end{aligned} \tag{10}$$

and, similarly,

$$E(N_2) = \frac{np_2\big(1-(1-p_1)^{n-1}\big)}{1-(1-p_1)^n-(1-p_2)^n+p_3^n}, \tag{11}$$

respectively.

• Given $N_1 = r_1$, the random variable $V_1 = \sum_{i=1}^{N_1} X_{i:n}$ may be considered as the sum of the random sample $X_1, \ldots, X_{r_1}$ from the pmf

$$h_1(x) = \frac{g_1(x)}{G_1(\tau)} = \frac{\theta_1(1-\theta_1)^{x-1}}{1-(1-\theta_1)^\tau}, \quad x = 1, 2, \ldots, \tau,$$

where $g_1(\cdot)$ and $G_1(\cdot)$ are as defined in (2) and (3), respectively. Therefore, the conditional moment generating function of $V_1$, given both $N_1 = r_1$ and $B$, is

$$\begin{aligned} M_{V_1}(v) &= E\left(e^{vV_1} | N_1 = r_1, B\right) \\ &= E\left(e^{v\sum_{i=1}^{r_1} X_i}\right) \\ &= \left\{\frac{\theta_1 e^v\big(1-((1-\theta_1)e^v)^\tau\big)}{\big(1-(1-\theta_1)^\tau\big)\big(1-(1-\theta_1)e^v\big)}\right\}^{r_1}. \end{aligned} \tag{12}$$

From (12), it is deduced that

$$E(V_1) = E\big(E(V_1|N_1)\big) = E(N_1)\left\{1 - \frac{\tau(1-\theta_1)^\tau}{1-(1-\theta_1)^\tau} + \frac{1-\theta_1}{\theta_1}\right\}.$$

Now, using (5), the expected value of $D_1$, when the event $B$ occurs, is as follows

$$E(D_1) = n\tau + \big\{\frac{1-\theta_1}{\theta_1} - \frac{\tau}{1-(1-\theta_1)^\tau}\big\} E(N_1), \tag{13}$$

where $E(N_1)$ is as defined in (10).

• Given $N_1 = r_1$, $N_2 = r_2$ and $B$, the random variable $V_2 = \sum_{j=N_1+1}^{N_2+N_1} X_{i:n}$ is identical in distribution to the sum of the random sample $Y_1, \ldots, Y_{r_2}$ from the pmf

$$h_2(x) = \frac{g_2(x)}{G_2(\eta) - G_1(\tau)} = \frac{\theta_2(1-\theta_2)^{x-(\tau+1)}}{1-(1-\theta_2)^{\eta-\tau}}, \quad x = \tau+1, \ldots, \eta,$$

where $g_2(\cdot)$ and $G_i(\cdot)$ $(i = 1, 2)$ are as defined in (2) and (3), respectively. Hence, the conditional moment generating function of $V_2$, given $N_1 = r_1$, $N_2 = r_2$ and $B$ can be obtained as follows

$$\begin{aligned} M_{V_2}(v) &= E\left(e^{vV_2} | N_1 = r_1, N_2 = r_2, B\right) \\ &= \left\{ \frac{\theta_2 e^{v(\tau+1)}\left(1 - ((1-\theta_2)e^v)^{\eta-\tau}\right)}{\left(1-(1-\theta_2)^{\eta-\tau}\right)\left(1-(1-\theta_2)e^v\right)} \right\}^{r_2}. \end{aligned} \tag{14}$$

Since $h_2(x)$ is free of $\theta_1$, the function $M_{V_2}(v)$ also does not depend on $\theta_1$. Using (6) and (14), it can be shown that the expected value of $D_2$, given the event $B$, is

$$E(D_2) = (\eta-\tau)(n-E(N_1)) + \left\{ \frac{1-\theta_2}{\theta_2} - \frac{(\eta-\tau)}{1-(1-\theta_2)^{\eta-\tau}} \right\} E(N_2), \tag{15}$$

where $E(N_1)$ and $E(N_2)$ are as defined in (10) and (11), respectively.

Using the above results, one may compute the Fisher information (FI) matrix which is a main equipment in this paper to find an optimal simple SSALT. The FI matrix plays a key role in the parameter estimation. It is a measure of the information content of the data relative to the parameters being estimated. Let us denote the FI matrix of the parameters $\theta_1$ and $\theta_2$ by $I(\theta_1, \theta_2)$. By definition, the elements of $I(\theta_1, \theta_2)$ are the negative expectations of the second partial and mixed partial derivatives of the log-likelihood function of the parameters $\theta_1$ and $\theta_2$ (see, for example, Lehmann and Casella, 1998, p. 125). Using (4), it is easy to show that $E\left(\frac{\partial^2 \log L(\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2}\right) = 0$. Therefore, we get

$$I(\theta_1, \theta_2) = \begin{pmatrix} a_1(\tau) & 0 \\ 0 & a_2(\tau) \end{pmatrix}, \tag{16}$$

such that

$$\begin{aligned} a_j(\tau) &= -E\left(\frac{\partial^2 \log L(\theta_1,\theta_2)}{\partial\theta_j^2}\right) \\ &= \frac{E(N_j)}{\theta_j^2} + \frac{E(D_j)}{(1-\theta_j)^2}, \quad j = 1, 2, \end{aligned} \tag{17}$$

where $E(N_1), E(N_2), E(D_1)$ and $E(D_2)$ are as defined in (10), (11), (13) and (15), respectively.

# 4 Optimization Criteria

In this section, we present three different optimality criteria. The first one is obtained by minimizing the asymptotic variance of a function of $\hat{\theta}_1$ and $\hat{\theta}_2$, which were presented in (7). Two other approaches are derived by maximizing the determinant of the FI matrix and minimizing the probability of non-existence of the parameter estimates.

## 4.1 AV-Optimality

In the SSALT plans, minimizing the asymptotic variance (AV) of the MLE of any parameter of interest such as the mean life or some percentile life at a specified level of stress may be considered as a commonly used optimization criterion. Since the reliability function and the mean time to failure (MTTF) are related together as MTTF $= \sum_{x=0}^{\infty} R(x)$, where $R(x)$ stands for the reliability function at $x$, the optimization criterion could also be defined as a function of reliability. Here, we minimize the AV of the reliability estimate at time $x$ under normal operating conditions to get the optimal value of the change time. Toward this end, we assume that $S_0$ is the stress level under normal operating conditions and the failure times are geometrically distributed with success probability $\theta_0$. Since there is not any data set under the stress level $S_0$, we cannot give a direct estimator for $\theta_0$. So, we have to consider a relationship between the parameters and the levels of stress to get a suitable estimator for $\theta_0$ on the basis of $\hat{\theta}_1$ and $\hat{\theta}_2$. Suppose that the parameter $1-\theta_i$ is a log-linear function of the stress level $S_i$ $(i = 0, 1, 2)$. That is,

$$\log(1-\theta_i) = \gamma_0 + \gamma_1 S_i, \quad i = 0, 1, 2, \tag{18}$$

where $\gamma_0$ and $\gamma_1$ are unknown parameters which are depend on the nature of test. The log-linear function is a common choice for the lifestress relationship, because, it includes both the power law and the Arrhenius law as special cases (see, for example, Wu *et al.*, 2008). Using (1) and (18), by taking $\omega = \frac{S_1-S_0}{S_2-S_0}$, the reliability at time $x$, where $x$ is an integer number, under stress level $S_0$ is given by

$$R_{S_0}(x) = \exp\Big\{\frac{x}{1-\omega}\big(\log(1-\theta_1) - \omega\log(1-\theta_2)\big)\Big\}.$$

Using the invariance property of the MLEs, the MLE of $R_{S_0}(x)$ is

$$\hat{R}_{S_0}(x) = \exp\Big\{\frac{x}{1-\omega}\big(\log(1-\hat{\theta}_1) - \omega\log(1-\hat{\theta}_2)\big)\Big\},$$

where $\hat{\theta}_1$ and $\hat{\theta}_2$ are as defined in (7). The AV of the MLE of $R_{S_0}(x)$ is given by

$$\phi(\tau) = H' I^{-1}(\theta_1, \theta_2) H, \tag{19}$$

where $I^{-1}(\theta_1, \theta_2)$ is the inverse of the FI matrix of $\theta_1$ and $\theta_2$ as defined in (16) and $H'$ stands for transpose of $H$ such that

$$H = \left(\frac{\partial R_{S_0}(x)}{\partial\theta_1}, \frac{\partial R_{S_0}(x)}{\partial\theta_2}\right)'$$

(see, for example, Lawless, 2003, p. 549). By some algebraic calculations, it can be shown that

$$\frac{\partial R_{S_0}(x)}{\partial\theta_1} = \frac{-x}{(1-\omega)(1-\theta_1)} R_{S_0}(x),$$

and

$$\frac{\partial R_{S_0}(x)}{\partial\theta_2} = \frac{\omega x}{(1-\omega)(1-\theta_2)} R_{S_0}(x).$$

Therefore, the objective function in (19) reduces to

$$\phi(\tau) = \left(\frac{x R_{S_0}(x)}{1-\omega}\right)^2 \left(\frac{1}{(1-\theta_1)^2 a_1(\tau)} + \frac{\omega^2}{(1-\theta_2)^2 a_2(\tau)}\right), \tag{20}$$

where for $j = 1, 2$, $a_j(\tau)$ is as defined in (17). The AV-optimal $\tau$ (viz., $\tau^*_{AV}$) is the one that minimizes $\phi(\tau)$. From (20), it is also obvious that the value of $\tau^*_{AV}$ does not depend on $x$.

## 4.2 D-Optimality

Another optimal criterion is based on maximizing the determinant of the FI matrix. It can be statistically shown that maximizing the determinant of the FI matrix is the same as minimizing the determinant of the covariance matrix. Therefore, the optimal SSALT plan that maximizes the determinant of the FI matrix will provide the smallest standard error and it is called a D-optimal plan. Moreover, the determinant of $I(\theta_1, \theta_2)$ is proportional to the reciprocal of the volume of the asymptotic joint confidence region for the parameters so that maximizing the determinant is equivalent to minimizing the volume of confidence region. In other words, a larger value of the determinant of the FI matrix would correspond to higher joint precision of the estimators of $\theta_1$ and $\theta_2$ (see, Wu *et al.*, 2008). Therefore, using (16), the D-optimal $\tau$ (viz., $\tau_D^*$) is obtained by maximizing

$$\delta(\tau) = a_1(\tau)a_2(\tau), \tag{21}$$

where for $j = 1, 2$, $a_j(\tau)$ is as defined in (17).

### 4.3 P-Optimality

In the simple SSALT plans, the estimates of the parameters of interest may not exist due to the absence of failure times either before or after the stress change time. For this reason, we choose the change time of the stress levels so as to minimize the probability of non-existence of the parameter estimates (see, for example, Kateri *et al.*, 2011). Therefore, by use of the (8), the criterion function is defined to be

$$\begin{aligned}
\pi(\tau) &= 1 - P(B) \\
&= (1-\theta_1)^{\tau n} + \big(1 - (1-\theta_1)^{\tau} + (1-\theta_1)^{\tau}(1-\theta_2)^{\eta-\tau}\big)^n \\
&\quad -(1-\theta_1)^{n\tau}(1-\theta_2)^{n(\eta-\tau)}.
\end{aligned} \tag{22}$$

The P-optimal $\tau$ (viz. $\tau_P^*$) is obtained by minimizing the function $\pi(\tau)$.

For more investigations, let us denote the events of non-existence of $\hat{\theta}_1$ and $\hat{\theta}_2$ by $B_1$ and $B_2$, respectively. Then, it is obvious that the event $B$ is complement of the union of $B_1$ and $B_2$, denoted by $B_1 \cup B_2$. That is, in fact we have $\pi(\tau) = P(B_1 \cup B_2)$. On the other hand, it can be shown that

$$P(B_1) = P(X_{1:n} > \tau) = (1 - G_1(\tau))^n = (1-\theta_1)^{\tau n}. \tag{23}$$

Similarly, the probability of the intersection of $B_1$ and $B_2$, denoted by $B_1 \cap B_2$, is given by

$$\begin{aligned}
P(B_1 \cap B_2) &= P(X_{1:n} > \eta) \\
&= (1 - G_2(\eta))^n \\
&= (1-\theta_1)^{n\tau}(1-\theta_2)^{n(\eta-\tau)}.
\end{aligned} \tag{24}$$

Furthermore, using (22)–(24), we have

$$P(B_2) = \big(1 - (1-\theta_1)^{\tau} + (1-\theta_1)^{\tau}(1-\theta_2)^{\eta-\tau}\big)^n.$$

**Remark 4.1.** From (23), it is observed that the probability of non-existence of $\hat{\theta}_1$ does not depend on the censoring scheme and it is also a decreasing function of $\tau$. Therefore, if minimizing $P(B_1)$ is considered as the optimization criterion, the optimum change time is obtained to be $\tau_P^* = \eta - 1$. Similarly, if minimizing $P(B_1 \cap B_2)$ in (24) is considered as the optimization criterion, $\tau_P^* = 1$ is the optimum change time, since $P(B_1 \cap B_2)$ is an increasing function in $\tau$ for $\theta_1 < \theta_2$.

# 5 Numerical Illustrations

In this section, we first present an artificial example to illustrate the proposed procedure in obtaining the optimal simple SSALT plan. Then, we perform a sensitivity analysis.

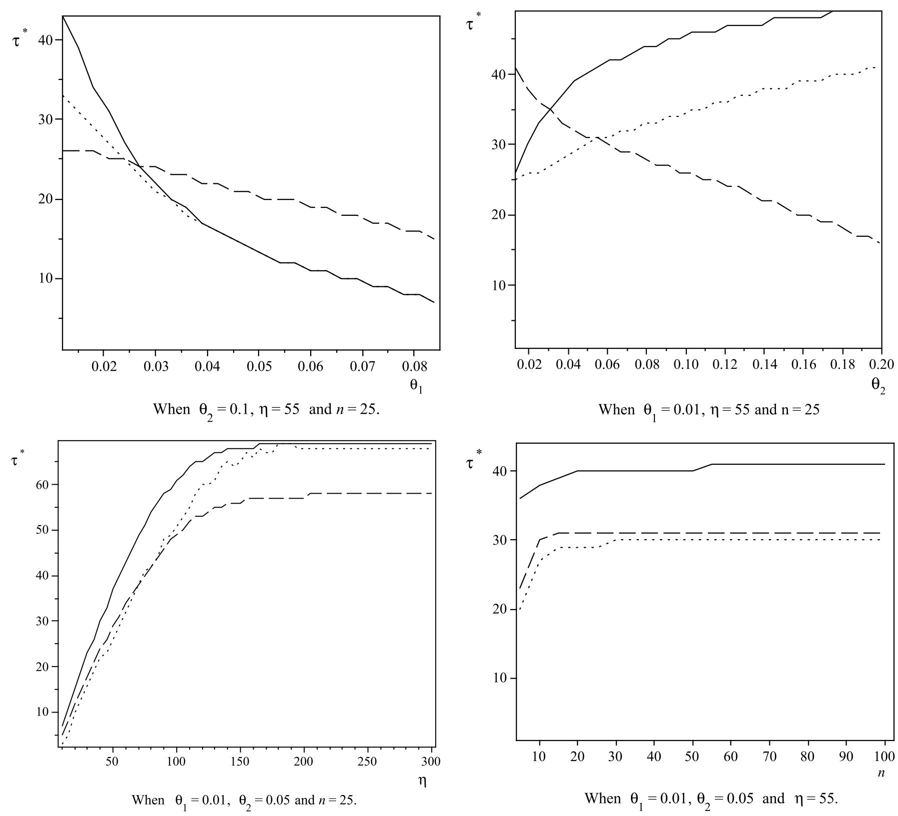
## 5.1 Example

Suppose that the number of switches on and off the electronic units receive until they fail is subject to examination in a simple SSALT for which the lowest and highest levels of stress applied to test units are 24 kilovolt (KV) and 36 KV of voltage, respectively. Moreover, assume that the test is terminated when the units have received the number of $\eta = 55$ switches on and off. If from a previous experience based on similar data or based on a preliminary test we know that the failure times of the units follow a geometric distribution with parameters $\theta_1 = 0.03$ and $\theta_2 = 0.08$, then by maximizing the criterion function in (21) and minimizing the function in (22), we get $\tau_D^* = 20$ and $\tau_P^* = 22$, respectively.

As mentioned in Section 4, the value of $\tau_{AV}^*$ does not depend on $x$. So, if the estimation of the reliability at any arbitrary time $x$ at the design voltage of 20 KV be the problem of interest, we define $\omega = \frac{S_1 - S_0}{S_2 - S_0}$, while $S_0 = 20$ KV, $S_1 = 24$ KV and $S_2 = 36$ KV. So, $\omega = 0.25$. By minimizing the criterion function in (20), the optimum change time of stress levels is determined to be $\tau_{AV}^* = 26$.

## 5.2 Sensitivity Analysis

To examine the effect of changes in the initial parameters $\theta_1$, $\theta_2$ and the values of $\eta$ and $n$ on the optimal value of $\tau$, a sensitivity analysis is performed. Toward this end, the values of $\tau_{AV}^*$, $\tau_D^*$ and $\tau_P^*$ are derived when one of the objectives $\theta_1, \theta_2, \eta$ or $n$ changes and the others are




Dash, point and solid lines show the values of $\tau^*_{AV}$, $\tau^*_D$ and $\tau^*_P$, respectively.

Figure 1: Optimal $\tau$ changes in $\theta_1$, $\theta_2$, $\eta$ and $n$.

fixed. To determine the value of $\tau^*_{AV}$ in this analysis, we assume that $\omega = 0.25$. The results are presented in Figure 1. Notice that in this figure, the dash, point and solid lines show the values of $\tau^*_{AV}$, $\tau^*_D$ and $\tau^*_P$, respectively.
From Figure 1, it is observed that:

1. For fixed $\theta_2, \eta$ and $n$, the optimal value of change time $\tau$ in terms of AV-, D- and P-optimality criteria decreases, when $\theta_1$ closes to $\theta_2$. However, the P-optimality is more sensitive criterion than two others, the AV-optimality is approximately stable (see the top left-hand corner of the Figure 1).

2. For pre-determined $\theta_1, \eta$ and $n$, the optimal value of $\tau$ increases in terms of D- and P-optimality criteria, when $\theta_2$ moves away from $\theta_1$, while it is decreasing in terms of AV-optimality criterion (see the

top right-hand corner of the Figure 1). In this case, the different criteria give quite different results. This is because each criterion can be used for a special purpose. For example, the AV criterion focuses on estimation of the reliability under normal working conditions while the other criteria play an important role in the situations in which the estimation of the parameters $\theta_1$ and $\theta_2$ is the main goal without relating back to normal working conditions.

3. For pre-specified $\theta_1, \theta_2$ and $n$, the optimal value of $\tau$ increases when the censoring time $\eta$ increases, moreover, for a large value of $\eta$ (as $\eta > 130$), the optimal value of $\tau$ is quite stable (see the bottom left-hand corner of the Figure 1).

4. For fixed $\theta_1, \theta_2$ and $\eta$, the optimal value of $\tau$ increases when the sample size $n$ increases, specially, for a moderate value of $n$ (as $n > 10$), the optimal value of $\tau$ is quite stable (see the bottom right-hand corner of the Figure 1).

# 6 Simulation Study

As seen in Section 4, all the criteria used in the paper to find the optimal simple SSALT plan depend on the parameters $\theta_1$ and $\theta_2$. Hence, when the parameters are unknown they must be estimated based on a preliminary test (without knowing the optimal $\tau$) and plugged into the formulas. Therefore, the estimation error will be a main concern. In this section, we present a simulation study to investigate the robustness of the criteria with respect to estimation of the parameters. Toward this end, the following algorithm has been used:

1. It is assumed that $\theta_1 = 0.02$ and $\theta_2 = 0.05$.

2. The optimal values of $\tau$ are derived as $\tau^*_{AV} = 32$, $\tau^*_D = 24$ and $\tau^*_P = 27$, which are computed by minimizing (20), maximizing (21) and minimizing (22), respectively. To derive the value of $\tau^*_{AV}$, it is also assumed that $\omega = 0.25$.

3. By considering all optimal values of $\tau$ derived in the Step 2, a random sample of size $n = 20$ is separately taken from the cdf in (3).

4. Assuming $\eta = 55$, the values of $\hat{\theta}_1$ and $\hat{\theta}_2$ are obtained using (7).

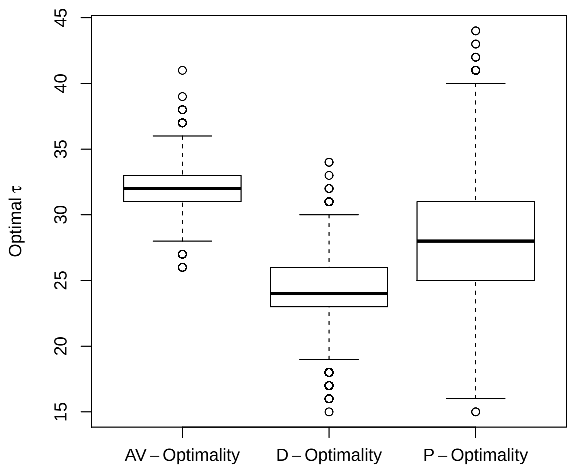



Figure 2: Box plot of optimal $\tau$ for different optimality criteria.

5. By substituting the values of $\hat{\theta}_1$ and $\hat{\theta}_2$ in the objective functions (20), (21) and (22), the associated optimal values of $\tau$ are computed again by the same way as expressed in the Step 2.

6. The Steps 3–5 are repeated $10^4$ times and the box plots of optimal change times are obtained for all criteria.

The results are presented in Figure 2 and it is observed that the AV-optimality criterion is quite stable, but the D- and P-optimality criteria are sensitive to estimation error of the parameters. Note that the functional type of the log-linear relationship considered between the parameters and stress levels may be the main reason of the fact that the AV-optimality criterion is less influenced by estimation error than the D- and P-optimality criteria. This implies that use of the log-linear function in (18) is a good idea to estimate the reliability under the normal operating conditions.

## 7 Conclusions

In this paper, a simple SSALT plan was considered in a discrete set-up for which the failure times at each level of stress are geometrically distributed. It was also assumed that the data are sampled under Type-I censoring scheme. To find the optimal change time of the low to high stress levels, three different criteria were considered, namely, the AV-, D- and P-optimality criteria. The AV-optimality criterion determined the optimal value of change time such that the asymptotic variance of the reliability estimate under the stress level $S_0$, which is a function of $\hat{\theta}_1$ and $\hat{\theta}_2$, is minimized. Maximizing the determinant of the FI matrix and the minimizing the probability of non-existence of the parameter estimates lead to the D- and P-optimal change times, respectively. Since each criterion can be used for a special purpose, it was also seen that in some situations the different criteria give quite different results. Furthermore, all the optimality criteria used in the paper were depended on the parameters of interest, hence, a simulation study was performed and it was seen that the AV-optimality criterion was less influenced by estimation error than the D- and P-optimality criteria. The proposed procedure can be extended to some other cases:

- To derive the AV-optimal plan, a log-linear relationship between the parameters and stress levels was considered. Some other choices may also be studied in future works.
- All results of the paper were derived by assuming that the data come from the geometric distribution. Notice that for applications of the proposed procedure in a practical setting it is important to verify that the assumptions for using the geometric distribution actually hold. Independence and equal probability in each trial are assumptions that have to be fulfilled for the geometric distribution to apply.

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## References

Arefi, A. and Razmkhah, M. (2013), Point and interval estimation for a simple step-stress model with Type-I censored data from the geometric distributions. Submitted.

Arefi, A., Razmkhah, M., and Mohtashami Borzadaran, G. R. (2011), Bayes estimation for a simple step-stress model with Type-I censored data from the geometric distribution. J. Statist. Res. Iran, **8**, 149-169.

Bagdonavicius, V. and Nikulin, M. (2002), Accelerated Life Models: Modeling and Statistical Analysis. Chapman & Hall/CRC Press, Boca Raton, FL.

Bai, D. S., Kim, M. S., and Lee, S. H. (1989), Optimum simple step-stress accelerated life tests with censoring. IEEE Trans. Reliability, **38**, 528–532.

Balakrishnan, N., Cramer, E., and Dembińska, A. (2011), Characterizations of geometric distribution through progressively Type-II right-censored order statistics. Statistics, **45**, 559–573.

Balakrishnan, N. and Han, D. (2009), Optimal step-stress testing for progressively Type-I censored data from exponential distribution. J. Statist. Plann. Inference, **139**, 1782–1798.

Davarzani, N. and Parsian, A. (2011), Statistical inference for discrete middle-censored data. J. Statist. Plann. Inference, **141**, 1455–1462.

Fard, N. and Li, C. (2009), Optimal simple step stress accelerated life test design for reliability prediction. J. Statist. Plann. Inference, **139**, 1799–1808

Gan, G. and Bain, L. J. (1995), Distribution of order statistics for discrete parents with applications to censored sampling. J. Statist. Plann. Inference, **44**, 37–46.

Gouno, E., Sen, A., and Balakrishnan, N. (2004), Optimal step-stress test under progressive Type-I censoring. IEEE Trans. Reliability, **53**, 383–393.

Kateri, M., Kamps, U., and Balakrishnan, N. (2011), Optimal allocation of change points in simple step-stress experiments under Type-II censoring. Comput. Statist. Data Analysis, **55**, 236–247.

Khamis, I. H. (1997), Optimum $m$-step step-stress test with $k$ stress variables. Commun. Statist. Simul. Comput., **26**, 1301–1313.

Khamis, I. H. and Higgins, J. J. (1996), Optimum 3-step step-stress tests. IEEE Trans. Reliability, **45**, 341–345.

Khamis, I. H. and Higgins, J. J. (1998), A new model for step-stress testing. IEEE Trans. Reliability, **47**, 131–134.

Lawless, J. L. (2003), Statistical Models and Methods for Lifetime Data. Second Edition, New York: John Wiley.

Lehmann, E. L. and Casella, G. (1998), Theory of Point Estimation. Second Edition, New York: Springer.

Ling, L., Xu, W., and Li, M. (2011), Optimal bivariate step-stress accelerated life test for Type-I hybrid censored data. J. Statist. Comput. Simul., **81**, 1175–1186.

Meeker, W. Q. and Escobar, L. A. (1998), Statistical Methods for Reliability Data. New York: Wiley.

Miller, R. and Nelson, W. (1983), Optimum simple step-stress plans for accelerated life testing. IEEE Trans. Reliability, **32**, 59–65.

Nagaraja, H. N. (1992), Order statistics from discrete distributions. Statistics, **23**, 189–216.

Nelson, W. (1990), Accelerated Testing: Statistical Models, Test Plans, and Data Analysis. New York, Wiley.

Rezaei, A. H. and Arghami, N. R. (2002), Right censoring in a discrete life model. Metrika, **55**, 151–160.

Wang, B. X. and Yu, K. (2009), Optimum plan for step-stress model with progressive Type-II censoring. Test, **18**, 115–135.

Wang, R., Xu, X., Pan, R., and Sha, N. (2012), On parameter inference for step-stress accelerated life test with geometric distribution. Comm. Statist. Theory Methods, **41**, 1796–1812.

Wu, S. J., Lin, Y. P., and Chen, S. T. (2008), Optimal step-stress test under Type-I progressive group-censoring with random removals. J. Statist. Plann. Inference, **138**, 817–826.

Xu, X., Wang R., Sun, Z., and Ping, L. (2010), The statistical analysis of geometric distribution under the step-stress accelerated life testing. The proceeding of 2010 international conference on probability and statistics of the international institute for general system studies, 287–290.