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Outer and Inner Confidence Intervals Based on Extreme Order Statistics in a Proportional Hazard Model

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Abstract. Let M_i and $M_i^{'}$ be the maximum and minimum of the ith sample from k independent sample with different sample sizes, respectively. Suppose that the survival distribution function of the ith sample is $\bar{F}_i = \bar{F}^{\alpha_i}$, where α_i is known and positive constant. It is shown that how various exact non-parametric inferential procedures can be developed on the basis of M_i's and $M_i^{'}$'s for distribution function F without any assumptions about it other than F is continuous. These include outer and inner confidence intervals for quantile intervals and upper and lower confidence limits for quantile differences. Three schemes have been investigated and in each case, the associated confidence coefficients are obtained. A numerical example is given in order to illustrate the proposed procedure.

1 Introduction

The population quantile ξ_p of order p $(0 < p < 1)$ of cumulative distribution function (cdf) F is defined by $\xi_p = \inf\{x : F(x) \geq p\}$.

Key words and phrases: Coverage probability, proportional hazard model, Quantile difference, Quantile interval, tolerance interval.

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The order statistics play an important role in the inferences related to the quantiles; interested readers may refer to the books of Serfling (1980), Arnold *et al.* (1992) and David and Nagaraja (2003). In recent years, several articles have been published on nonparametric confidence intervals for quantiles based on usual order statistics, ranked set sampling and record statistics. See for example, Hutson (1999), Zielinski and Zielinski (2005), Chen (2000), Ozturk and Deshpande (2006), Balakrishnan and Li (2006), Deshpande *et al.* (2006), Gulati and Padgett (1994), Ahmadi and Arghami (2003), Ahmadi and Balakrishnan (2004) and references therein.

For $0 < p < q < 1$, let $F(\xi_p) = p$ and $F(\xi_q) = q$, Wilks (1962) proposed the random intervals $(X_{i:n}, X_{j:n})$, $j > i$, as the *outer* and *inner* confidence intervals for the quantile intervals (ξ_p, ξ_q), where $X_{i:n}$ and $X_{j:n}$ are the ith and jth smallest observations in a random sample of size n from $F(x)$. Krewski (1976) obtained upper and lower bounds for the confidence coefficients of outer confidence intervals. Reiss and Ruschendorf (1976) improved the results of Krewski (1976). Sathe and Lingras (1981) obtained even sharper bounds by using properties of convex functions. Meyer (1987) found the analogous results of Wilks (1962) for finite populations. Ahmadi and Balakrishnan (2005) obtained the outer and inner confidence intervals for quantile intervals in terms of record statistics.

There are many experiments in which only the lowest or highest observations are recorded. Some experiments which have been done in different periods of times, only maxima are applied, for example scientific competitions and in the other experiments only minima are exerted, like speed sports. Sometimes, both minima and maxima are used, say the lowest and highest temperature during a weak or a month. Therefore, it is worthwhile to use the extreme order statistics in a multisampling plan to do inference about various characteristics of the parent distribution, i.e., mean, standard deviation, quantiles and so on. Constructing nonparametric outer and inner confidence intervals for quantile intervals on the basis of extreme order statistics from k independent random samples is the main aim of this paper. With this in mind, let $X_{i,j}$ $(1 \leq i \leq k, 1 \leq j \leq n_i)$ be independent random variables. Moreover for a fixed i, $X_{i,j}$'s, $(1 \leq j \leq n_i)$ are identically distributed with cdf

$$F_i(x) = 1 - [1 - F(x)]^{\alpha_i} = 1 - [\bar{F}(x)]^{\alpha_i}, \quad \alpha_i > 0, \tag{1}$$

where $F(x)$ is an absolutely continuous distribution function. The aforementioned identity is well-known in the literature as the pro-

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portional hazard rate model (see for example Lawless, 2003). Let M_i $(1 \leq i \leq k)$ be the maximum of a random sample of size n_i; that is, $M_i = \max\{X_{i,1}, X_{i,2}, \ldots, X_{i,n_i}\}$ and $M_i^{'}$ is the corresponding minimum. Then M_i's $(1 \leq i \leq k)$ are independent random variables with cdf

$$F_{M_i}(x) = \left(1 - [\bar{F}(x)]^{\alpha_i}\right)^{n_i}, \quad i = 1, 2, \ldots, k, \tag{2}$$

also, $M_i^{'}$'s $(1 \leq i \leq k)$ are independent random variables with cdf

$$F_{M_i^{'}}(x) = 1 - [\bar{F}(x)]^{n_i \alpha_i}, \quad i = 1, 2, \ldots, k. \tag{3}$$

No previous work seems to have been done on quantile estimation of F in (1). In Section 2, we will attempt to construct nonparametric confidence intervals for quantile intervals of F based on M_i's and $M_i^{'}$'s. Various cases have been studied and in each case, the exact nonparametric confidence intervals are obtained and the exact expressions for the confidence coefficients of these confidence intervals are derived. Such intervals are exact and distribution-free in that the corresponding coverage probabilities are known exactly without any assumptions about the distribution F other than F is continuous. An example to illustrate the proposed procedure is given in Section 3. At the end, some conclusions and discussions are presented.

2 Outer confidence intervals for quantile intervals

Let p and q be any given real numbers satisfying $0 < p < q < 1$, and let (ξ_p, ξ_q) be the interval $\{x|\ p \leq F(x) \leq q\}$. The interval (L, U) is called an outer confidence interval if containing an interval of parent quantiles. Now, we show that how extreme order statistics from k independent random samples come into the picture here in the form of L and U. Toward this end, we consider three schemes.

2.1 Outer confidence intervals based on minima and maxima

Let us arrange M_i's and $M_i^{'}$'s $(1 \leq i \leq k)$ in ascending order and denote the jth order statistic of the set $\{M_1^{'}, M_1, \ldots, M_k^{'}, M_k\}$ by $V_{j:2k}$. In order to construct confidence interval for (ξ_p, ξ_q) in terms of $V_{i:2k}$'s, first we obtain the joint cdf of $V_{i:2k}$ and $V_{2k:2k}$, which is stated in the following.

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Theorem 2.1.1. *Let $M_r^{'}$ and M_r be corresponding minimum and maximum of the rth random sample from distribution F_r $(r = 1, \ldots, k)$, respectively and $V_{i:2k}$ be the ith order statistic of the set*

$$\{M_1^{'}, M_1, \ldots, M_k^{'}, M_k\}.$$

Then, the joint cdf of $(V_{i:2k}, V_{2k:2k})$, $1 \leq i \leq 2k-1$, is as follows

$$\begin{aligned} & P(V_{i:2k} \leq x, V_{2k:2k} \leq y) \\ = & \sum_{r=i}^{2k} \sum_{j=[\frac{r+1}{2}]}^{min(r,k)} \sum_{A_{r-j,j,k}} \Big\{ \prod_{s=1}^{r-j} [F_{t_s}(x)]^{n_{t_s}} \prod_{s=r-j+1}^{j} \big\{ [F_{t_s}(y)]^{n_{t_s}} \\ - & [F_{t_s}(x)]^{n_{t_s}} - [F_{t_s}(y) - F_{t_s}(x)]^{n_{t_s}} \big\} \prod_{s=j+1}^{k} [F_{t_s}(y) - F_{t_s}(x)]^{n_{t_s}} \big\}, \quad (4) \end{aligned}$$

where $[u]$ stands for the integer part of u and $A_{i_1,i_2,k}$ extends over all permutations of $(t_1, \ldots, t_k)$ from $\{1, \ldots, k\}$ such that $t_1 < \cdots < t_{i_1}$, $t_{i_1+1} < \cdots < t_{i_2}$ and $t_{i_2+1} < \cdots < t_k$.

Proof. For $k = 1$ the following identity is obvious

$$P\big(V_{1:2} \leq x, V_{2:2} \leq y\big) = [F_1(y)]^{n_1} - [F_1(y) - F_1(x)]^{n_1},$$

which is also confirmed by (4), so we prove (4) for $k \geq 2$. We can write

$$\begin{aligned} & P\big(V_{i:2k} \leq x, V_{2k:2k} \leq y\big) \\ & = P(\text{at least } i \text{ of } M_j \text{ or } M_j^{'}, j = 1, \ldots, k \text{ are at most } x \\ & \qquad \text{and at least } 2k \text{ of them are at most } y) \\ & = \sum_{r=i}^{2k} \eta_k(r, x, y), \end{aligned}$$

where

$$\begin{aligned} \eta_k(r, x, y) \;=\; & P(\text{exactly } r \text{ of } M_j \text{ or } M_j^{'}, j = 1, \ldots, k \text{ are at most } x \\ & \qquad \text{and exactly } 2k \text{ of them are at most } y). \end{aligned}$$

It may be noted that under the assumptions, there are $2k$ statistics as $M_1^{'} < M_1$, $M_2^{'} < M_2, \ldots, M_k^{'} < M_k$ which are extracted from k

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independent random samples. Let $k = 2$, then we derive an exact expression for $\eta_2(r, x, y)$, $r = 1, \ldots, 4$. We get

$$\begin{aligned}
\eta_2(1,x,y) &= \sum_{A_{0,1,2}} P(M'_{t_1} \le x, x < M_{t_1} \le y) P(x < M'_{t_2}, M_{t_2} \le y), \\
\eta_2(2,x,y) &= \sum_{A_{1,0,2}} P(M_{t_1} \le x) P(x < M'_{t_2}, M_{t_2} \le y) \\
&\quad + \prod_{j=1}^{2} P(M'_j \le x, x < M_j \le y), \\
\eta_2(3,x,y) &= \sum_{A_{1,2,2}} P(M_{t_1} \le x) P(M'_{t_2} \le x, x < M_{t_2} \le y), \\
\eta_2(4,x,y) &= \prod_{j=1}^{2} P(M_j \le x).
\end{aligned}$$

By a careful scrutiny of the details for other values of k and a simplification of computations, for $k \ge 2$ one can deduce

$$\begin{aligned}
\eta_k(r,x,y) &= \sum_{j=[\frac{r+1}{2}]}^{min(r,k)} \sum_{A_{r-j,j,k}} \Bigg\{ \prod_{s=1}^{r-j} P\big(M_{t_s} \le x\big) \\
&\quad \times \prod_{s=r-j+1}^{j} P\big(M'_{t_s} \le x, x < M_{t_s} \le y\big) \\
&\quad \times \prod_{s=j+1}^{k} P\big(M'_{t_s} > x, M_{t_s} \le y\big) \Bigg\}. \qquad (5)
\end{aligned}$$

It is not difficult to verify that

$$P(M'_{t_s} \le x, x < M_{t_s} \le y) = [F_{t_s}(y)]^{n_{t_s}} - [F_{t_s}(x)]^{n_{t_s}} - [F_{t_s}(y) - F_{t_s}(x)]^{n_{t_s}} \qquad (6)$$

and

$$P(M'_{t_s} > x, M_{t_s} \le y) = [F_{t_s}(y) - F_{t_s}(x)]^{n_{t_s}}. \qquad (7)$$

By substituting Eqs. (6) and (7) in (5), the proof is complete. □

From Theorem 2.1.1, we immediately have the following result.

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Corollary 2.1.2 *The marginal cdf of $V_{i:2k}$ is given by*

$$P(V_{i:2k} \leq x) = \sum_{r=i}^{2k} \sum_{j=[\frac{r+1}{2}]}^{min(r,k)} \sum_{A_{r-j,j,k}} \Big\{ \prod_{s=1}^{r-j} [F_{t_s}(x)]^{n_{t_s}} \prod_{s=r-j+1}^{j} \big\{1 - [F_{t_s}(x)]^{n_{t_s}} - [1 - F_{t_s}(x)]^{n_{t_s}} \big\} \prod_{s=j+1}^{k} [1 - F_{t_s}(x)]^{n_{t_s}} \Big\}. \tag{8}$$

The problem is to determine the coverage probability of the event $(V_{i:2k} \leq \xi_p \leq \xi_q \leq V_{j:2k})$, for $j > i$ and $q > p$. We consider two cases:

Case I. j = 2k

Notice that for $q > p$,

$$\begin{aligned} \gamma(i, 2k; p, q) &= P\big(V_{i:2k} \leq \xi_p \leq \xi_q \leq V_{2k:2k}\big) \\ &= P\big(V_{i:2k} \leq \xi_p\big) - P\big(V_{i:2k} \leq \xi_p, V_{2k:2k} \leq \xi_q\big) \\ &= \alpha(i; p) - \beta(i, 2k; p, q), \end{aligned} \tag{9}$$

where from (1) and (8),

$$\begin{aligned} \alpha(i; p) &= P\big(V_{i:2k} \leq \xi_p\big) \\ &= \sum_{r=i}^{2k} \sum_{j=[\frac{r+1}{2}]}^{min(r,k)} \sum_{A_{r-j,j,k}} \prod_{s=1}^{r-j} [1 - (1-p)^{\alpha_{t_s}}]^{n_{t_s}} \\ &\quad \times \prod_{s=r-j+1}^{j} \big\{1 - (1-p)^{\alpha_{t_s} n_{t_s}} - \big[1 - (1-p)^{\alpha_{t_s}}\big]^{n_{t_s}} \big\} \\ &\quad \times (1-p)^{\sum_{s=j+1}^{k} \alpha_{t_s} n_{t_s}}. \end{aligned} \tag{10}$$

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Also, by Theorem 2.1.1 we have,

$$\begin{aligned}
\beta(i,2k;p,q) &= P\big(V_{i:2k} \le \xi_p, V_{2k:2k} \le \xi_q\big) \\
&= \sum_{r=i}^{2k} \sum_{j=[\frac{r+1}{2}]}^{min(r,k)} \sum_{A_{r-j,j,k}} \Bigg\{ \prod_{s=1}^{r-j} [1-(1-p)^{\alpha_{t_s}}]^{n_{t_s}} \\
&\quad \times \prod_{s=r-j+1}^{j} \big\{ [1-(1-q)^{\alpha_{t_s}}]^{n_{t_s}} \\
&\quad -[1-(1-p)^{\alpha_{t_s}}]^{n_{t_s}} - [(1-p)^{\alpha_{t_s}} - (1-q)^{\alpha_{t_s}}]^{n_{t_s}} \big\} \\
&\quad \times \prod_{s=j+1}^{k} [(1-p)^{\alpha_{t_s}} - (1-q)^{\alpha_{t_s}}]^{n_{t_s}} \Bigg\}. \qquad (11)
\end{aligned}$$

Thus, from Eqs. (9) – (11) we have a confidence interval $(V_{i:2k}, V_{2k:2k})$, $1 \le i \le 2k-1$, for quantile interval (ξ_p, ξ_q), $q > p$, whose confidence coefficient is free of F.

Remark 1. *It can be shown that in the special case that $\alpha_i = 1$ and $n_i = n$ $(i = 1, \ldots, k)$,*

$$\max_{i,j} \gamma(i,j;p,q) = 1-(1-p)^{nk} - q^{nk} + (q-p)^{nk}.$$

Case II. j < 2k

In this case it is very intractable to calculate the exact expression for the confidence coefficient of $(V_{i:2k}, V_{j:2k})$ for $j > i$ $(i, j = 1, \ldots, 2k-1)$ as an outer confidence interval for (ξ_p, ξ_q), because the calculation of the joint cdf of $(V_{i:2k}, V_{j:2k})$ is too complicated, so we find confidence level instead of confidence coefficient. From (11), we have

$$\begin{aligned}
\beta(i,j;p,q) &= P\big(V_{i:2k} \le \xi_p, V_{j:2k} \le \xi_q\big) \\
&= P\big(V_{i:2k} \le \xi_p, V_{2k:2k} \le \xi_q\big) \\
&\quad + P\big(V_{i:2k} \le \xi_p, V_{j:2k} \le \xi_q < V_{2k:2k}\big) \\
&\le P\big(V_{i:2k} \le \xi_p, V_{2k:2k} \le \xi_q\big) + P\big(V_{j:2k} \le \xi_q < V_{2k:2k}\big) \\
&= \beta(i,2k;p,q) + \alpha(j,q) - \alpha(2k,q).
\end{aligned}$$

Thus, from (9) the upper and lower bounds for $\gamma(i,j;p,q) = P\big(V_{i:2k} \le \xi_p \le \xi_q \le V_{j:2k}\big)$ are as follows

$$\gamma(i,2k;p,q) + \alpha(2k,q) - \alpha(j,q) \le \gamma(i,j;p,q) \le \gamma(i,2k;p,q), \quad (12)$$

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where $\alpha(\cdot,\cdot)$ is defined in (10).

If p, q, α_r's, n_r's and the desired confidence level γ are specified, we can choose i and j so that $\gamma(i,j;p,q)$ achieve to γ. Note that the choice of i and j is not unique, the one that minimizes the expected length of the confidence interval appears reasonable. Because of the fact that $E(V_{j:2k}-V_{i:2k})$ is a step function of $j-i$, one can choose i and j as close together as possible to accomplish a specified confidence coefficient. It is obvious that for fixed i, the expected length of $(V_{i:2k},V_{j:2k})$ is minimized by minimizing j.

2.2 Outer confidence intervals based on maxima

Denote the rth order statistic of the set $\{M_1,M_2,\ldots,M_k\}$ by $M_{r:k}$. In this subsection, we obtain the outer confidence intervals for (ξ_p,ξ_q), $q>p$, in terms of $M_{r:k}$'s. First, we present the expressions for the probability of the events $\{M_{i:k}\le\xi_p\}$ and $\big\{M_{i:k}\le\xi_p, M_{j:k}\le\xi_q, j>i, q>p\big\}$, respectively. From David and Nagaraja (2003, p. 96), we have

$$\begin{aligned}
\alpha_{max}(i;p) &= P\big(M_{i:k}\le\xi_p\big)\\
&= \sum_{r=i}^{k}\sum_{\Gamma_{r,k}}\prod_{s=1}^{r}F_{M_{t_s}}(\xi_p)\prod_{s=r+1}^{k}\bar{F}_{M_{t_s}}(\xi_p)\\
&= \sum_{r=i}^{k}\sum_{\Gamma_{r,k}}\prod_{s=1}^{r}[1-(1-p)^{\alpha_{t_s}}]^{n_{t_s}}\\
&\quad\times\prod_{s=r+1}^{k}\big\{1-\big[1-(1-p)^{\alpha_{t_s}}\big]^{n_{t_s}}\big\}, \qquad (13)
\end{aligned}$$

where the summation index $\Gamma_{r,k}$ extends over all permutations $(t_1,\ldots,t_k)$ of $\{1,\ldots,k\}$ for which $t_1<\cdots<t_r$ and $t_{r+1}<\cdots<t_k$. Also, from David and Nagaraja (2003, p.113, Exe. 5.2.3), we find

$$\begin{aligned}
\beta_{max}(i,j;p,q) &= P\big(M_{i:k}\le\xi_p, M_{j:k}\le\xi_q\big)\\
&= \sum_{S_{r_1,r_2,r_3}}\frac{1}{r_1!r_2!r_3!}\sum_{\Delta_k}\Big\{\prod_{s=1}^{r_1}F_{M_{t_s}}(\xi_p)\\
&\quad\times\prod_{s=r_1+1}^{r_1+r_2}[F_{M_{t_s}}(\xi_q)-F_{M_{t_s}}(\xi_p)]\prod_{s=r_1+r_2+1}^{k}\bar{F}_{M_{t_s}}(\xi_q)\Big\}
\end{aligned}$$

$$= \sum_{S_{r_1,r_2,r_3}} \frac{1}{r_1!r_2!r_3!} \sum_{\Delta_k} \Big\{ \prod_{s=1}^{r_1} [1-(1-p)^{\alpha_{t_s}}]^{n_{t_s}}$$
$$\times \prod_{s=r_1+1}^{r_1+r_2} \big\{ [1-(1-q)^{\alpha_{t_s}}]^{n_{t_s}} - [1-(1-p)^{\alpha_{t_s}}]^{n_{t_s}} \big\}$$
$$\times \prod_{s=r_1+r_2+1}^{k} \big\{ 1-[1-(1-q)^{\alpha_{t_s}}]^{n_{t_s}} \big\} \Big\}, \tag{14}$$

where the summation index S_{r_1,r_2,r_3} extends over all values of (r_1, r_2, r_3) such that $r_1 \geq i$, $r_1 + r_2 \geq j$, $r_1 + r_2 + r_3 = k$ and the summation index Δ_k extends over all permutations $(t_1, \ldots, t_k)$ of $\{1, \ldots, k\}$.

So, from (13) and (14), one can readily obtain the confidence coefficient of the outer confidence interval $(M_{i:k}, M_{j:k})$, $j > i$, for (ξ_p, ξ_q), $q > p$,

$$\begin{aligned} \gamma_{max}(i,j;p,q) &= P\big(M_{i:k} \leq \xi_p \leq \xi_q \leq M_{j:k}\big) \\ &= \alpha_{max}(i;p) - \beta_{max}(i,j;p,q). \end{aligned} \tag{15}$$

Remark 2. *In the special case $\alpha_i = 1$ and $n_i = n$ $(i = 1, \ldots, k)$, we have*

$$\max_{i,j} \gamma_{max}(i,j;p,q) = 1 - (1-p^n)^k - q^{nk} + (q^n - p^n)^k.$$

2.3 Outer confidence intervals based on minima

It is obvious that the minima contain more information than maxima about the left tail of the distributions. So, for constructing confidence intervals for (ξ_p, ξ_q), whenever $p < q < 0.5$, based on only minima or maxima, it is better to use minima instead of maxima. Let $M'_{r:k}$ denote the rth order statistic of the set $\{M'_1, M'_2, \ldots, M'_k\}$. Similar to the previous subsection, the confidence coefficient of $(M'_{i:k} \leq \xi_p \leq \xi_q \leq M'_{j:k})$ can be obtained as follows.

$$\gamma_{min}(i,j;p,q) = \alpha_{min}(i;p) - \beta_{min}(i,j;p,q), \tag{16}$$

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where

$$\begin{aligned}
\alpha_{min}(i;p) &= P\big(M'_{i:k} \le \xi_p\big) \\
&= \sum_{r=i}^{k}\sum_{\Gamma_{r,k}}\prod_{s=1}^{r} F_{M'_{t_s}}(\xi_p)\prod_{s=r+1}^{k}\bar{F}_{M'_{t_s}}(\xi_p) \\
&= \sum_{r=i}^{k}\sum_{\Gamma_{r,k}}\prod_{s=1}^{r}[1-(1-p)^{\alpha_{t_s}n_{t_s}}](1-p)^{\sum_{s=r+1}^{k}\alpha_{t_s}n_{t_s}}
\end{aligned}$$

and

$$\begin{aligned}
\beta_{min}(i,j;p,q) &= P\big(M'_{i:k} \le \xi_p, M'_{j:k} \le \xi_q\big) \\
&= \sum_{S_{r_1,r_2,r_3}} \frac{1}{r_1!r_2!r_3!}\sum_{\Delta_k}\Bigg\{\prod_{s=1}^{r_1}[1-(1-p)^{\alpha_{t_s}n_{t_s}}] \\
&\quad \times \prod_{s=r_1+1}^{r_1+r_2}\big[(1-p)^{\alpha_{t_s}n_{t_s}} - (1-q)^{\alpha_{t_s}n_{t_s}}\big] \\
&\quad \times (1-q)^{\sum_{s=r_1+r_2+1}^{k}\alpha_{t_s}n_{t_s}}\Bigg\}.
\end{aligned}$$

Thus, we have a confidence interval $(M'_{i:k}, M'_{j:k})$, $j > i$ $(i, j = 1, \ldots, k)$, for quantile interval (ξ_p, ξ_q), $q > p$, whose confidence coefficient is free of F.

Remark 3. *For the special case $\alpha_i = 1$ and $n_i = n$ $(i = 1, \ldots, k)$, we get*

$$\max_{i,j}\gamma_{min}(i,j;p,q) = 1-(1-p)^{nk}-\big(1-(1-q)^n\big)^k+\big((1-p)^n-(1-q)^n\big)^k.$$

3 Numerical computations

To illustrate the results of this paper, we assume $F(x) = 1 - e^{-x}$ and $k = 5$. For given n_i's and α_i's $(i = 1, \ldots, 5)$, presented in Table 1, a random sample of size n_i from $\bar{F}_i(x) = e^{-\alpha_i x}$ (see Eq. (1)) is generated. Minimum and maximum of each sample are extracted and the results summarized in Table 1.

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Table 1. Summary description of the generated data for given n_i's and α_i's.

	sample 1	sample 2	sample 3	sample 4	sample 5
n_i	17	20	25	30	37
α_i	0.561	0.815	1.112	1.459	2.053
$M_i^{'}$	0.006	0.114	0.004	0.022	0.012
M_i	9.133	4.631	3.719	1.513	2.096

From Eq. (12) and using the data in Table 1, we obtain the values of lower bounds of $\gamma(i, j; p, q)$ for given p and q and some choices of i and j. The results are tabulated in Table 2 and help us to choose the appropriate outer confidence interval for given pth and qth quantile interval.

Table 2. Values of lower bounds of $\gamma(i, j; p, q)$ using (12) and Table 1.

		$p = 0.1$				$p = 0.25$			$p = 0.5$		$p = 0.75$
i	j	$q = 0.25$	$q = 0.5$	$q = 0.75$	$q = 0.9$	$q = 0.5$	$q = 0.75$	$q = 0.9$	$q = 0.75$	$q = 0.9$	$q = 0.9$
1	6	0.999	0.999	0.876	0.152	0.999	0.875	0.152	0.876	0.152	0.152
	7	0.999	0.999	0.998	0.654	0.999	0.998	0.654	0.998	0.654	0.654
	8	0.999	0.999	0.999	0.955	0.999	0.999	0.955	0.999	0.955	0.955
	10	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999
2	6	0.999	0.999	0.876	0.152	0.999	0.876	0.152	0.876	0.152	0.152
	7	0.999	0.999	0.998	0.654	0.999	0.998	0.654	0.998	0.654	0.654
	8	0.999	0.999	0.999	0.955	0.999	0.999	0.955	0.999	0.955	0.955
	10	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999
3	6	**0.996**	**0.996**	0.871	0.148	0.999	0.876	0.152	0.876	0.152	0.152
	7	0.996	0.996	**0.994**	0.650	0.999	0.998	0.654	0.998	0.654	0.654
	8	0.996	0.996	0.996	**0.951**	0.999	0.999	0.955	0.999	0.955	0.955
	10	0.996	0.996	0.996	0.996	0.999	0.999	0.999	0.999	0.999	0.999
4	6	0.908	0.908	0.783	0.060	**0.999**	0.875	0.152	0.876	0.152	0.152
	7	0.908	0.908	0.906	0.562	0.999	**0.997**	0.654	0.998	0.654	0.654
	8	0.908	0.908	0.908	0.863	0.999	0.999	**0.954**	0.999	0.955	0.955
	10	0.908	0.908	0.908	0.908	0.999	0.999	0.999	0.999	0.999	0.999
5	6	0.487	0.487	0.363	0.000	0.927	0.802	0.079	0.874	0.151	0.152
	7	0.487	0.487	0.487	0.141	0.927	0.925	0.581	**0.997**	0.653	0.654
	8	0.487	0.487	0.487	0.442	0.927	0.927	0.882	0.999	**0.953**	**0.955**
	10	0.487	0.487	0.487	0.487	0.927	0.927	0.927	0.999	0.999	0.999

Considering the shortest interval length as an optimality criterion, from Tables 1 and 2, the outer confidence intervals with confidence level 95% for (ξ_p, ξ_q) are readily obtained and these results are presented in Table 3 for some choices of p and q. Note that for $j = 10$, the values of $\gamma(i, 10; p, q)$ in Table 2 are exact confidence coefficients of outer confidence intervals $(V_{i:10}, V_{10:10})$ for (ξ_p, ξ_q).

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Table 3. The outer CIs for (ξ_p, ξ_q) with confidence level 95%.

p	q	(i, j)	$(V_{i:10}, V_{j:10})$	γ
0.1	0.25	(3, 6)	(0.012, 1.513)	0.996
	0.50	(3, 6)	(0.012, 1.513)	0.996
	0.75	(3, 7)	(0.012, 2.096)	0.994
	0.90	(3, 8)	(0.012, 3.719)	0.951
0.25	0.50	(4, 6)	(0.022, 1.513)	0.999
	0.75	(4, 7)	(0.022, 2.096)	0.997
	0.90	(4, 8)	(0.022, 3.719)	0.954
0.5	0.75	(5, 7)	(0.114, 2.096)	0.997
	0.90	(5, 8)	(0.114, 3.719)	0.953
0.75	0.90	(5, 8)	(0.114, 3.719)	0.954

As pointed in Section 2, the choice of i and j is not unique. For example, from Tables 1 and 2, it is observed that each of $(0.012, 2.096)$ and $(0.022, 2.096)$ can be considered as an outer confidence interval with confidence level 95% for $(\xi_{0.25}, \xi_{0.75})$, whereas by considering the shortest interval length criterion, the second one is accepted.

In order to compare three schemes proposed in Section 2, in general it is too intractable to distinguish among them, theoretically. But, intuitively, the confidence coefficients of the confidence intervals on the basis of the minima and maxima jointly are greater than others (It is confirmed by Table 4). Here, we give a numerical comparison.

Using the data in Table 1 and Eqs. (9), (15) and (16), one can obtain the values of $\gamma(i, 2k; p, q)$, $\gamma_{max}(i, j; p, q)$ and $\gamma_{min}(i, j; p, q)$ for given p and q and some choices of i, j and k. Their maximum values are tabulated in Table 4.

Table 4. Values of $\gamma_{max}(1, 5; p, q)$, $\gamma_{min}(1, 5; p, q)$ and $\gamma(1, 10; p, q)$.

p	q	$\gamma_{max}(1, 5; p, q)$	$\gamma_{min}(1, 5; p, q)$	$\gamma(1, 10; p, q)$
0.1	0.25	0.000	0.073	0.999
	0.50	0.000	0.001	0.999
	0.75	0.000	0.000	0.999
	0.9	0.000	0.000	0.999
0.25	0.50	0.001	0.001	0.999
	0.75	0.001	0.000	0.999
	0.90	0.001	0.000	0.999
0.5	0.75	0.034	0.000	0.999
	0.90	0.034	0.000	0.999
0.75	0.90	0.230	0.000	0.999

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4 Concluding Remarks

In this paper, we developed the nonparametric inferential procedures based on extreme order statistics from k independent random samples, where the observations of the ith sample coming from $\bar{F}_i = (\bar{F})^{\alpha_i}$ such that α_i is known and positive constant. We obtained outer confidence intervals for quantile intervals of F in three cases. These intervals are all exact and distribution-free in that the corresponding coverage probabilities are known exactly without any assumptions about the distribution F other than that F is continuous. The proposed procedure can be used for constructing inner confidence intervals for quantile intervals as well as upper and lower confidence limits for quantile differences:

- It may be noted that an outer confidence interval $(V_{i:2k}, V_{j:2k})$ for the quantile interval (ξ_p, ξ_q) may also be viewed as a distribution-free tolerance interval which specifies, with probability at least $\gamma(i, j; p, q)$, that no more than a proportion p of the population F is below the lower limit $V_{i:2k}$, and simultaneously, no more than a proportion $(1-q)$ of the population F is above the upper limit $V_{j:2k}$.

- Let p and q be any given real numbers satisfying $0 < p < q < 1$, and let $\left(\xi_p, \xi_q\right)$ be the interval $\{x|\ p < F(x) < q\}$. Sometimes, we may also be interested in the inner confidence intervals for the quantile intervals. In this case it is desired to determine the distribution-free coverage probability of the event $\left(\xi_p \leq L^{'} < U^{'} \leq \xi_q\right)$. Then, $\left(L^{'}, U^{'}\right)$ may be called an inner confidence interval for the quantile interval (ξ_p, ξ_q). Notice that

$$\begin{aligned} P(\xi_p < L^{'} < U^{'} < \xi_q) &= 1 - P(L^{'} < \xi_p \cup U^{'} > \xi_q) \\ &= P(L^{'} < \xi_p < \xi_q < U^{'}) \\ &\quad + P(U^{'} < \xi_q) - P(L^{'} < \xi_p). \end{aligned}$$

  Therefore by using the results obtained in Section 2, the coverage probabilities of the inner confidence intervals for quantile intervals can be obtained in the different cases, while the statistics $V_{i:2k}$, $M_{i:k}$ or $M^{'}_{i:k}$ are used in place of $L^{'}$ and $U^{'}$.

- Another subject that is important in this field is to determine the confidence limits for the quantile differences, $\xi_q - \xi_p$. Suppose

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that $1 \leq i < j < r < s \leq 2k$, then we have

$$\begin{aligned} P\big(V_{r:2k} - V_{j:2k} \leq \xi_q - \xi_p\big) &\geq P\big(V_{r:2k} \leq \xi_q, V_{j:2k} \geq \xi_p\big) \\ &= P\big(\xi_p < V_{j:2k} < V_{r:2k} < \xi_q\big) \end{aligned}$$

and

$$\begin{aligned} P\big(V_{s:2k} - V_{i:2k} \geq \xi_q - \xi_p\big) &\geq P\big(V_{s:2k} \geq \xi_q, V_{i:2k} \leq \xi_p\big) \\ &= P\big(V_{i:2k} \leq \xi_p < \xi_q \leq V_{s:2k}\big). \end{aligned}$$

Therefore, we get

$$\begin{aligned} &P\big(V_{r:2k} - V_{j:2k} \leq \xi_q - \xi_p \leq V_{s:2k} - V_{i:2k}\big) \\ &\quad = P\big(V_{r:2k} - V_{j:2k} \leq \xi_q - \xi_p, V_{s:2k} - V_{i:2k} \geq \xi_q - \xi_p\big) \\ &\quad = P\big(V_{r:2k} - V_{j:2k} \leq \xi_q - \xi_p\big) \\ &\qquad -P\big(V_{s:2k} - V_{i:2k} \leq \xi_q - \xi_p\big) \\ &\quad = P\big(V_{r:2k} - V_{j:2k} \leq \xi_q - \xi_p\big) \\ &\qquad +P\big(V_{s:2k} - V_{i:2k} \geq \xi_q - \xi_p\big) - 1 \\ &\quad \geq P\big(\xi_p < V_{j:2k} < V_{r:2k} < \xi_q\big) \\ &\qquad +P\big(V_{i:2k} \leq \xi_p < \xi_q \leq V_{s:2k}\big) - 1. \end{aligned}$$

The similar relation may be obtained in terms of just $M_{i:k}$'s or $M'_{i:k}$'s.

- Let $Y_{i,j}$ $(1 \leq i \leq k, 1 \leq j \leq n_i)$ be independent random variables. Moreover for a fixed i, $Y_{i,j}$'s, $(1 \leq j \leq n_i)$ are identically distributed with cdf $G_i(x) = [G(x)]^{\beta_i}$, where $G(x)$ is an absolutely continuous distribution function and β_i is known and positive constant. The aforementioned identity is well-known in the lifetime experiments literature as the proportional reversed hazard model (see for example Lawless, 2003). In this case the results of this paper hold with obvious modifications for the quantiles of G.

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