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Bayesian Prediction Intervals for Future Order Statistics from the Generalized Exponential Distribution

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Abstract. Let $X_1, X_2, ..., X_r$ be the first r order statistics from a sample of size n from the generalized exponential distribution with shape parameter θ. In this paper, we consider a Bayesian approach to predicting future order statistics based on the observed ordered data. The predictive densities are obtained and used to determine prediction intervals for unobserved order statistics for one-sample and two-sample prediction plans. A numerical study is conducted to illustrate the prediction procedures.

1 Introduction

Let $X_1, X_2, ..., X_n$ denote the order statistics of a sample of size n from generalized exponential (GE) distribution with probability density function (pdf)

$$f(x;\theta) = \theta(1 - e^{-x})^{\theta-1}e^{-x}; \qquad x > 0, \quad \theta > 0. \tag{1}$$

and cumulative distribution function (cdf)

$$F(x;\theta) = (1 - e^{-x})^{\theta}; \qquad x > 0, \quad \theta > 0, \tag{2}$$

where θ is a shape parameter. When $\theta = 1$, the GE distribution reduces to the standard exponential distribution. When θ is an integer, the GE distribution is the distribution of the maximum of a sample of size n from the standard exponential distribution. The GE distribution has a unique mode and its median is –ln(1-$(0.5)^{1/\theta}$), where ln denotes the natural logarithm. Gupta and Kundu (1999) used the above given distribution for analyzing skewed data. Gupta and Kundu (2001a) showed that the GE distribution can be used as a good alternative to the gamma or the Weibull models. They observed that this distribution has more similarities to gamma family than to Weibull family in terms of hazard function. It has an increasing hazard function if $\theta > 1$ and decreasing hazard function if $\theta < 1$. The density function varies significantly depending on the shape parameter. Therefore this distribution can also be used in a situation where the course of disease is such that mortality reaches a peak after some finite period, and then slowly decline. For example, in a study of curability of breast cancer, Langlands et al. (1979) found that the peak of mortality occurred after three years. It is therefore, important to analyze such data sets with appropriate models like gamma, Weibull or GE distributions. The GE distribution has many properties that are quite similar to those of the gamma distribution, but it has a distribution function similar to that of the Weibull distribution which can be computed simply. The GE family has likelihood ratio ordering on the shape parameter; so it is possible to construct uniformly most powerful test for testing one-sided hypothesis on the shape parameter, when the scale and location parameters are known. Gupta and Kundu (2003) used the ratio of the maximized likelihoods in discriminating between the Weibull and the GE distributions. Raqab and Ahsanullah (2001) and Raqab (2002) obtained the estimation of the location and scale parameters of the GE distributions based on order statistics and record values, respectively. Recently Raqab and Madi (2005) used importance sampling techniques in the Bayesian estimation and prediction for the GE distribution.

Let $X_1, X_2, ..., X_n$ be the order statistics from a sample of size n from GE distribution. Let $\mathbf{X} = (X_1, X_2, ..., X_r)$, $r \leq n$, be the censored sample. Prediction problem does arise naturally in the context of order statistics. Two prediction scenarios are considered; first, given the r observed order statistics $x_1 \leq x_2 \leq ... \leq x_r$, we predict the remaining order statistics $x_{r+1}, x_{r+2}, ..., x_n$. This is referred to as one-sample prediction. The second scenario, known as the two-sample prediction, consists of predicting the first m order statistics in a future sample. Prediction intervals for different statistics of future observations are discussed in the literature. Ahsanullah (1980) developed the best linear unbiased predictors (BLUP's) of the future record statistics from the exponential distribution. Raqab (1997) obtained the modified maximum likelihood predictors of future order statistics from normal samples. Other prediction problems can be found in Lawless (1973), Kaminsky and Nelson (1974), Evans and Ragab (1983) and Sartawi and Abu-Salih (1991).

In the context of prediction, we say that $(L(\mathbf{X}), U(\mathbf{X}))$ is a $100(1-\alpha)\%$ prediction interval for a future random variable Y if

$$P(L(\mathbf{X}) < Y < U(\mathbf{X})) = 1 - \alpha,$$

where $L(\mathbf{X})$ and $U(\mathbf{X})$ are lower and upper prediction limits for the random variable Y, and $1 - \alpha$ is called the confidence prediction coefficient.

In this paper, we use Bayesian statistical analysis to predict future order statistics from GE distribution on the basis of some ordered data. In Section 2, we obtain prediction intervals for order statistics using a one-sample prediction plan. In Section 3, we present prediction intervals for future order data based on a two-sample prediction plan. Section 4 includes an illustration of the proposed methods using a simulated data set and different choices of prior parameters.

2 Bayesian Prediction Interval for the $(r+s)^{th}$ Order Statistic: One-Sample Case

The likelihood for θ of the given type II censored sample $\mathbf{X} = (X_1, X_2, \ldots, X_r)$ is given by:

$$f(\mathbf{x}|\ \theta) = \frac{n!}{(n-r)!}\theta^r \prod_{i=1}^{r} e^{-x_i} \prod_{i=1}^{r}(1-e^{-x_i})^{\theta-1}\left[1-\left(1-e^{-x_r}\right)^{\theta}\right]^{n-r} \quad (3)$$

$$0 \le x_1 \le ... \le x_r, \theta > 0.$$

We assume that θ follows a Gamma distribution with density function

$$\pi(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}, \theta > 0. \quad (4)$$

From (3) and (4), we get the posterior density of θ as

$$\pi(\theta|\ \mathbf{x}) = \frac{\theta^{a+r-1}e^{-b\theta}v^{\theta}(1-\omega^{\theta})^{n-r}}{\sum_{k=0}^{n-r}\binom{n-r}{k}(-1)^k(b-\ln(\omega^k v))^{-a-r}\ \Gamma(a+r)}, \quad (5)$$

where

$$v = \prod_{i=1}^{r}(1-e^{-x_i}) \text{ and } \omega = 1-e^{-x_r}.$$

Let $X_{r+1,}X_{r+2,},...,X_n$ be the future remaining order statistics. The extended likelihood function is

$$\begin{aligned} f(x_1, x_2, .., x_{r+s}, .., x_n|\theta) &= n!f(x_1|\theta)...f(x_{r+s}|\theta)...f(x_n|\theta), \\ &\quad 0 \le x_1 \le ... \le x_r \le ... \le x_n \end{aligned} \quad (6)$$

By integrating (6), with respect $x_{r+1}, x_{r+2}, .., x_{r+s-1}, x_{r+s+1} .., x_n$, we have

$$\begin{aligned} &f(x_1, x_2, .., x_r, x_{r+s}|\theta) = \\ &\frac{n!}{(s-1)!(n-s-r)!}\prod_{i=1}^{r} f(x_i)[F(x_{r+s})-F(x_r)]^{s-1} \\ &[1-F(x_{r+s})]^{n-r-s}\ f(x_{r+s}). \end{aligned} \quad (7)$$

On using (1) and (2), we obtain

$$\begin{aligned} &f(x_1, x_2, .., x_r, x_{r+s}|\theta) = \\ &\frac{n!}{(s-1)!(n-s-r)!}\theta^{r+1}\prod_{i=1}^{r}\left[e^{-x_i}(1-e^{-x_i})^{\theta-1}\right] \\ &\left[e^{x_{r+s}}(1-e^{-x_{r+s}})^{\theta-1}\right]\left[1-(1-e^{-x_{r+s}})^{\theta}\right]^{n-r-s} \\ &\sum_{i=1}^{s-1}\frac{(-1)^i}{i!(s-i-1)!}(1-e^{-x_r})^{\theta i}(1-e^{-x_{r+s}})^{\theta(s-i-1)} \end{aligned} \quad (8)$$

It follows from (3) and (8) that

$$\begin{aligned} f(x_{r+s}|\theta,\mathbf{x}) = \\ & s\binom{n-r}{s}\theta\left[e^{-x_{r+s}}(1-e^{-x_{r+s}})^{\theta-1}\right] \\ & \left[1-(1-e^{-x_{r+s}})^{\theta}\right]^{n-r-s}(1-\omega^{\theta})^{r-n} \\ & \sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}(1-e^{-x_i})^{\theta i}(1-e^{-x_{r+s}})^{\theta(s-i-1)} \end{aligned} \tag{9}$$

Forming the product of $f(x_{r+s}|\theta,\mathbf{x})$ and the posterior density of θ given in (5) and integrating out θ, it may be shown that for $1 \le s \le n-r$, the predictive density function of x_{r+s} given $\mathbf{x}$ is

$$\begin{aligned} p(x_{r+s}|\mathbf{x}) &= \frac{s\binom{n-r}{s}e^{-x_{r+s}}\sum_{i=0}^{s-1}\sum_{j=0}^{n-r-s}(-1)^{i+j}\binom{s-1}{i}\binom{n-r-s}{j}W_{ij}}{\sum_{k=0}^{n-r}\binom{n-r}{k}(-1)^k(b-\ln(\omega^k v))^{-a-r}\ \Gamma(a+r)}, \\ & x_{r+s} > 0, \end{aligned} \tag{10}$$

where

$$W_{ij} = \frac{(1-e^{-x_{r+s}})^{-1}\ \Gamma(a+r+1)}{\left\{b-\ln\left[v\ \omega^i(1-e^{-x_{r+s}})^{(j+s-i)}\right]\right\}^{a+r+1}}.$$

The predictive density $p(x_{r+s}|\mathbf{x})$ can be used to find the prediction bounds on x_{r+s}. Note that

$$\begin{aligned} P(X_{r+s} \ge y|\mathbf{x}) = \\ & \frac{s\binom{n-r}{s}}{T}\sum_{i=0}^{s-1}\sum_{j=0}^{n-r-s}\frac{(-1)^{i+j}\binom{s-1}{i}\binom{n-r-s}{j}}{(j-i+s)} \\ & [b-\ln(\omega^i v)]^{-(a+r)}\left\{1-\left[1-\frac{(j-i+s)\ln(1-e^{-y})}{b-\ln(\omega^i\ v)}\right]^{-(a+r)}\right\} \end{aligned} \tag{11}$$

where

$$T = \sum_{k=0}^{n-r}\binom{n-r}{k}(-1)^k(b-\ln(\omega^k\ v))^{-(a+r)}.$$

Let $L(\mathbf{x})$ and $U(\mathbf{x})$ be the lower and upper bounds for $100(1-\alpha)\%$ prediction interval, respectively. Then $100(1-\alpha)\%$ prediction bounds can be obtained by equating (11) to $1-\alpha/2$ for the lower limit and

$\alpha/2$ for the upper limit and solving the resulting equations for y using numerical techniques.

When $s = 1$ (we wish to predict the next failure time), Equation (11) becomes

$$P(X_{r+1} \geq y|\mathbf{x}) = \frac{n-r}{T} \sum_{j=0}^{n-r-1} \frac{(-1)^j \binom{n-r-j}{j}}{(j-i+s)} [b - \ln(v)]^{-(a+r)} \left\{ 1 - \left[1 - \frac{(j+1)\ln(1-e^{-y})}{b-\ln(v)} \right]^{-(a+r)} \right\}. \tag{12}$$

When $s = n - r$; that is, we predict the last failure time. In this case, Equation (11) reduces to

$$P(X_n \geq y|\mathbf{x}) = \frac{n-r}{T} \sum_{i=0}^{n-r-1} \frac{(-1)^i \binom{n-r-1}{i}}{(i+n-r)} \left[b - \ln(\omega^i v) \right]^{-(a+r)} \left\{ 1 - \left[1 - \frac{(n-r-i)\ln(1-e^{-y})}{b-\ln(\omega^i\, v)} \right]^{-(a+r)} \right\}. \tag{13}$$

from which prediction can be made about X_n. Consider the special case where $r = n - 1$, and $s = 1$. In this case, we predict the last failure time X_n based on observing $X_1, X_2, ..., X_{n-1}$. Substituting $r = n - 1$ in (13) and solving the equation:

$$P(X_n \geq y|\mathbf{x}) = \gamma, \tag{14}$$

where $\gamma = 1 - \alpha/2$ and $\gamma = \alpha/2$, we get a $100(1-\alpha)\%$ prediction interval for X_n as $(L(\mathbf{x}), U(\mathbf{x}))$, so that

$$L(\mathbf{x}) = -\ln \left[1 - \left(\frac{e^b}{v} \right)^{\xi_1} \right] \quad \text{and} \quad U(\mathbf{x}) = -\ln \left[1 - \left(\frac{e^b}{v} \right)^{\xi_2} \right],$$

where

$$\xi_1 = 1 - \left[1 - \frac{1}{1 - c\left(\frac{1+\alpha}{2}\right)} \right]^{\frac{1}{r+a}},$$

and

$$\xi_2 = 1 - \left[1 - \frac{1}{1 - c\left(\frac{1-\alpha}{2}\right)}\right]^{\frac{1}{r+a}},$$

with

$$c = 1 - \left[1 - \frac{\ln(\omega)}{b - \ln(v)}\right]^{-(r+a)}.$$

Prediction intervals for the remaining order statistics can also be found using The Gibbs Sampler. Let $\mathbf{Z} = (X_{r+1}, X_{r+2}, ..., X_n)$. By forming the product of the extended likelihood and the prior of θ, the full Bayesian model is expressed as

$$\pi(\theta, \mathbf{z}|\mathbf{x}) \propto \theta^{n+a-1} \exp\{-n\overline{x} - \theta(D+b) + D\},$$

where $D = -\sum_{i=1}^{n} \ln(1 - e^{-x_i})$ and $\overline{x} = \sum_{i=1}^{n} x_i/n$.

Setting $\mathbf{Z}_s = (X_{r+1}, ..., X_{s-1}, X_{s+1}, ...X_n)$, the full conditional distribution of X_s $(r+1 \le s \le n)$, is found to be

$$\pi(x_s|\mathbf{x}, \mathbf{z}_s, \theta) = \begin{cases} \frac{\theta e^{-x_s}(1-e^{-x_s})^{\theta-1} I[x_{s-1} < x_s < x_{s+1}]}{(1-e^{-x_{s+1}})^{\theta} - (1-e^{-x_{s-1}})^{\theta}}, s = r+1, ..., n-1 \\ \frac{\theta e^{-x_n}(1-e^{-x_n})^{\theta-1} I_{(x_n > x_{n-1})}}{1-(1-e^{-x_{n-1}})^{\theta}}, \qquad s = n. \end{cases} \tag{15}$$

and the full conditional distribution of $\theta\,|\mathbf{x}, \mathbf{y}$ is $G(n+a, D+b)$.

Given an arbitrary set of starting values of θ and $\mathbf{z}$, we generate values from (15) using the inverse cdf transformation method (Devroye, 1986) as follows:

$$X_j = -\ln\left\{1 - \left[(1 - e^{-X_{j-1}})^{\theta} + U\left[(1 - e^{-X_{j+1}})^{\theta} - (1 - e^{-X_{j-1}})^{\theta}\right]\right]^{\theta^{-1}}\right\}, \quad for \quad j = r+1, ..., n-1,$$

$$X_n = -\ln\left\{1 - \left[(1 - e^{-X_{j-1}})^{\theta} + U\left[1 - (1 - e^{-X_{n-1}})^{\theta}\right]\right]^{\theta^{-1}}\right\} \tag{16}$$

where $U \sim U(0,1)$. In fact, θ is generated directly from its standard full conditional distribution.

3 Bayesian Prediction Interval for Future order Statistics: Two-Sample Case

Let $\mathbf{Y} = (Y_1, Y_2, ..., Y_m)$ be a future ordered random sample independent of $\mathbf{X}$ from the GE distribution with density (3). It can be shown that the predictive density of Y_k given $\mathbf{x}$ is expressed as

$$p(y_k|\ \mathbf{x}) = \frac{k\binom{m}{k}e^{-y_k}(1-e^{-y_k})^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{m-k}\binom{n-r}{i}\binom{m-k}{j}(-1)^{i+j}\ V_{ij}}{Q(\mathbf{x})}$$

where

$$Q(\mathbf{x}) = \sum_{k=0}^{n-r}\binom{n-r}{k}(-1)^k(b-\ln(\omega^k v))^{-a-r}\ \Gamma(a+r)$$

and

$$V = \frac{\Gamma(a+r+1)}{\{b-\ln[\omega^i\ v(1-e^{-y_k})^{k+j}]\}^{a+r+1}}$$

It follows that

$$P(Y_k \ \ \geq \ \ y|\mathbf{x}) = \frac{k\binom{m}{k}}{T}\sum_{i=0}^{n-r}\sum_{j=0}^{m-k}\frac{\binom{n-r}{i}\binom{m-k}{j}(-1)^{i+j}[b-\ln(\omega^i v)]^{-a-r}}{(k+j)}$$
$$.\left\{1-\left[1-\frac{(k+j)\ln(1-e^{-y})}{b-\ln(\omega^k\ v)}\right]^{-a-r}\right\}, \tag{17}$$

where

$$T = \sum_{k=0}^{n-r}\binom{n-r}{k}(-1)^k[b-\ln(\omega^i\ v)]^{-a-r}.$$

Since (17) does not permit explicit solution for the prediction bounds on y_k, numerical methods have to be employed. The $100(1-\alpha)\%$ prediction bounds for the k^{th} order statistic Y_k can be obtained by equating (17) to $1-\alpha/2$ for the lower limit and $\alpha/2$ for the upper limit and solving the resulting equations for y using numerical techniques.

For example, to predict Y_m, we need to solve

$$1-\frac{1}{T}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i[(b-\ln(\omega^i\ v))-m\ln(1-e^{-y})]^{-a-r} = \gamma \tag{18}$$

and if $r = n$ (prediction of Y_m based on a complete sample), (18) reduces to

$$1-\left[1-\frac{m\ln(1-e^{-y})}{b-\ln v}\right]^{-a-n} = \gamma$$

and results in

$$L(\mathbf{x}) = -\ln\left[1-\left(\frac{e^b}{v}\right)^{\delta_1}\right] \ \text{ and } \ U(\mathbf{x}) = -\ln\left[1-\left(\frac{e^b}{v}\right)^{\delta_2}\right],$$

where

$$\delta_1 = m^{-1}[1-(\frac{2}{2-\alpha})^{\frac{1}{a+n}} \quad \text{and} \quad \delta_2 = 1-(\frac{2}{\alpha})^{\frac{1}{a+n}}.$$

When $r = n$ and $k = 1$, we predict Y_1 based on a complete sample of size n. Then we have to solve

$$m\sum_{j=0}^{m-1}\frac{\binom{m-1}{j}(-1)^j}{(1+j)}\left\{1-\left[1-\frac{(1+j)\ln(1-e^{-y})}{b-\ln v}\right]^{-a-n}\right\} = \gamma \quad (19)$$

A special case of (19) is to predict a single future failure. Setting $m = 1$ in (19), we get

$$1-\left[1-\frac{\ln(1-e^{-y})}{b-\ln v}\right]^{-a-n} = \gamma.$$

The resulting $100(1-\alpha)\%$ prediction limits are

$$L(\mathbf{x}) = -\ln\left[1-\left(\frac{e^b}{v}\right)^{\kappa_1}\right] \quad \text{and} \quad U(\mathbf{x}) = -\ln\left[1-\left(\frac{e^b}{v}\right)^{\kappa_2}\right].$$

where

$$\kappa_1 = 1-(\frac{2}{\alpha})^{\frac{1}{a+n}} \quad \text{and} \quad \kappa_2 = 1-(\frac{2}{2-\alpha})^{\frac{1}{a+n}}.$$

Prediction intervals for the future order statistics can also be found via MCMC. By forming the product of the extended likelihood and the prior of θ, the full Bayesian model is expressed as

$$\begin{aligned}\pi(\theta,\mathbf{y}|\mathbf{x}) \quad \propto \quad & \theta^{m+r+a-1}\exp\{-m\overline{y}-\theta(D_m+b)+D_m\} \\ & \sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\exp\{-\theta[D_r+iT_r]\},\end{aligned}$$

where $D_m = -\sum_{i=1}^m \ln(1-e^{-y_i})$, $\overline{y} = \sum_{i=1}^m y_i/m$, $D_r = -\sum_{i=1}^r \ln(1-e^{-x_i})$ and $T_r = -\ln(1-e^{-x_r})$.

Setting $\mathbf{Y}_k = (Y_1, ..., Y_{k-1}, Y_{k+1}, ...Y_m)$, the full conditional distribution of Y_k $(1 \le k \le m)$, is found to be

$$\pi(y_k|\mathbf{x},\mathbf{y}_k,\theta) = \begin{cases} \frac{\theta\, e^{-y_k}(1-e^{-y_k})^{\theta-1}\, I[y_{k-1}<y_k<y_{k+1}]}{(1-e^{-y_{k+1}})^{\theta}-(1-e^{-y_{k-1}})^{\theta}}, & k=1,...,m-1 \\ \frac{\theta e^{-y_m}\,(1-e^{-y_m})^{\theta-1} I_{(y_m>y_{m-1})}}{1-(1-e^{-y_{m-1}})^{\theta}}, & k=m, \end{cases} \quad (20)$$

and the full conditional distribution of $\theta\,|\mathbf{x},\mathbf{y}$ is given by

$$\pi(\theta\,|\mathbf{x},\mathbf{y}) \propto \theta^{m+r+a-1}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i \exp\{-\theta[D_r + D_m + iT_r + b]\}$$

Using the Gibbs sampler to estimate the posterior distribution requires being able to sample from the full conditional distributions for each quantity involved. This is the case for Y_k but not for θ. Consequently, Metropolis-Hastings (M-H) steps are introduced into the Gibbs sampler so that Y_k is sampled directly from its full conditional distribution via the the inverse cdf transformation method, whereas θ is updated via a M-H step as explained in Tierney (1994), using $G(m+r+a, D_r+D_m+b)$ as a proposal distribution. The M-H step proceeds as follows:

Given $\theta^{(i-1)}$,
(i) Sample y from $G[m+r+a, D_r+D_m+b]$ and u from $U(0,1)$
(ii) If $u < \min(1,\vartheta)$ then let $\theta^{(i)} = y$ else go to (i), where

$$\vartheta = \frac{\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}e^{-(iyT_r)}}{\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}e^{-(i\theta T_r)}}.$$

4 Data Analysis

In this section, we illustrate the procedures by presenting a complete analysis for a simulated data set. The following data sample was generated from GE distribution $G(3,\ 1)$. Suppose that $r = 15$ observed order statistics are available from a sample of size $n = 20$.These observations are as follows:

0.65306, 0.67631, 0.68341, 1.05645, 1.46194, 1.71555, 1.73903, 1.78940, 1.79847, 1.82522, 1.95587, 2.16530, 2.35033, 2.38706, 2.39005

We present some results to compare the performance of the classical and Bayesian approaches for different choices of prior parameters. All computations are performed via Mathematica 5.0 and Fortran-90.

We use the iterative alogirthm to find the root y that solves $P(X_{r+s} > y) = \gamma = 0.975$ and $\gamma = 0.025$ and the iterative process stops when the difference between two consecutive iterates are less than 10^{-10}. This allows us to compute 95% confidence intervals. Tables 1 and 3 present 95% Bayesian prediction intervals of

$X_i(i = 16, 17, ..., 20)$ and $Y_i(i = 1, 2, ..., 5)$ for prior parameters $(a = 4,\ b = 2)$, $(a = b = 0.25)$, $(a = 3,\ b = 1)$ and $(a = 1,\ b = 0)$. Further, we apply the Gibbs and Metropolis samplers to determine the Bayesian prediction intervals. After setting initial values of θ and $\mathbf{x}$ for the one-sample prediction and θ and $\mathbf{y}$ for the 2-sample prediction, a sampler single chain with pre-determined number of iterations is run and used as input in Raftery & Lewis Fortran program (Raftery and Lewis, 1992) to determine the required number of iterations needed to attain convergence. Subsequent to convergence, 5,000 draws of equally spaced variates were collected for the parameter θ as well as $\mathbf{x}$ and $\mathbf{y}$. Tables 2 and 4 present 95% MCMC prediction intervals for the remaining and future order statistics.

Although, both mehods provide close lower and upper limits of 95% prediction intervals, it is observed that the prediction intervals tend be wider when s and k increase. This is a natural, since the prediction of the future order statistic that is far a way from the last observed value has less accuracy than that of other future order statistics.

Table 1: Bayesian Prediction Intervals: One-Sample Case

			$a=4,\ b=2$		$a=b=0.25$		$a=3,\ b=1$		$a=1,\ b=0$	
n	r	s	L	U	L	U	L	U	L	U
20	15	1	2.396	3.197	2.396	3.212	2.396	3.210	2.396	3.223
		2	2.337	3.743	2.335	3.762	2.336	3.759	2.334	3.776
		3	2.569	4.418	2.573	4.440	2.573	4.437	2.576	4.456
		4	2.127	5.450	2.119	5.475	2.120	5.471	2.113	5.492
		5	3.103	7.801	3.116	7.826	3.114	7.826	3.126	7.844

Table 2: MCMC Bayesian Prediction Intervals: One-Sample Case

			$a=4,\ b=2$		$a=b=0.25$		$a=3,\ b=1$		$a=1,\ b=0$	
n	r	s	L	U	L	U	L	U	L	U
20	15	1	2.397	3.184	2.396	3.209	2.396	3.210	2.396	3.212
		2	2.452	3.752	2.457	3.792	2.458	3.741	2.456	3.757
		3	2.566	4.442	2.575	4.447	2.577	4.444	2.572	4.461
		4	2.764	5.474	2.773	5.503	2.755	5.524	2.758	5.503
		5	3.113	7.994	3.127	7.954	3.088	7.968	3.112	7.891

Table 3: Bayesian Prediction Intervals: Two-Sample Case

			$a=4, b=2$		$a=b=0.25$		$a=3, b=1$		$a=1, b=0$	
n	r	k	L	U	L	U	L	U	L	U
20	15	1	0.222	1.854	0.273	1.991	0.273	1.969	0.318	2.073
		2	0.521	2.475	0.597	2.617	0.596	2.595	0.661	2.702
		3	0.815	3.187	0.906	3.324	0.904	3.309	0.979	3.417
		4	1.153	4.235	1.255	4.377	1.252	4.357	1.335	4.466
		5	1.630	6.582	1.743	6.723	1.738	6.705	1.828	6.813

Table 4: MCMC Bayesian Prediction Intervals: Two-Sample Case

			$a=4, b=2$		$a=b=0.25$		$a=3, b=1$		$a=1, b=0$	
n	r	k	L	U	L	U	L	U	L	U
20	15	1	0.233	1.989	0.338	2.203	0.383	2.095	0.420	2.222
		2	0.540	2.468	0.717	2.825	0.777	2.749	0.718	2.847
		3	0.846	3.280	1.083	3.486	1.005	3.512	1.119	3.637
		4	1.166	4.288	1.470	4.457	1.342	4.496	1.481	4.694
		5	1.788	6.397	1.990	6.806	1.862	7.040	2.036	7.565

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