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On the Average Height of b-Balanced Ordered Trees

Rainer Kemp

Johann Wolfgang Goethe-Universität, Fachbereich Biologie und Informatik (15), Institut für Informatik, D - 60054 Frankfurt am Main, Germany.

Abstract. An ordered tree with height h is b-balanced if all its leaves have a level ℓ with $h - b \leq \ell \leq h$, where at least one leaf has a level equal to $h - b$. For large n, we shall compute asymptotic equivalents to the number of all b-balanced ordered trees with n nodes and of all such trees with height h. Furthermore, assuming that all b-balanced ordered trees with n nodes are equally likely, we shall determine the exact asymptotic behaviour of the average height of such a tree together with the variance.

1 Introduction

The concept of balanced trees such as AVL-trees, B-trees, or 2-3-trees is used in many types of data structures appearing in sorting and searching algorithms. Although the algorithmic importance of such trees is out of the question, only few results are known on the enumeration of such trees or on the exact average behaviour of impor-

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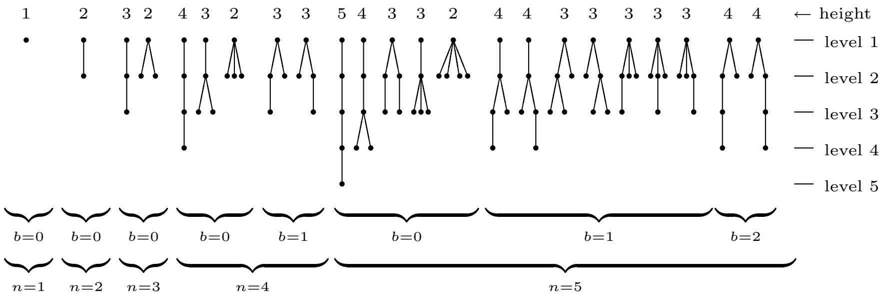

Figure 1: All b-balanced ordered trees with less than six nodes.

tant parameters defined on these trees. A conspicuous general result in the relevant sense has been presented in [12] for balanced 2-3-trees with a specified number of leaves. This enumeration result also includes the counting of a variety of balanced trees such as B-trees with a specified order.

In this article we shall continue with the study of the class of 0-balanced ordered trees introduced in [5] by extending the notion '0-balanced': Given an ordered tree T with the set of leaves L, such a tree is called *b-balanced*, $b \in \mathbb{N}_0$, if

$$\max\{\ell ev(x)|x \in L\} - \min\{\ell ev(x)|x \in L\} = b;$$

here, the *level* $\ell ev(x)$ of a node x is equal to the number of nodes appearing on the simple path from the root to the node x including the root and node x. Thus, an ordered tree with *height* $h := \max\{\ell ev(x)|x \in L\}$ is b-balanced if all its leaves have a level ℓ with $h - b \leq \ell \leq h$, where at least one leaf has the level equal to $h - b$. All b-balanced ordered trees with less than six nodes are drawn in Figure 1. Clearly, $b + 2 \leq h < n + \delta_{b,0}$, $n \geq 2$, where $\delta_{n,k}$ denotes Kronecker's delta. Note that all leaves appearing in a 0-balanced ordered tree have the same level. The parameter $b \in \mathbb{N}_0$ appearing in the definition of a b-balanced ordered tree with $n \geq 3$ nodes obviously describes the transition from all totally balanced n-node trees characterized by $b = 0$ to all (unbalanced) ordered trees with n nodes characterized by $b \leq n - 3$.

The 0-balanced ordered trees are a fundamental constructive device in combinatorial considerations on enumerating specified nodes appearing in block code trees [6] or in other classes of trees [8, Remark 1]. The enumeration of various types of 0-balanced ordered trees together with an average case analysis of important parameters defined on these trees (e.g. the height; the path length; the root-degree) have been presented in [5]; aspects with respect to the generation of such trees have been discussed in [7, Section 3.8]. In the former paper, the author has been successful in proving that the number of all 0-balanced n-node ordered trees of height h is given by $F_{n-2}^{(h-1)}$, $2 \leq h \leq n$, where $F_n^{(p)}$ is the pth order FIBONACCI number defined by $F_n^{(p)} = \delta_{n,p-1}$ for $0 \leq n < p$, and by $F_n^{(p)} = \sum_{1 \leq k \leq p} F_{n-k}^{(p)}$ for $n \geq p$ ([9, p. 77]). This result essentially reflects a one-to-one correspondence between 0-balanced n-node ordered trees of height h and the ordered partitions of the integer $n-h$ ([7, Section 3]; [9, p. 287]). Assuming that all 0-balanced n-node ordered trees are equally likely, further detailed considerations (cf. [5]) imply that

– the number $t_0(n)$ of all 0-balanced ordered trees with n nodes is asymptotically given by $t_0(n) \sim 2^{n-1}\, n^{-1}\, [1 + f(n)]$, $n \to \infty$;

– the average height $\underline{h}_0(n)$ of a 0-balanced ordered tree with n nodes has the asymptotic behaviour $\underline{h}_0(n) \sim \log_2(n) + \frac{\gamma}{\ln(2)} + \chi(n)$ with the variance $\sigma_0^2(n) \sim \frac{\pi^2}{6[\ln(2)]^2} + \varphi(n)$, $n \to \infty$.

Here, $\gamma = .577\,215\ldots$ is EULER's constant and $h(n)$, $h \in \{f, \chi, \varphi\}$, are bounded oscillating functions with a very small amplitude satisfying the equality $h(n) = h(2n)$.

In this paper we shall generalize these results to b-balanced n-node ordered trees with $b \geq 1$. For large n, we shall first derive an asymptotic equivalent to the number of all b-balanced ordered trees of height h with n nodes (Theorem 4.1(a)). Then, assuming that all b-balanced ordered trees with n nodes are equally likely, we shall find for fixed b that

– the number $t_b(n)$ of all b-balanced ordered trees with n nodes is asymptotically given by $t_b(n) \sim -\frac{1-r_{b+2}}{\ln(r_{b+2})}\, r_{b+2}^{-n}\, n^{-1}\, [1 + \eta_{0,b}(n)]$, $n \to \infty$ (Theorem 4.1(b));

– the average height $\underline{h}_b(n)$ of a b-balanced ordered tree with n nodes has the asymptotic behaviour

$\underline{h}_b(n) \sim -\frac{1}{\ln(r_{b+2})}\left[\ln(n) + \ln\left(e^{\gamma}\frac{(1-r_{b+2})(4r_{b+2}-1)}{(b+4)r_{b+2}^{b+3}}\right)\right] + \chi_b(n)$

with the variance $\sigma_b^2(n) \sim \frac{\pi^2}{6[\ln(r_{b+2})]^2} + \varphi_b(n)$, $n \to \infty$ (Theorem 4.2).

Here, $r_w := \left[4\cos^2\left(\frac{\pi}{w+2}\right)\right]^{-1}$ and $h_b(n)$, $h_b \in \{\eta_{0,b}, \chi_b, \varphi_b\}$, are bounded oscillating functions with $h_b(n) = h_b(r_{b+2}\,n)$ possessing a small amplitude not exceeding .04.

The general structure of this paper is as follows: In Section 2 we shall present basic enumeration results with respect to b-balanced ordered trees of height h with n nodes. It turns out that we have to focus our further considerations on a specific rational function which is discussed in Section 3. The determination of the dominant poles of this function (Subsection 3.2) requires subtle analytical investigations using some technical lemmata presented in Subsection 3.1. Exploiting all this information, we are able to compute the exact asymptotic behaviour of the TAYLOR coefficients of the rational function in discussion by standard methods (Subsection 3.3). Finally in Section 4, we shall use this result in order to prove the asymptotical relations for $t_b(n)$ and $\underline{h}_b(n)$ pointed out above.

2 Basic Enumeration Results

We begin our study by reviewing some known results. Let $a_h(n,m)$ be the number of all 0-balanced ordered trees with n nodes, m leaves and height h, and let

$$A_h(z,y) := \sum_{n\geq 0}\sum_{m\geq 0} a_h(n,m) z^n y^m$$

be the corresponding generating function. It is well known ([5]) that $A_h(z,y)$ is explicitly given by

$$A_h(z,y) = \frac{yz^h(1-z)}{1-z(y+1)+yz^h}, \; h \geq 1. \tag{1}$$

Now, let $t_{h,b}(n)$ be the number of all n-node ordered trees of height less than or equal to h with leaves appearing at a level greater than or equal to $h-b$. We obtain such a tree

– by taking a 0-balanced ordered tree τ_0 of height $h-b$ with n_0 nodes and m_0 leaves ℓ_i, $1 \le i \le m_0$, and

– by replacing the leaves ℓ_i by ordered trees τ_i with n_i nodes and height less than or equal to $b+1$ such that $n_0 + \sum_{1\le i\le m_0} n_i = n$.

Hence, the generating function of the numbers $t_{h,b}(n)$ is given by

$$T_{h,b}(z) := \sum_{n\ge 0} t_{h,b}(n) z^n = A_{h-b}(z, z^{-1} F_{b+1}(z)),\ 0 \le b < h, \tag{2}$$

where $F_k(z)$ is the generating function of the number $\phi_k(n)$ of all ordered trees with n nodes and height less than or equal to k. The classical paper [1] tells us that

$$F_k(z) := \sum_{n\ge 0} \phi_k(n) z^n = z \frac{p_k(z)}{p_{k+1}(z)}, \tag{3}$$

where

$$\begin{aligned} p_k(z) &:= \frac{1}{2^k \epsilon(z)}[(1+\epsilon(z))^k - (1-\epsilon(z))^k], \\ \epsilon(z) &:= \sqrt{1-4z},\ k \ge 0, \end{aligned} \tag{4}$$

is the kth Fibonacci polynomial satisfying the linear recurrence

$$p_0(z) = 0,\ p_1(z) = 1,\ p_k(z) = p_{k-1}(z) - z\, p_{k-2}(z),\ k \ge 2. \tag{5}$$

Now, inserting (1) and (3) into (2) and applying (5) to the denominator of the resulting expression, we obtain $T_{h,b}(z) = G_{h+1,b+1}(z)$, where

$$G_{h,b}(z) := \frac{z^{h-b}\,(1-z)\, p_b(z)}{p_{b+3}(z) + z^{h-b}\, p_b(z)},\ 1 \le b < h. \tag{6}$$

Note that $G_{h,0}(z) = 0$ and we set $G_{h,b}(z) := 0$ for $b < 0$. Introducing the number $d_{h,b}(n)$ of all n-node ordered trees of height h with leaves appearing at a level greater than or equal to $h-b$, a similar construction as presented above for the generating function $T_{h,b}(z)$ immediately yields

$$\begin{aligned} D_{h,b}(z) &:= \sum_{n\ge 0} d_{h,b}(n) z^n \\ &= A_{h-b}(z, z^{-1} F_{b+1}(z)) - A_{h-b}(z, z^{-1} F_b(z)) \\ &= T_{h,b}(z) - T_{h-1,b-1}(z) \\ &= G_{h+1,b+1}(z) - G_{h,b}(z),\ 0 \le b < h. \end{aligned} \tag{7}$$

	$b=0$										$b=1$						
$n \setminus h$	1	2	3	4	5	6	7	8	9	10	3	4	5	6	7	8	9
1	1																
2		1															
3		1	1														
4		1	1	1							2						
5		1	2	1	1						5	2					
6		1	3	2	1	1					12	7	2				
7		1	5	4	2	1	1				26	20	7	2			
8		1	8	7	4	2	1	1			55	55	22	7	2		
9		1	13	13	8	4	2	1	1		114	143	63	22	7	2	
10		1	21	24	15	8	4	2	1	1	234	363	175	65	22	7	2

	$b=2$					$b=3$				$b=4$			$b=5$		$b=6$
$n \setminus h$	4	5	6	7	8	5	6	7	8	6	7	8	7	8	8
5	2														
6	9	2				2									
7	33	11	2			13	2			2					
8	107	45	11	2		61	15	2		17	2		2		
9	326	166	47	11	2	247	77	15	2	97	19	2	21	2	2

Table 1: The number $\ell_{h,b}(n)$ of all b-balanced ordered trees with $n \leq 9$ nodes and height h.

Obviously, the quantity $\ell_{h,b}(n) := d_{h,b}(n) - d_{h,b-1}(n)$, $0 \leq b < h$, is the number of all n-node ordered trees of height h with leaves appearing at a level greater than or equal to $h-b$ such that at least one leaf has the level $h-b$. Hence, $\ell_{h,b}(n)$ is the number of all b-balanced ordered trees of height h with n nodes and the corresponding generating function is given by

$$\begin{aligned} L_{h,b}(z) &:= \sum_{n \geq 0} \ell_{h,b}(n) z^n = D_{h,b}(z) - D_{h,b-1}(z) \\ &= [G_{h+1,b+1}(z) - G_{h+1,b}(z)] - [G_{h,b}(z) - G_{h,b-1}(z)], \quad (8) \end{aligned}$$

where $G_{h,b}(z)$ is explicitly given by (6). The first few values of $\ell_{h,b}(n)$ are summarized in Table 1. Clearly, since $p_0(z) = 0$, $p_1(z) = 1$ and $p_s(z) = 1 - (s-2)z$, $2 \leq s \leq 4$, we rediscover the known result $L_{h,0}(z) = G_{h+1,1}(z) = A_h(z,1)$ (see [5]). Using (5), another straightforward computation yields the equalities $G_{h,h-2}(z) + z = G_{h,h-1}(z) = z\,\frac{p_{h-1}(z)}{p_h(z)} = F_{h-1}(z)$. These relations together with (8) immediately imply the evident equality $L_{h,h-1}(z) = 0$ because a leaf cannot appear at level one of the root. Furthermore, we find

$$\begin{aligned}\sum_{0\le b\le h-2} L_{h,b}(z) &\overset{(8)}{=} G_{h+1,h-1}(z) - G_{h,h-2}(z) \\ &= z\,\frac{p_h(z)}{p_{h+1}(z)} - z\,\frac{p_{h-1}(z)}{p_h(z)} \\ &\overset{(3)}{=} F_h(z) - F_{h-1}(z), \end{aligned} \qquad (9)$$

because all b-balanced ordered trees with $0 \le b \le h-2$, are all ordered trees of height h.

In summary, an inspection of (7) and (8) shows that we first have to consider the enumerator $G_{h,b}(z)$, $0 \le b < h$, defined in (6) more detailed in order to derive explicit asymptotical enumeration and distribution results.

3 The Enumerator $G_{h,b}(z)$

3.1 Technical Lemmata

In this subsection we shall present some basic results on the Fibonacci polynomials $p_k(z)$ introduced in (5) and on the polynomial $P_{h,k}(z) := p_{k+3}(z) + z^{h-k}\,p_k(z)$ appearing in the denominator of the enumerator $G_{h,k}(z)$ given in (6).

Lemma 3.1. *Let* $\alpha_w(j) := 4\cos^2\left(\frac{\pi j}{w}\right)$ *and* $r_w := [\alpha_{w+2}(1)]^{-1}$, $(w,j) \in \mathbb{N}^2$. *The* Fibonacci *polynomials* $p_k(z)$ *fulfill the following properties:*

(a) The polynomial $p_k(z)$ *has the* $\lfloor \frac{1}{2}(k-1)\rfloor$ *simple roots* $r_{k-2} = [\alpha_k(1)]^{-1} < [\alpha_k(j)]^{-1} < [\alpha_k(j+1)]^{-1}$, $2 \le j < \lfloor \frac{1}{2}(k-1)\rfloor$, $k \ge 0$.

(b) The polynomial $p_k(z)$ *has the representation*

$$p_k(z) = \prod_{1\le j\le\lfloor\frac{1}{2}(k-1)\rfloor} (1-\alpha_k(j)\,z), \;\; k\ge 0.$$

(c) The derivatives $p_k^{(s)}([4\cos^2(\upsilon)]^{-1})$, $0 \le s \le 3$, $k \ge 0$, *are given by*

$$p_k^{(s)}([4\cos^2(\upsilon)]^{-1}) = \frac{\vartheta_k^{[s]}(\upsilon)}{2^{k-1}\,\sin^{2s+1}(\upsilon)\,\cos^{k-2s-1}(\upsilon)},$$

where

$$
\begin{aligned}
\vartheta_k^{[0]}(\upsilon) &= \sin(k\,\upsilon),\\
\vartheta_k^{[1]}(\upsilon) &= -k\,\sin((k-2)\,\upsilon) + (k-2)\,\sin(k\,\upsilon),\\
\vartheta_k^{[2]}(\upsilon) &= k\,(k-1)\,\sin((k-4)\,\upsilon) - 2\,k\,(k-4)\,\sin((k-2)\,\upsilon)\\
&\quad + (k-3)\,(k-4)\,\sin(k\,\upsilon),\\
\vartheta_k^{[3]}(\upsilon) &= -k\,(k-1)\,(k-2)\,\sin((k-6)\,\upsilon)\\
&\quad + 3\,k\,(k-1)\,(k-6)\,\sin((k-4)\,\upsilon)\\
&\quad - 3\,k\,(k-5)\,(k-6)\sin((k-2)\,\upsilon)\\
&\quad + (k-4)\,(k-5)\,(k-6)\sin(k\,\upsilon).
\end{aligned}
$$

(d) *The polynomial $p_k(z)$ and its derivatives take the following special values:*

$$
\begin{aligned}
&p_k(r_{k-4}) = -r_{k-4}^{\frac{1}{2}(k-2)}, \qquad p_k(r_k) = r_k^{\frac{1}{2}(k-2)},\\
&p_k(r_{k-3}) = -r_{k-3}^{\frac{1}{2}(k-1)}, \qquad p_k(r_{k+1}) = (1-r_{k+1})\,r_{k+1}^{\frac{1}{2}(k-3)},\\
&p_k(r_{k-1}) = r_{k-1}^{\frac{1}{2}(k-1)},
\end{aligned}
$$

$$
\begin{aligned}
&p_k'(r_{k-4}) = \tfrac{-(k-2)}{4\,r_{k-4}-1}\,r_{k-4}^{\frac{1}{2}(k-2)}, \quad p_k'(r_{k-1}) = \tfrac{2\,(k-1)\,r_{k-1}-k}{4\,r_{k-1}-1}\,r_{k-1}^{\frac{1}{2}(k-3)},\\
&p_k'(r_{k-3}) = \tfrac{-2\,(k-1)}{4\,r_{k-3}-1}\,r_{k-3}^{\frac{1}{2}(k-1)}, \quad p_k'(r_k) = \tfrac{(3k-2)\,r_k-k}{4\,r_k-1}\,r_k^{\frac{1}{2}(k-4)},\\
&p_k'(r_{k-2}) = \tfrac{-k}{4\,r_{k-2}-1}\,r_{k-2}^{\frac{1}{2}(k-2)}, \quad p_k'(r_{k+1}) = \tfrac{-2\,(k-1)\,r_{k+1}^2+2\,(2k-1)\,r_{k+1}-k}{4\,r_{k+1}-1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times r_{k+1}^{\frac{1}{2}(k-5)},
\end{aligned}
$$

$$
\begin{aligned}
&p_k''(r_{k-2}) = -k\,\tfrac{2\,(2\,k-5)\,r_{k-2}-k+1}{(4\,r_{k-2}-1)^2}\,r_{k-2}^{\frac{1}{2}(k-4)},\\
&p_k''(r_{k+1}) = -\tfrac{4\,(k-3)\,(k-1)\,r_{k+1}^3-(13\,k^2-37\,k+12)\,r_{k+1}^2+k\,(7\,k-13)\,r_{k+1}-k\,(k-1)}{(4\,r_{k+1}-1)^2}\\
&\qquad\qquad\quad \times r_{k+1}^{\frac{1}{2}(k-7)},
\end{aligned}
$$

$$
p_k'''(r_{k-2}) = -k\,\frac{12\,(k^2-7\,k+11)\,r_{k-2}^2-(k-1)\,(7\,k-26)\,r_{k-2}+(k-1)\,(k-2)}{(4\,r_{k-2}-1)^3}\,r_{k-2}^{\frac{1}{2}(k-6)}.
$$

(e) *The polynomial $p_k(z)$, $k \geq 3$, is strictly monotonically decreasing in the interval $[0, r_{k-2}[$ and the inequality $p_k(z) > 0$ holds for all z in that interval.*

(f) *The polynomial $p'_k(z)$, $k \geq 5$, is strictly monotonically increasing in the interval $[r_{k+1}, r_k]$ and the inequality $p'_k(z) < 0$ holds for all z in that interval.*

(g) *The inequality $p''_k(z) > 0$ holds for all $z \in [r_{k+1}, r_k]$, $k \geq 5$.*

Proof: **(a)**, **(b)**: These relations are well known (see [1]). They reflect corresponding relationships fulfilled by the CHEBYSHEV polynomials $U_k(z)$ of the second kind because the identity

$$p_k(z) = z^{\frac{1}{2}(k-1)} U_{k-1}(\frac{1}{2\sqrt{z}})$$

holds.

(c): The explicit expression for $p_k([4\cos^2(\upsilon)]^{-1})$ can be found in [1], too. It follows from (4) by setting $z := [4\cos^2(\upsilon)]^{-1}$ and by using EULER's formula $e^{iz} = \cos(z) + i\,\sin(z)$, $i^2 := -1$. The expressions for the higher derivatives are obtained by taking successively the derivatives on both sides of the equation for $p_k([4\cos^2(\upsilon)]^{-1})$.

(d): Choosing $z \in \{r_s | k-4 \leq s \leq k+1\}$ in part (c) and applying elementary trigonometric relations as well as the identity $\cos(\frac{\pi}{k+2}) = \frac{1}{2}\, r_k^{-\frac{1}{2}}$, the explicit expressions for the special values can be derived by a straightforward lengthy computation.

(e): Taking the derivative on both sides of the equality presented in (b), we find

$$p'_k(z) = -\sum_{1 \leq j \leq \lfloor \frac{1}{2}(k-1) \rfloor} \alpha_k(j) \prod_{\substack{1 \leq \lambda \leq \lfloor \frac{1}{2}(k-1) \rfloor \\ \lambda \neq j}} (1 - \alpha_k(\lambda)\, z).$$

For $z \in [0, r_{k-2}[$, we have $1 - \alpha_k(\lambda)\, z > 1 - \frac{\alpha_k(\lambda)}{\alpha_k(1)} > 0$ (resp. $= 0$) if $\lambda \geq 2$ (resp. $\lambda = 1$). Hence, $p'_k(z) < 0$ for $0 \leq z < r_{k-2}$, and $p_k(z)$ is strictly monotonically decreasing in $[0, r_{k-2}[$. Therefore, we have $1 \geq p_k(0) \geq p_k(z) > p_k(r_{k-2}) = 0$.

(f), **(g)**: First, we shall show that $p''_k(z)$, $k \geq 5$, is positive for $z \in [r_{k+1}, r_k]$. Instead of considering the function $p''_k(z)$ for $z \in [r_{k+1}, r_k]$, we change the variable and consider the function $f_k(\upsilon) := p''_k([4\cos^2(\upsilon)]^{-1})$ for $\upsilon \in \mathcal{I}_k := [\frac{\pi}{k+3}, \frac{\pi}{k+2}]$. Using part (c) with $s = 2$, we obtain $f_k(\upsilon) = \vartheta_k^{[2]}(\upsilon)\, [2^{k-1} \sin^5(\upsilon) \cos^{k-5}(\upsilon)]^{-1}$. Since $\sin(\upsilon)$

(resp. $\cos(\upsilon)$) is positive and monotonically increasing (resp. decreasing) in $\mathcal{I}_k$, $k \geq 5$, the denominator $2^{k-1} \sin^5(\upsilon) \cos^{k-5}(\upsilon)$ is positive. Using elementary trigonometric formulae, the nominator $\vartheta_k^{[2]}(\upsilon)$ can be transformed into

$$\begin{aligned}\vartheta_k^{[2]}(\upsilon) &= -4\,k\,(k-1)\,\sin^2(\upsilon)\,\sin((k-2)\,\upsilon)\\ &\quad -12\,k\,\sin(\upsilon)\,\cos((k-1)\,\upsilon) + 12\,\sin(k\,\upsilon).\end{aligned}$$

Now, a moment's reflection shows that the functions $\sin(k\,\upsilon)$, $\sin((k-2)\,\upsilon)$ and $\cos((k-1)\,\upsilon)$ are monotonically decreasing in $\mathcal{I}_k$, $k \geq 7$, and that $\sin(k\,\upsilon)$ and $\sin((k-2)\,\upsilon)$ (resp. $\cos((k-1)\,\upsilon)$) are positive (resp. negative) in that interval. Therefore,

$$\begin{aligned}\vartheta_k^{[2]}(\upsilon) &> -4\,k\,(k-1)\,\sin^2(\tfrac{\pi}{k+2})\,\sin(\tfrac{k-2}{k+3}\,\pi)\\ &\quad -12\,k\,\sin(\tfrac{\pi}{k+3})\,\cos(\tfrac{k-1}{k+3}\,\pi) + 12\,\sin(\tfrac{k}{k+2}\,\pi)\\ &= -4\,k\,(k-1)\,\sin^2(\tfrac{\pi}{k+2})\,\sin(\tfrac{5\pi}{k+3}) + 12\,k\,\sin(\tfrac{\pi}{k+3})\\ &\quad -24\,k\,\sin(\tfrac{\pi}{k+3})\,\sin^2(\tfrac{2\pi}{k+3}) + 12\,\sin(\tfrac{2\pi}{k+2}).\end{aligned}$$

Using the inequality $\sin(x) < x - \frac{1}{6}x^3 + \frac{1}{120}x^5$, $x > 0$, for the factor $\sin(\frac{5\pi}{k+3})$ and the inequalities $x - \frac{1}{6}x^3 < \sin(x) < x$, $x > 0$, for the other factors, we further obtain by a lengthy computation

$$\vartheta_k^{[2]}(\upsilon) > \tfrac{\pi}{6\,(k+2)^3\,(k+3)^5}\Big(c + (k-12)\sum_{0\leq j\leq 7} c_j\,k^j\Big),$$

where c and c_j, $0 \leq j \leq 7$, are positive constants given by

$$\begin{aligned}
&c_0 = -2\,(48\,125\,\pi^6 - 8\,662\,500\,\pi^4 + 652\,327\,128\,\pi^2 - 5\,893\,959\,168),\\
&c_1 = -8\,125\,\pi^6 + 1\,444\,500\,\pi^4 - 108\,716\,040\,\pi^2 + 982\,291\,536,\\
&c_2 = -625\,\pi^6 + 120\,500\,\pi^4 - 9\,051\,216\,\pi^2 + 81\,811\,296,\\
&c_3 = 4\,(2\,375\,\pi^4 - 186\,543\,\pi^2 + 1\,695\,546),\\
&c_4 = 4\,(125\,\pi^4 - 14\,472\,\pi^2 + 137\,016),\\
&c_5 = -12\,(299\,\pi^2 - 3\,360),\\
&c_6 = -24\,(5\,\pi^2 - 96),\\
&c_7 = 72,\\
&c = -600\,(1\,925\,\pi^6 - 346\,500\,\pi^4 + 26\,093\,124\,\pi^2 - 235\,758\,600).
\end{aligned}$$

Hence, $\vartheta_k^{[2]}(\upsilon) > 0$ for $k \geq 12$. For $5 \leq k \leq 11$, we find directly the explicit expressions

$$\vartheta_5^{[2]}(\upsilon) = 32\,\sin^5(\upsilon),$$

$$\begin{aligned}
\vartheta_6^{[2]}(\upsilon) &= 192\,\cos(\upsilon)\,\sin^5(\upsilon),\\
\vartheta_7^{[2]}(\upsilon) &= 96\,[4\,\cos(2\,\upsilon)+3]\,\sin^5(\upsilon),\\
\vartheta_8^{[2]}(\upsilon) &= 128\,[5\,\cos(3\,\upsilon)+9\cos(\upsilon)]\,\sin^5(\upsilon),\\
\vartheta_9^{[2]}(\upsilon) &= 192\,[5\,\cos(4\,\upsilon)+10\,\cos(2\,\upsilon)+6]\,\sin^5(\upsilon),\\
\vartheta_{10}^{[2]}(\upsilon) &= 192\,[7\,\cos(5\,\upsilon)+15\,\cos(3\,\upsilon)+20\,\cos(\upsilon)]\,\sin^5(\upsilon),\\
\vartheta_{11}^{[2]}(\upsilon) &= 64\,[28\,\cos(6\,\upsilon)+63\,\cos(4\,\upsilon)+90\,\cos(2\,\upsilon)+50]\,\sin^5(\upsilon).
\end{aligned}$$

Evidently, the function $\vartheta_k^{[2]}(\upsilon)$, $5 \le k \le 11$, takes no negative value in the interval $\mathcal{I}_k$ because the arguments appearing in the sin- and cos-functions are less than $\frac{\pi}{2}$. It is easily verified that $\vartheta_k^{[2]}(\upsilon) \ge \vartheta_k^{[2]}(\frac{\pi}{k+3}) > 0$, $5 \le k \le 11$.

In summary, we have shown that $p_k''(z) > 0$ for all $z \in [r_{k+1}, r_k]$, $k \ge 5$. Hence, $p_k'(z)$ is strictly monotonically increasing in that interval and we have $p_k'(z) < p_k'(r_k)$. The value of $p_k'(r_k)$ given in part (d) is clearly negative for $k \ge 3$. $\square$

Lemma 3.2. *Let* $\alpha_w(j) := 4\cos^2\left(\frac{\pi j}{w}\right)$, $r_w := [\alpha_{w+2}(1)]^{-1}$, $(w, j) \in \mathbb{N}^2$, $p_k(z)$ *be the* FIBONACCI *polynomial defined in (4) and* $P_{h,k}(z) := p_{k+3}(z) + z^{h-k} p_k(z)$.

(a) The polynomial $P_{h,k}(z)$, $h \ge k+1$, $k \ge 2$, *is strictly monotonically decreasing in* $[r_{k+1}, r_k]$ *and the inequality* $P_{h,k}'(z) < -\frac{(k-1)\,r_{k+1}}{4\,r_{k+1}-1}\, r_k^{\frac{1}{2}k} < 0$ *holds for all* z *in that interval.*

(b) The inequality $P_{h,k}''(z) > 0$ *holds for all* $z \in [r_{k+1}, r_k]$, $h \ge k+1+\delta_{k,1}$, $k \ge 1$.

Proof: (a): Since the relations $p_k'(z) = -(k-2)$, $2 \le k \le 4$, (resp. $p_k'(z) < 0$ for $z \in [r_{k+1}, r_k]$, $k \ge 5$) hold by the definition of $p_k(z)$ (resp. by Lemma 3.1(f)), we have $P_{h,k}'(z) \le \widehat{\eta}_{h,k}(z) := p_{k+3}'(z) + (h-k)\, z^{h-k-1} p_k(z)$ for $r_{k+1} \le z \le r_k$. Instead of considering the function $\widehat{\eta}_{h,k}(z)$ for $z \in [r_{k+1}, r_k]$, we change the variable and deal with the function $\eta_{h,k}(\upsilon) := \widehat{\eta}_{h,k}([4\cos^2(\upsilon)]^{-1})$ for $\upsilon \in \mathcal{I}_k := [\frac{\pi}{k+3}, \frac{\pi}{k+2}]$, $k \ge 2$. Using Lemma 3.1(c) with $s \in \{0, 1\}$, we find

$$\eta_{h,k}(\upsilon) = \frac{\widetilde{\varphi}_{h,k}(\upsilon)}{2^{k+2}\,\sin^3(\upsilon)\,\cos^k(\upsilon)}, \tag{10}$$

where

$$\begin{aligned}\widetilde{\varphi}_{h,k}(\upsilon) &= -(k+3)\,\sin((k+1)\,\upsilon) + (k+1)\,\sin((k+3)\,\upsilon)\\ &\quad + 4\,(h-k)\,[4\cos^2(\upsilon)]^{-(h-k-1)}\sin(k\,\upsilon)\sin(2\,\upsilon)\sin(\upsilon).\end{aligned}$$

Obviously, the denominator appearing in (10) is positive for $\upsilon \in \mathcal{I}_k$, $k \geq 2$, because $0 < \upsilon \leq \frac{\pi}{4} < \frac{\pi}{2}$. By the monotonicity of the functions $\sin(\upsilon)$ and $\cos(\upsilon)$ in $\mathcal{I}_k$, we obtain

$$\begin{aligned}2^{k+2}\,\sin^3(\upsilon)\,\cos^k(\upsilon) &\geq 2^{k+2}\,\sin^3(\tfrac{\pi}{k+3})\,\cos^k(\tfrac{\pi}{k+2})\\ &= \tfrac{1}{2}\,(4\,r_{k+1}-1)^{\frac{3}{2}}\,r_{k+1}^{-\frac{3}{2}}\,r_k^{-\frac{1}{2}k}. \qquad (11)\end{aligned}$$

Next, let us turn to the nominator $\widetilde{\varphi}_{h,k}(\upsilon)$ appearing in (10). It is easily verified that the functions $\sin((k+1)\,\upsilon)$ and $\sin((k+3)\,\upsilon)$, $k \geq 2$, and $\sin(k\,\upsilon)$, $k \geq 3$, are monotonically decreasing in $\mathcal{I}_k$; the functions $\sin(\upsilon)$, $\sin(2\,\upsilon)$ and $\cos^{-2}(\upsilon)$ are monotonically increasing in $\mathcal{I}_k$. Hence, we obtain

$$\begin{aligned}\widetilde{\varphi}_{h,2}(\upsilon) &\leq -5\,\sin(\tfrac{3\pi}{4}) + 4\,(h-2)\,[4\cos^2(\tfrac{\pi}{4})]^{-(h-3)}\,\sin^2(\tfrac{\pi}{2})\,\sin(\tfrac{\pi}{4})\\ &= \tfrac{1}{\sqrt{2}}\Big(-5+\tfrac{h-2}{2^{h-5}}\Big) \leq -\tfrac{1}{\sqrt{2}} < -\sin(\tfrac{\pi}{5}), \qquad (12)\end{aligned}$$

and for $k \geq 3$

$$\begin{aligned}\widetilde{\varphi}_{h,k}(\upsilon) &\leq -(k+3)\,\sin(\tfrac{k+1}{k+2}\,\pi)\\ &+\; 4\,(h-k)\,[4\cos^2(\tfrac{\pi}{k+2})]^{-(h-k-1)}\,\sin(\tfrac{k}{k+3}\,\pi)\,\sin(\tfrac{2\,\pi}{k+2})\,\sin(\tfrac{\pi}{k+2})\\ &=\; \sin(\tfrac{\pi}{k+2})\Big[-(k+3)+4\,(h-k)\underbrace{\sin(\tfrac{3\pi}{k+3})}_{\leq 1}\,\underbrace{\sin(\tfrac{2\pi}{k+2})}_{<1}\,r_k^{h-k-1}\Big]\\ &<\; -(k-1)\,\sin(\tfrac{\pi}{k+2}) < -(k-1)\,\sin(\tfrac{\pi}{k+3}), \qquad (13)\end{aligned}$$

because $(h-k)\,r_k^{h-k} \leq r_k$ for $h \geq k+1$. Now, combining (10), (11), (12) and (13), we find the inequality stated in part (a) of the lemma implying that $P_{h,k}(z)$ is strictly monotonically decreasing in the interval $[r_{k+1}, r_k]$.

(b) Using additionally Lemma 3.1(g), the stated inequality can be proved in a similar way. The details are left to the reader. □

Lemma 3.3. *Let* $\alpha_w(j) := 4\cos^2\left(\frac{\pi j}{w}\right)$, $r_w := [\alpha_{w+2}(1)]^{-1}$, $(w,j) \in \mathbb{N}^2$ *and* $\chi(x,\varphi) := 1 + x^2 - 2\,x\,\cos(\varphi)$, $0 \leq \varphi < 2\pi$. *We have:*

(a) $\chi(x,\varphi) = 0$ *iff* $(\varphi, x) \in \{(0,1), (\pi, -1)\}$, *and* $\chi(x,\varphi) > 0$, *otherwise. Furthermore,* $\chi(x,\varphi)$, $x > 0$, *has a minimum (resp. maximum) at* $\varphi = 0$ *(resp.* $\varphi = \pi$*) and the inequalities* $(1-x)^2 \leq \chi(x,\varphi) \leq (1+x)^2$ *hold.*

(b) $\chi(r_w\, \alpha_{w+3}(j), \varphi) > 0$, *and* $\chi(r_w\, \alpha_w(j), \varphi) > 0$, *for all* $\varphi \in [0, 2\pi[$, *and all* $1 \leq j \leq \lfloor \frac{1}{2}(w+2) \rfloor$.

(c) Let $q_w(j,\varphi) := \frac{\chi(r_w\, \alpha_{w+3}(j),\varphi)}{\chi(r_w\, \alpha_w(j),\varphi)}$. *The inequality*

$$f_w(\varphi) := \prod_{1 \leq j \leq \lfloor \frac{1}{2}(w+2) \rfloor} q_w(j,\varphi) > \tfrac{1}{4}\, r_w^4$$

holds for all $w \geq 3$ *and* $\varphi \in [0, 2\pi[$.

Proof: **(a)**: An elementary discussion of the function $\chi(x,\varphi)$ yields the established statements.

(b): Since $j \in \mathbb{N}$, it is easily verified that $0 < r_w\, \alpha_{w+3}(j) \neq 1$ and $0 < r_w\, \alpha_w(j) \neq 1$ for $1 \leq j \leq \lfloor \frac{1}{2}(w+2) \rfloor$. Hence, the stated inequalities follow from part (a).

(c): We have

$$\frac{\partial}{\partial \varphi}\, q_w(j,\varphi) = \frac{2\, r_w\, [\alpha_w(j) - \alpha_{w+3}(j)]\, [r_w^2\, \alpha_w(j)\, \alpha_{w+3}(j) - 1]}{[1 + r_w^2\, [\,\alpha_w(j)\,]^2 - 2\, r_w\, \alpha_w(j)\, \cos(\varphi)]^2}\, \sin(\varphi).$$

Thus, the function $q_w(j,\varphi)$ has relative extreme values at $\varphi \in \{0, \pi\}$. We find

$$\frac{\partial^2}{\partial \varphi^2}\, q_w(j,\varphi)\Big|_{\varphi=0} = \frac{2\, r_w\, [\alpha_w(j) - \alpha_{w+3}(j)]\, [r_w^2\, \alpha_w(j)\, \alpha_{w+3}(j) - 1]}{[1 - r_w\, \alpha_w(j)]^4}$$

and

$$\frac{\partial^2}{\partial \varphi^2}\, q_w(j,\varphi)\Big|_{\varphi=\pi} = -\frac{2\, r_w\, [\alpha_w(j) - \alpha_{w+3}(j)]\, [r_w^2\, \alpha_w(j)\, \alpha_{w+3}(j) - 1]}{[1 + r_w\, \alpha_w(j)]^4}.$$

Hence, $q_w(j,\varphi)$, $1 \leq j \leq \lfloor \frac{1}{2}(w+2) \rfloor$ has a minimum at

- $\varphi = 0$ iff $\alpha_w(j) \gtrless \alpha_{w+3}(j) \;\wedge\; r_w^2\, \alpha_w(j)\, \alpha_{w+3}(j) \lessgtr 1$,
- $\varphi = \pi$ iff $\alpha_w(j) \gtrless \alpha_{w+3}(j) \;\wedge\; r_w^2\, \alpha_w(j)\, \alpha_{w+3}(j) \gtrless 1$.

Now, elementary computations show, for $w \geq 3$ and $1 \leq j \leq \lfloor \frac{1}{2}(w+2) \rfloor$,

$$\alpha_w(j) - \alpha_{w+3}(j) \begin{cases} > 0, & \text{if } w \equiv 0 \bmod 2 \ \wedge \ j = \lfloor \frac{1}{2}(w+2) \rfloor; \\ = 0, & \text{if } (w,j) = (3,2); \\ < 0, & \text{otherwise,} \end{cases} \tag{14}$$

and $r_w^2 \, \alpha_w(j) \, \alpha_{w+3}(j) < 1$. Thus, $q_w(j, \varphi)$, $1 \leq j \leq \lfloor \frac{1}{2}(w+2) \rfloor$, $w \geq 3$, has a minimum at $\varphi = \pi$ iff $w \equiv 0 \bmod 2 \ \wedge \ j = \lfloor \frac{1}{2}(w+2) \rfloor$, and at $\varphi = 0$ otherwise with the exception of $(w, j) = (3, 2)$. Note that $q_3(2, \varphi) \equiv 1$. Therefore, we obtain, for $w \geq 3$,

$$\begin{aligned} f_w(\varphi) &= q_w(\lfloor \tfrac{1}{2}(w+2) \rfloor, \varphi) \prod_{1 \leq j \leq \lfloor \frac{1}{2} w \rfloor} q_w(j, \varphi) \\ &\geq \ q_w(\lfloor \tfrac{1}{2}(w+2) \rfloor, \pi \, \delta_{0, w \bmod 2}) \prod_{1 \leq j \leq \lfloor \frac{1}{2} w \rfloor} q_w(j, 0) \\ &= \left(\frac{1 + \xi_w \, r_w \, \alpha_{w+3}(\lfloor \frac{1}{2}(w+2) \rfloor)}{1 + \xi_w \, r_w \, \alpha_w(\lfloor \frac{1}{2}(w+2) \rfloor)} \right)^2 \prod_{1 \leq j \leq \lfloor \frac{1}{2} w \rfloor} \left(\frac{1 - r_w \, \alpha_{w+3}(j)}{1 - r_w \, \alpha_w(j)} \right)^2, \end{aligned} \tag{15}$$

where $\xi_w := (-1)^{w \bmod 2}$. Now, using Lemma 3.1(b), we find

$$\prod_{1 \leq j \leq \lfloor \frac{1}{2} w \rfloor} \frac{1 - r_w \, \alpha_{w+3}(j)}{1 - r_w \, \alpha_w(j)} = \frac{1}{1 - r_w \, \alpha_{w+3}(\lfloor \frac{1}{2}(w+2) \rfloor)} \, \frac{p_{w+3}(r_w)}{p_w(r_w)}.$$

The quotient $\frac{p_{w+3}(r_w)}{p_w(r_w)}$ can be explicitly computed by the corresponding formulae presented in Lemma 3.1(d). We immediately find $\frac{p_{w+3}(r_w)}{p_w(r_w)} = -r_w^2$. Hence, the lower bound for $f_w(\varphi)$ presented in (15) can be simplified to $f_w(\varphi) \geq \mu_w^2 \, r_w^4$, where

$$\mu_w := \frac{1}{1 + \xi_w \, r_w \, \alpha_w(\lfloor \frac{1}{2}(w+2) \rfloor)} \, \frac{1 + \xi_w \, r_w \, \alpha_{w+3}(\lfloor \frac{1}{2}(w+2) \rfloor)}{1 - r_w \, \alpha_{w+3}(\lfloor \frac{1}{2}(w+2) \rfloor)}.$$

The latter factor appearing on the right-hand side is clearly equal to one for odd w, and greater than one for even w because $r_w \, \alpha_{w+3}(\lfloor \frac{1}{2}(w+2) \rfloor) > 0$. Since $0 < r_w \, \alpha_w(\lfloor \frac{1}{2}(w+2) \rfloor) \leq \frac{2}{3}$, we further obtain $\mu_w \geq [1 + \xi_w \, r_w \ \alpha_w(\lfloor \frac{1}{2}(w+2) \rfloor)]^{-1} \geq \frac{3}{5} > \frac{1}{2}$. This completes the proof. $\square$

Lemma 3.4. *Let* $\alpha_w(j) := 4 \cos^2 \left(\frac{\pi j}{w} \right)$, $r_w := [\alpha_{w+2}(1)]^{-1}$, $(w, j) \in \mathbb{N}^2$ *and* $p_k(z)$ *be the* FIBONACCI *polynomial defined in (4). For* $|z| =$

r_w, the function $\Delta_w(z) := r_w^3 - \frac{|p_{w+3}(z)|}{|p_w(z)|}$, $w \geq 2$, satisfies the inequality $\Delta_w(z) \leq -\Upsilon\, r_w^2 < 0$, where $\Upsilon := \frac{3}{2\sqrt{2}} - 1 \approx .060\,660\ldots > 0$.

Proof: By Lemma 3.1(a), the positive root of minimum modulus of the polynomial $p_w(z)$ is at $z = r_{w-2} > r_w$. Hence, $\Delta_w(z)$ is well defined on the circle $|z| = r_w$. Now, using Lemma 3.1(b) and the definition of the function χ introduced in Lemma 3.3, we obtain for $z := r_w\, e^{i\,\varphi}$, $i^2 = -1$, $0 \leq \varphi < 2\pi$,

$$\begin{aligned}
\Delta_w(z) &= r_w^3 - \Big(\frac{p_{w+3}(z)\, p_{w+3}(\overline{z})}{p_w(z)\, p_w(\overline{z})}\Big)^{\frac{1}{2}} \\
&= r_w^3 - \prod_{1 \leq j \leq \lfloor \frac{1}{2}(w+2) \rfloor} [\,\chi(r_w\, \alpha_{w+3}(j), \varphi)]^{\frac{1}{2}} \\
&\qquad \times \prod_{1 \leq j \leq \lfloor \frac{1}{2}(w-1) \rfloor} [\,\chi(r_w\, \alpha_w(j), \varphi)]^{-\frac{1}{2}}.
\end{aligned}$$

Since $r_w\, \alpha_w(\lfloor \frac{1}{2}(w-1) \rfloor + 1) = 0$, for even $w \geq 2$, we have $\chi(r_w\, \alpha_w(\lfloor \frac{1}{2}(w-1) \rfloor + 1), \varphi) = 1$ by Lemma 3.3(a), and therefore for $z := r_w\, e^{i\,\varphi}$, $i^2 = -1$, $0 \leq \varphi < 2\pi$,

$$\begin{aligned}
\Delta_w(z) &= r_w^3 - [\,\chi(r_w\, \alpha_w(\lfloor \tfrac{1}{2}(w+2) \rfloor), \varphi)]^{\frac{1}{2}} \\
&\qquad \times \prod_{1 \leq j \leq \lfloor \frac{1}{2}(w+2) \rfloor} \Big(\frac{\chi(r_w\, \alpha_{w+3}(j), \varphi)}{\chi(r_w\, \alpha_w(j), \varphi)}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Now, we shall successively consider the four cases $w \in \{2, 3, 4\}$ and $w \geq 5$.

– Since $w = 2$, we immediately obtain for $z := r_2\, e^{i\,\varphi}$, $0 \leq \varphi < 2\pi$,

$$\begin{aligned}
\Delta_2(z) &= r_2^3 - [\chi(r_2\, \alpha_5(1), \varphi)]^{\frac{1}{2}}\, [\chi(r_2\, \alpha_5(2), \varphi)]^{\frac{1}{2}} \\
&= \underbrace{r_2^3}_{=\frac{1}{8}} - \Big[\tfrac{1}{4}(\underbrace{1 - \cos(\varphi)}_{\geq 0})\,(\underbrace{11 - 4\,\cos(\varphi)}_{>0} + \tfrac{1}{16})\Big]^{\frac{1}{2}} \\
&\leq \tfrac{1}{8} - \tfrac{1}{4} = -\tfrac{1}{2}\, r_2^2 < -\Upsilon\, r_2^2.
\end{aligned}$$

– Since $w = 3$, we find for $z := r_3\, e^{i\,\varphi}$, $0 \leq \varphi < 2\pi$,

$$\Delta_3(z) = r_3^3 - \Big(\frac{\chi(r_3\, \alpha_6(1), \varphi)\, \chi(r_3\, \alpha_6(2), \varphi)}{\chi(r_3\, \alpha_3(1), \varphi)}\Big)^{\frac{1}{2}}$$

$$
\begin{aligned}
&= r_3^3 - \Big[\tfrac{24}{(1+\sqrt{5})^2}\,(\underbrace{1-\cos(\varphi)}_{\geq 0}) + \Big(\underbrace{\tfrac{12}{(1+\sqrt{5})^2}-1}_{=3\,r_3-1>0}\Big)^2\Big]^{\frac{1}{2}} \\
&\leq r_3^3 - 3\,r_3 + 1 = -\tfrac{2}{1+\sqrt{5}}\,r_3^2 < -\Upsilon\, r_3^2.
\end{aligned}
$$

– Since $w = 4$, we obtain for $z := r_4\, e^{i\,\varphi}$, $0 \leq \varphi < 2\pi$,

$$
\begin{aligned}
\Delta_4(z) &= r_4^3 - \Big(\frac{\chi(r_4\,\alpha_7(1),\varphi)\,\chi(r_4\,\alpha_7(2),\varphi)\,\chi(r_4\,\alpha_7(3),\varphi)}{\chi(r_4\,\alpha_4(1),\varphi)}\Big)^{\frac{1}{2}} \\
&= \underbrace{r_4^3}_{=\frac{1}{27}} - \Big[\tfrac{1}{81} + \underbrace{\tfrac{4\,(1-\cos(\varphi))}{27\,(13-12\,\cos(\varphi))}}_{\geq 0} \\
&\qquad \times (\underbrace{26 + 3\,(1-\cos(\varphi))\,(47-6\cos(\varphi)}_{>0})\Big]^{\frac{1}{2}} \\
&\leq \tfrac{1}{27} - \tfrac{1}{9} = -\tfrac{2}{3}\,r_4^2 < -\Upsilon\, r_4^2.
\end{aligned}
$$

– Finally, let $w \geq 5$. Using the inequalities established in part (a) and (c) of Lemma 3.3, we find for $z := r_w\, e^{i\,\varphi}$, $0 \leq \varphi < 2\pi$,

$$
\begin{aligned}
\Delta_w(z) &= r_w^3 - \underbrace{[\,\chi(r_w\,\alpha_w(\lfloor\tfrac{1}{2}(w+2)\rfloor),\varphi)]^{\frac{1}{2}}}_{\geq 1 - r_w\,\alpha_w(\lfloor\frac{1}{2}(w+2)\rfloor)}\,[\,\underbrace{f_w(\varphi)}_{>\frac{1}{4}\,r_w^4}\,]^{\frac{1}{2}} \\
&< r_w^2\,(\underbrace{r_w - \tfrac{1}{2}\,[1 - r_w\,\alpha_w(\lfloor\tfrac{1}{2}(w+2)\rfloor)]}_{:=\lambda(w)}).
\end{aligned}
$$

It is not hard to show that $\lambda(2\,w+1)$ (resp. $\lambda(2\,w)$) is monotonically decreasing for growing $w \geq 2$ (resp. $w \geq 3$). Since $\lambda(5) = -\frac{1}{16}\,(8 - [\,7-\sqrt{5}\,]\,\cos^{-2}(\frac{\pi}{7})) \approx -.133\,202\ldots < -\Upsilon$ and $\lambda(6) = -\Upsilon$, we have $\lambda(w) \leq \lambda(6)$ for $w \geq 5$.

This completes the proof. $\square$

3.2 The Dominant Poles of $G_{h,b}(z)$

In this subsection we shall determine the root of minimum modulus of the polynomial $P_{h,k}(z) = p_{k+3}(z) + z^{h-k}\,p_k(z)$ appearing in the denominator of the enumerator $G_{h,k}(z)$ introduced in (6).

Theorem 3.1. *Let* $\alpha_w(j) := 4\cos^2\left(\frac{\pi j}{w}\right)$ *and* $r_w := [\alpha_{w+2}(1)]^{-1}$, $(w,j) \in \mathbb{N}^2$. *We have:*

(a) *The enumerator* $G_{h,b}(z)$ *has a dominant singularity* $z_{h,b}$*, for* $1 \le b < h$, $h \ge 3$, *which is a simple pole.*

(b) *The dominant singularities* $z_{h,b}$, $1 \le b < h$, $h \ge 3$, *satisfy the following chain of inequalities*

$$1 = r_1 > z_{h,1} > r_2 > z_{h,2} > r_3 > \ldots > r_b > z_{h,b} > r_{b+1} > \ldots > r_{h-3} > z_{h,h-3} > r_{h-2} = z_{h,h-2} = z_{h,h-1} > \tfrac{1}{4}.$$

(c) *The dominant singularities* $z_{h,b}$ *satisfy the following chain of inequalities*

$$\left\{ \begin{array}{ll} r_1 = z_{3,1} & \text{if } b = 1 \\ r_{b-1} = z_{b+1,b} > r_b = z_{b+2,b} & \text{if } b \ge 2 \end{array} \right\} > z_{b+3,b} > \ldots > z_{h,b} > z_{h+1,b} > \ldots > r_{b+1}.$$

(d) *The dominant singularity* $z_{b+3,b}$, $b \ge 1$, *satisfies the inequality* $z_{b+3,b} < \widehat{r}_b := [4 \cos^2(\frac{2\pi}{2\,b+5})]^{-1} < r_b$.

Proof: (a), **(b)**: Clearly, the inequalities $\frac{1}{4} < r_{j-1} < r_j < r_1 = 1$ hold for $j \ge 2$ by definition.

As pointed out at the end of Section 2, the enumerator $G_{h,b}(z)$, $b \in \{h-1, h-2\}$, has the representation $G_{h,b}(z) = z\,\frac{p_{h-1}(z)}{p_h(z)} - \delta_{h-2,b}\,z$, where $p_k(z)$ denotes a FIBONACCI polynomial. Hence, $G_{h,h-1}(z)$ (resp. $G_{h,h-2}(z)$) has a simple pole at $z_{h,h-1} = r_{h-2}$ (resp. $z_{h,h-2} = r_{h-2}$), $h \ge 3$, by Lemma 3.1(b). Therefore, it is sufficient to restrict our further considerations to $1 \le b \le h-3$, $h \ge 4$.

Since $G_{h,b}(z)$ is an enumerator, the dominant singularity $z_{h,b}$ is the positive root of minimum modulus of the polynomial $P_{h,b}(z) := p_{b+3}(z) + z^{h-b}\,p_b(z)$ appearing in the denominator of (6) such that the nominator $N_{h,b}(z) := (1-z)\,z^{h-b}\,p_b(z)$ is unequal to zero for $z = z_{h,b}$. Note that the polynomial $P_{h,b}(z)$, $b \ge 1$, has the degree $\max\{\lfloor \frac{1}{2} b \rfloor + 1, h - 1 - \lfloor \frac{1}{2} b \rfloor\}$.

For $b = 1$, i.e. $h \ge 4$, we obtain $P_{h,1}(z) = 1 - 2\,z + z^{h-1}$. This polynomial has already been discussed in [5]. It has exactly one simple positive root $z_{h,1}$ inside the circle $|z| = 1 = r_1$. Since $0 < z_{h,1}^{h-1} = 2\,z_{h,1} - 1$, the inequality $z_{h,1} > \frac{1}{2} = r_2$ holds. Obviously, $N_{h,1}(z_{h,1}) = (1 - z_{h,1})\,z_{h,1}^{h-1}\,p_1(z_{h,1}) \ne 0$. Hence, $G_{h,1}(z)$, $h \ge 4$, has a simple pole at $z_{h,1}$ with $r_2 < z_{h,1} < r_1$. Note that the exact expression

$$z_{h,1} = \sum_{j \ge 0} \frac{1}{(h-1)\,j+1} \binom{(h-1)\,j+1}{j} 2^{-(h-1)\,j-1} \tag{16}$$

has been computed in [5].

Now, let $b \geq 2$, i.e. $h \geq 5$. Setting $\varphi_{h,b}(z) := z^{h-b}\,p_b(z)$ and $\psi_b(z) := p_{b+3}(z)$ and using the definition of the function χ (resp. Δ_w) introduced in Lemma 3.3(a) (resp. Lemma 3.4), we can make the following computations for $z = r_b\,e^{i\,\varphi}$, $0 \leq \varphi < 2\pi$,

$$
\begin{aligned}
|\,\psi_b(z)\,| &= (\,|p_{b+3}(z)|\,|p_{b+3}(\overline{z})|\,)^{\frac{1}{2}} \\
&\overset{Lem.3.1(b)}{=} \prod_{1\leq j\leq\lfloor\frac{1}{2}(b+2)\rfloor}[\underbrace{(1-r_b\,\alpha_{b+3}(j)\,e^{i\,\varphi})\,(1-r_b\,\alpha_{b+3}(j)\,e^{-i\,\varphi})}_{=\chi(r_b\,\alpha_{b+3}(j),\varphi)}]^{\frac{1}{2}} \\
&\overset{Lem.3.3(b)}{>} 0,
\end{aligned}
$$

and

$$
|\varphi_{h,b}(z)| - |\psi_b(z)| = |p_b(z)|\left[\underbrace{|z|^{h-b}}_{\leq r_b^3} - \frac{|p_{b+3}(z)|}{|p_b(z)|}\right] \overset{Lem.3.4}{<} 0. \tag{17}
$$

Thus, the relations $\psi_b(z) \neq 0$ and $|\varphi_{h,b}(z)| < |\psi_b(z)|$ hold for $|z| = r_b$, $b \geq 2$. Hence, ROUCHÉ's theorem applies and the polynomials $\psi_b(z)$ and $\psi_b(z) + \varphi_{h,b}(z) = P_{h,b}(z)$ have the same number of zeros inside the circle $|z| = r_b$. By Lemma 3.1(a), the roots of $\psi_b(z) = p_{b+3}(z)$ are at $z_j = [\,\alpha_{b+3}(j)\,]^{-1}$, $1 \leq j \leq \lfloor\frac{1}{2}(b+2)\rfloor$. Clearly, we have $z_1 < r_b$ and $z_j > r_b$ for $2 \leq j \leq \lfloor\frac{1}{2}(b+2)\rfloor$. Therefore, $P_{h,b}(z)$ has exactly one root $\widehat{z}_{h,b}$ with $\widehat{z}_{h,b} < r_b$, $2 \leq b \leq h-3$.

Now, assume that $\widehat{z}_{h,b}$ is not a simple root of $P_{h,b}(z)$, i.e. the equations $P_{h,b}(\widehat{z}_{h,b}) = 0$ and $P'_{h,b}(\widehat{z}_{h,b}) = 0$ hold. The former equation implies $\widehat{z}_{h,b}^{\,h-b} = -\frac{p_{b+3}(\widehat{z}_{h,b})}{p_b(\widehat{z}_{h,b})}$. By Lemma 3.1(e), the quotient $\frac{p_{b+3}(z)}{p_b(z)}$ is positive for $z \in [0, r_{b+1}[$, $b \geq 3$. Since $p_2(z) = 1$, this fact holds for $b = 2$, too. Hence, the inequality $\widehat{z}_{h,b} \geq r_{b+1}$ is valid because $p_b(r_{b+1}) \neq 0$ by Lemma 3.1(d). The assumption $\widehat{z}_{h,b} = r_{b+1}$ yields the contradiction $r_{b+1}^{h-b} = -\frac{p_{b+3}(r_{b+1})}{p_b(r_{b+1})} = 0$ by Lemma 3.1(a). Therefore, we have $r_{b+1} < \widehat{z}_{h,b} < r_b$, $2 \leq b \leq h-3$, and Lemma 3.2(a) yields the contradiction $P'_{h,b}(\widehat{z}_{h,b}) < 0$. Thus, the polynomial $P_{h,b}(z)$ has exactly one positive simple root $\widehat{z}_{h,b}$ with $r_{b+1} < \widehat{z}_{h,b} < r_b$. Since $r_{b-2} > r_b$, we have $p_b(\widehat{z}_{h,b}) > 0$, $b \geq 3$, by Lemma 3.1(e). For $b = 2$, this inequality is clearly valid. Therefore, $N_{h,b}(\widehat{z}_{h,b}) = (1 - \widehat{z}_{h,b})\,\widehat{z}_{h,b}^{\,h-b}\,p_b(\widehat{z}_{h,b}) \neq 0$ because $0 < \widehat{z}_{h,b} < r_b \leq r_2 < 1$, Hence, the enumerator $G_{h,b}(z)$, $h \geq b+3 \geq 5$, has a simple pole at $z_{h,b} := \widehat{z}_{h,b}$ with $r_{b+1} < z_{h,b} < r_b$.

(c) The equalities $z_{3,1} = r_1$ and $z_{b+1,b} = r_{b-1}$, $b \geq 2$, have already been verified at the beginning of the proof of the parts (a) and (b) of this lemma.

Now, assume that there is a tuple (h, b), $h \geq b+1$, $b \geq 2$, with $z_{h,b} \leq z_{h+1,b}$. By part (b) of this lemma, we know that both, $z_{h,b}$ and $z_{h+1,b}$, lie in the interval $]\, r_{b+1}, r_b \,[\, \subseteq \,]\, \frac{1}{4}, \frac{1}{2}\, [$. Since $p_2(z) = 1 > 0$ and $p_b(z_{h,b}) > 0$, $b \geq 3$, by Lemma 3.1(e), the assumption $\widetilde{z}_{h,b} := z_{h,b} = z_{h+1,b}$ immediately yields $0 = P_{h+1,b}(\widetilde{z}_{h,b}) - P_{h,b}(\widetilde{z}_{h,b}) = \widetilde{z}_{h,b}^{\,h-b}\,(\widetilde{z}_{h,b} - 1)\, p_b(\widetilde{z}_{h,b})$, i.e. we have the contradiction $\widetilde{z}_{h,b} \in \{0, 1\}$. Assuming that $z_{h,b} < z_{h+1,b}$, Lemma 3.2(a) yields

$$\begin{aligned} 0 &= P_{h+1,b}(z_{h+1,b}) < P_{h+1,b}(z_{h,b}) = p_{b+3}(z_{h,b}) + z_{h,b}^{h-b+1}\, p_b(z_{h,b}) \\ &< p_{b+3}(z_{h,b}) + z_{h,b}^{h-b}\, p_b(z_{h,b}) = P_{h,b}(z_{h,b}) = 0, \end{aligned}$$

which is another contradiction.

We are left with the case $b = 1$. Since $(h-1)\,j + 1 < h\,j + 1$ and $hj - \lambda + 2 < 2\,[(h-1)\,j - \lambda + 2]$, $1 \leq \lambda \leq j$, we immediately obtain $z_{h+1,1} < z_{h,1}$, $h \geq 3$, by the explicit expression for $z_{h,1}$ established in (16).

(d) Since $\frac{\pi}{b+3} < \frac{2\pi}{2\,b+5} < \frac{\pi}{b+2} < \frac{\pi}{2}$, $b \geq 1$, we have $r_{b+1} < \widehat{r}_b < r_b$.

First, for $b = 1$, we find $P_{4,1}(z) = 1 - 2\,z + z^3 = (z-1)\,(z + \phi)\,(z + \phi^{-1})$, where $\phi := \frac{1}{2}(1 + \sqrt{5}\,)$ is the 'golden ratio'. Hence, $z_{4,1} = \phi^{-1} = .618\,033\ldots < [\,4\cos^2(\frac{2\pi}{7})\,]^{-1} = \widehat{r}_1 \approx .643\,104\ldots$.

Next, let $b \geq 2$. An inspection of Lemma 3.2(a) shows that $P_{h,b}(z)$, $b \geq 2$, is strictly monotonically decreasing in $[r_{b+1}, r_b]$. Since $z_{b+3,b}$ is the unique root of $P_{b+3,b}(z)$ in $[r_{b+1}, r_b]$, it is sufficient to prove that $P_{b+3,b}(\widehat{r}_b) < 0$. For this purpose we change the variable and consider the function

$$\begin{aligned} \nu_b(\upsilon) &:= P_{b+3,b}([4\cos^2(\upsilon)]^{-1}) \\ &= p_{b+3}([4\cos^2(\upsilon)]^{-1}) + [4\cos^2(\upsilon)]^{-3}\, p_b([4\cos^2(\upsilon)]^{-1}) \end{aligned}$$

in the interval $[\frac{\pi}{b+3}, \frac{\pi}{b+2}]$. Setting $\upsilon := \frac{2\pi}{2\,b+5}$ and using Lemma 3.1(e), elementary trigonometric transformations yield $P_{b+3,b}(\widehat{r}_b) =$

$\nu_b(\frac{2\pi}{2b+5})$, where

$$\nu_b(\tfrac{2\pi}{2b+5}) = -\frac{1}{2^{b+6}\underbrace{\cos(\frac{\pi}{2b+5})}_{>0}\underbrace{\cos^{b+5}(\frac{2\pi}{2b+5})}_{>0}}\underbrace{[4\cos^2(\tfrac{\pi}{2b+5})-3)]^2}_{>0}$$
$$\times\underbrace{[4\cos^2(\tfrac{\pi}{2b+5})-1]}_{>0}.$$

Evidently, the latter expression is negative. This completes the proof. $\square$

Remark 3.1. **(a)** The preceding lemma presents a detailed survey of the dominant singularities $z_{h,b}$ of the enumerators $G_{h,b}(z)$, $1 \le b < h$, $h \ge 3$. Arranging the values of $z_{h,b}$ in an array, this array has the following structure:

$$\begin{array}{c|ccccccccccccccccc}
h \setminus b & 1 & & 2 & & 3 & & \cdots & & b & & \cdots & & h-2 & & h-1 & & h \\
\hline
3 & \substack{z_{3,1}\\=r_1} & = & \substack{z_{3,2}\\=r_1} & & & & & & & & & & & & & & \\
 & > & & > & & & & & & & & & & & & & & \\
4 & z_{4,1} & > & \substack{z_{4,2}\\=r_2} & = & \substack{z_{4,3}\\=r_2} & & & & & & & & & & & & \\
 & > & & > & & > & & & & & & & & & & & & \\
5 & z_{5,1} & > & z_{5,2} & > & \substack{z_{5,3}\\=r_3} & = & & & & & & & & & & & \\
 & > & & > & & > & & & & & & & & & & & & \\
6 \ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & & & & & & \\
 & > & & > & & > & & \cdots & & > & & \ddots & & > & & & & \\
h & z_{h,1} & > & z_{h,2} & > & z_{h,3} & > & \cdots & > & z_{h,b} & > & \cdots & > & \substack{z_{h,h-2}\\=r_{h-2}} & = & \substack{z_{h,h-1}\\=r_{h-2}} & & \\
 & > & & > & & > & & \cdots & & > & & \cdots & & > & & > & & \\
h+1 & z_{h+1,1} & > & z_{h+1,2} & > & z_{h+1,3} & > & \cdots & > & z_{h+1,b} & > & \cdots & > & z_{h+1,h-2} & > & \substack{z_{h+1,h-1}\\=r_{h-1}} & = & \substack{z_{h+1,h}\\=r_{h-1}} \\
 & > & & > & & > & & \cdots & & > & & \cdots & & > & & > & & > \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & > & & > & & > & & \cdots & & > & & \cdots & & > & & > & & > \\
 & r_2 & > & r_3 & > & r_4 & > & \cdots & > & r_{b+1} & > & \cdots & > & r_{h-1} & > & r_h & > & r_{h+1} \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\infty & r_2 & & r_3 & & r_4 & & \cdots & & r_{b+1} & & \cdots & & .25 & & .25 & & .25
\end{array}$$

In each row (resp. column), the entries are strictly monotonically decreasing from left to right (resp. top to bottom).

(b) We are left with the concrete problem of computing an approximation of the dominant singularity $z_{h,b}$ for $1 \leq b \leq h-3$, i.e. we have to find an approximation of the simple positive root of minimum modulus of the polynomial $P_{h,b}(z) = p_{b+3}(z) + z^{h-b}p_b(z)$. This problem can essentially be solved by NEWTON's method, i.e. we have to consider the recurrence

$$z_t = z_{t-1} - \frac{p_{b+3}(z_{t-1}) + z_{t-1}^{h-b}\, p_b(z_{t-1})}{p'_{b+3}(z_{t-1}) + z_{t-1}^{h-b-1}[\,(h-b)\,p_b(z_{t-1}) + z_{t-1}\, p'_b(z_{t-1})\,]}, \quad t \geq 1,$$

starting with a well chosen initial value z_0. We choose $z_0 := r_{b+1}$ because $p_{b+3}(r_{b+1}) = 0$ by Lemma 3.1(a), and $r_{b+1}^{h-b}\, p_b(r_{b+1}) = (1 - r_{b+1})\, r_{b+1}^{h-\frac{1}{2}(b+3)}$ by Lemma 3.1(d), becomes very small for large h, where $b = \circ(h)$.

Before we present an exact asymptotical evaluation of $z_{h,b}$, we briefly discuss some aspects concerning the convergence of the sequence z_t, $t \geq 0$. According to the proof of Theorem 3.1(a), (b), the polynomial $P_{h,b}(z)$ has exactly one simple root $z_{h,b}$ in the interval $[r_{b+1}, r_b]$. Hence, the equalities $P_{h,b}(r_{b+1})P_{h,b}(r_b) < 0$ and $P'_{h,b}(z) \neq 0$ hold in that interval. Indeed, we have $P_{h,b}(r_{b+1}) = (1 - r_{b+1})\, r_{b+1}^{h-\frac{1}{2}(b+3)} > 0$, and $P_{h,b}(r_b) = -r_b^{\frac{1}{2}b+1}(1 - r_b^{h-b-2}) < 0$ by Lemma 3.1(d) and $P'_{h,b}(z) < 0$ by Lemma 3.2(a). Since additionally $P_{h,b}(z)$ is convex in the interval $[r_{b+1}, r_b]$ by Lemma 3.2(b), the NEWTON-sequence $z_0 = r_{b+1}, z_1, z_2, \ldots$ is strictly monotonically increasing and it converges quadratically to the root $z_{h,b}$ in $[r_{b+1}, r_b]$, i.e. $\limsup_{t\to\infty} \frac{|z_{t+1} - z_{h,b}|}{|z_t - z_{h,b}|^2} = \alpha > 0$.

Now, let us turn to the asymptotical evaluation of $z_{h,b}$. We compute successively $z_0 := r_{b+1}, z_1, z_2, \ldots$. In each step we evaluate the approximant z_t around r_{b+1}^{h-b} and simplify the resulting expression by means of the explicit expressions presented in Lemma 3.1(d). Choosing this procedure, we obtain by a very lengthy computation, for $b = \circ(h)$ and large h:

$$\begin{aligned} z_{h,b} \;&=\; r_{b+1} + c_b^{(1)}\, r_{b+1}^{h-b-2} + c_{h,b}^{(2)}\, r_{b+1}^{2(h-b)-5} + c_{h,b}^{(3)}\, r_{b+1}^{3(h-b)-8} \\ &\quad + \mathcal{O}((h-b)^3\, r_{b+1}^{4(h-b)}), \end{aligned} \tag{18}$$

where[1]

$$
\begin{aligned}
c_b^{(1)} &= -\tfrac{1}{p'_{b+3}}\, r_{b+1}^2\, p_b = \tfrac{1}{b+3}\,(1-r_{b+1})(4\,r_{b+1}-1);\\
c_{h,b}^{(2)} &= \tfrac{1}{[\,p'_{b+3}\,]^2}\, r_{b+1}^4\, p_b^2\,(h-b) + \tfrac{1}{2\,[\,p'_{b+3}\,]^3}\, r_{b+1}^5\, p_b\,[\,2\,p'_b\,p'_{b+3} - p_b\,p''_{b+3}\,]\\
&= \tfrac{1}{2\,(b+3)^2}\,(1-r_{b+1})(4\,r_{b+1}-1)\,[\,\xi_b^{(1)}(h-b)+\xi_b^{(0)}\,],\\
&\quad \text{with } \xi_b^{(0)} := (2\,r_{b+1}+b-2)(3\,r_{b+1}-1),\\
&\quad \text{and } \xi_b^{(1)} := 2(1-r_{b+1})(4\,r_{b+1}-1);\\
c_{h,b}^{(3)} &= -\tfrac{3}{2\,[\,p'_{b+3}\,]^3}\, r_{b+1}^6\, p_b^3 (h-b)^2\\
&\quad + \tfrac{1}{2\,[\,p'_{b+3}\,]^4}\, r_{b+1}^6\, p_b^2\left(p_b\,p'_{b+3} - 6\,r_{b+1}\,p'_b\,p'_{b+3} + 3\,r_{b+1}\,p_b\,p''_{b+3}\right)(h-b)\\
&\quad + \tfrac{1}{6\,[\,p'_{b+3}\,]^5}\, r_{b+1}^8\, p_b\left(p_b^2\,p'_{b+3}\,p'''_{b+3} - 3\,p_b\,p''_b\,[\,p'_{b+3}\,]^2 - 3\,p_b^2\,[\,p''_{b+3}\,]^2\right.\\
&\qquad\qquad \left. + 9\,p_b\,p'_b\,p'_{b+3}\,p''_{b+3} - 6\,[\,p'_b\,]^2\,[\,p'_{b+3}\,]^2\right)\\
&= \tfrac{1}{6\,(b+3)^3}\,(1-r_{b+1})(4\,r_{b+1}-1)\\
&\quad [\,\eta_b^{(2)}(h-b)^2 + \eta_b^{(1)}(h-b) + \eta_b^{(0)}\,],\\
&\quad \text{with } \eta_b^{(0)} := 120\,r_{b+1}^4 - 2(b^2-57\,b+155)\,r_{b+1}^3\\
&\qquad\qquad + 9(2\,b^2-21\,b+30)\,r_{b+1}^2 - 3(b-6)(4\,b-5)\,r_{b+1}\\
&\qquad\qquad + 2(b-1)(b-5),\\
&\quad \text{and } \eta_b^{(1)} := 3(1-r_{b+1})(4\,r_{b+1}-1)\\
&\qquad\qquad \times[\,22\,r_{b+1}^2 + (9\,b-29)\,r_{b+1} - 3\,b+7],\\
&\quad \text{and } \eta_b^{(2)} := 9(1-r_{b+1})^2(4\,r_{b+1}-1)^2.
\end{aligned}
$$

In principle, the expansion for $z_{h,b}$ could be carried out as far as we like but the higher terms become more and more complicated. Without going into detail, another approach to the computation of the asymptotic approximation given in (18) is the bootstrapping technique starting with $z_0 := r_{b+1}$, and computing better estimates z_t, $t = 1, 2, 3, \ldots$. In this way, we find the same approximation. Notice that $r_2 = \frac{1}{2}$, and therefore $(c_1^{(1)}, c_{h,1}^{(2)}, c_{h,1}^{(3)}) = (\,\frac{1}{8}, \frac{1}{64}\,(h-1), \frac{1}{1024}\,(h-1)(3\,h-4)\,)$. Thus,

$$z_{h,1} \;=\; \tfrac{1}{2} + 2^{-h} + (h-1)\,2^{-(2h-1)} + (h-1)(3\,h-4)\,2^{-(3h-1)}$$

[1] The abbreviations p_b, p'_b, p''_b, and p'''_b stand for $p_b(r_{b+1})$, $p'_b(r_{b+1})$, $p''_b(r_{b+1})$ and $p'''_b(r_{b+1})$, respectively.

$$+\,\mathcal{O}(h^3\,2^{-4\,h}).$$

$h \setminus b$	1	2	3	4	5	6	7	8	9	10
3	$r_1 = 1$	$r_1 = 1$								
4	|.618 033 *.597 656*	r_2	r_2							
5	|.543 689 *.541 748*	|.414 213 *.413 487*	r_3	r_3						
6	|.518 790 *.518 600*	|.392 646 *.392 617*	|.347 296 *.347 253*	r_4	r_4					
7	|.508 660 *.508 642*	|.385 794 *.385 793*	|.337 666 *.337 665*	|.315 448 *.315 447*	r_5	r_5				
8	|.504 138 *.504 136*	.383 387 *.383 387*	.334 734 *.334 734*	.310 206 *.310 206*	|.297 411 *.297 412*	r_6	r_6			
9	|.502 017 *.502 016*	.382 501 *.382 501*	.333 794 *.333 794*	.308 654 *.308 654*	.294 206 *.294 206*	.286 081 *.286 081*	r_7	r_7		
10	.500 994 *.500 994*	.382 169 *.382 169*	.333 486 *.333 486*	.308 185 *.308 185*	.293 275 *.293 275*	.283 965 *.283 965*	.278 450 *.278 450*	r_8	r_8	
11	.500 493 *.500 493*	.382 043 *.382 043*	.333 384 *.333 384*	.308 042 *.308 042*	.293 005 *.293 005*	.283 358 *.283 358*	.276 973 *.276 973*	.273 045 *.273 045*	r_9	r_9
$\to$	$\to$	$\to$	$\to$	$\to$	$\to$	$\to$	$\to$	$\to$	$\to$	$\to$
∞	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}
	$=$	$=$	$=$	$=$	$=$	$=$	$=$	$=$	$=$	$=$
	.5	.381 966	.333 333	.307 978	.292 893	.283 118	.276 393	.271 554	.267 949	.265 187

Table 2: The dominant singularities $z_{h,b}$ for $3 \leq b \leq h$, $3 \leq h \leq 11$. The upper (resp. lower) number appearing in an entry corresponds to the exact (resp. approximative) value of $z_{h,b}$. The values are not rounded to the sixth decimal place. The exact and the approximate value coincide up to the sixth decimal place regarding the entries not marked with $|$.

This approximation coincides with the terms up to $j = 3$ in the explicit expression for $z_{h,1}$ established in (16). The exact and approximate values for $z_{h,b}$, $1 \le b \le h$, $3 \le h \le 12$, are presented in Table 2. Even for small h, the derived approximation yields very good values for the dominant singularity $z_{h,b}$. $\diamond$

3.3 An asymptotic Equivalent to the Taylor Coefficients of $G_{h,b}(z)$

In this subsection we shall determine the exact asymptotic behaviour of the coefficient at z^n in the expansion of the enumerator $G_{h,b}(z)$ around $z = 0$.

Theorem 3.2. *Let* $r_w := [4\cos^2\left(\frac{\pi}{w+2}\right)]^{-1}$, $w \in \mathbb{N}$, *and* $p_k(z)$ *be the* Fibonacci *polynomial defined in (4). Furthermore, let* $K_{h,h-1} = K_{h,h-2} := \frac{1}{h}\tan^2(\frac{\pi}{h})$, $h \ge 3$, *and*

$$K_{h,b} := \frac{1 - z_{h,b}}{z_{h,b}\left[\frac{p'_{b+3}(z_{h,b})}{p_{b+3}(z_{h,b})} - \frac{p'_b(z_{h,b})}{p_b(z_{h,b})}\right]} - (h-b),$$

for $1 \le b \le h-3$, *where* $z_{h,b}$ *is the dominant pole of the enumerator* $G_{h,b}(z)$. *An asymptotic equivalent to the coefficient* $\langle z^n; G_{h,b}(z)\rangle$[2] *is given by*

$$\langle z^n; G_{h,b}(z)\rangle = K_{h,b}\, z_{h,b}^{-n} + \mathcal{O}(b^3\, r_b^{-n}),\ n \to \infty.$$

Proof: First, let us consider the simple cases $b \in \{h-1, h-2\}$. It has been shown in the classical paper [1] that the n-th coefficient $\phi_h(n) = \langle z^n; F_h(z)\rangle$ of the enumerator $F_h(z) = z\frac{p_h(z)}{p_{h+1}(z)}$ introduced in (3) has the explicit representation

$$\phi_h(n) = \tfrac{1}{h+1} \sum_{1 \le j \le \lfloor \frac{1}{2}h \rfloor} \tan^2(\tfrac{\pi j}{h+1})\,[4\cos^2(\tfrac{\pi j}{h+1})]^n,\ n \ge 2.$$

Splitting up the sum appearing on the right-hand side and using the inequalities $\cos^2(\frac{\pi j}{h+1}) \le \cos^2(\frac{2\pi}{h+1})$ and $\tan^2(\frac{\pi j}{h+1}) \le \tan^2(\frac{\pi}{h+1}\lfloor\frac{h}{2}\rfloor) = \frac{2}{\pi^2}(5 + 3(-1)^h)\lfloor\frac{h}{2}\rfloor^2 + \mathcal{O}(h)$, $2 \le j \le \lfloor\frac{h}{2}\rfloor$, we further obtain for $h \ge 4$

$$\phi_h(n) = \tfrac{1}{h+1}\tan^2(\tfrac{\pi}{h+1})\, r_{h-1}^{-n} + \mathcal{O}(h^2\,[4\cos^2(\tfrac{2\pi}{h+1})]^n),\quad n \ge 2. \quad (19)$$

[2] The abbreviation $\langle z^n; f(z)\rangle$ denotes the coefficient of z^n in the expansion of $f(z)$ at $z = 0$.

Clearly, this estimate holds for $h \in \{2,3\}$, too. Now, as it was observed at the end of Section 2, we have $G_{h,h-2}(z) + z = G_{h,h-1}(z) = F_{h-1}(z)$ and therefore

$$\langle z^n; G_{h,h-2}(z)\rangle = \langle z^n; G_{h,h-1}(z)\rangle = \langle z^n; F_{h-1}(z)\rangle = \phi_{h-1}(n)$$

for $n \geq 3$. Combining this relation and the expression for $\phi_h(n)$ given in (19), we immediately find the asymptotic equivalent stated in the theorem because $z_{h,h-2} = z_{h,h-1} = r_{h-2}$ by Lemma 3.1(b) and $4\cos^2(\frac{2\pi}{h}) \leq r_{h-2}^{-1} < r_{h-1}^{-1}$, $h \geq 2$.

Next, let us consider the cases $1 \leq b \leq h-3$. For $b = 1$, we have $G_{h,1}(z) = \frac{(1-z)\,z^{h-1}}{1-2\,z+z^{h-1}}$, $h \geq 2$. This function is identical with the function $D_{h-1}(z)$ already introduced in [5, formula (6)]. In Theorem 3(b) of that paper it has been proved that $\langle z^n; D_h(z)\rangle = C_h\, q_h^{-n} + \mathcal{O}(1)$, where $C_h = \frac{(1-q_h)\,(2\,q_h-1)}{h-2(h-1)q_h}$ and $q_h = z_{h+1,1}$. Hence, $\langle z^n; G_{h,1}(z)\rangle = C_{h-1}\, z_{h,1}^{-n} + \mathcal{O}(1)$. Since $(p_1(z), p_4(z)) = (1, 1-2z)$, the expression for $K_{h,1}$ given in the theorem is identical with C_{h-1} and our result is valid for $b = 1$.

Now, let $2 \leq b \leq h-3$. By Theorem 3.1(a), the enumerator $G_{h,b}(z)$ has a dominant simple pole at $z_{h,b}$. Hence, the partial fraction expansion of $G_{h,b}(z)$ has the form

$$G_{h,b}(z) = \widehat{K}_{h,b}\,(1 - \tfrac{z}{z_{h,b}})^{-1} + R_{h,b}(z),$$

where

$$\widehat{K}_{h,b} := \lim_{z \to z_{h,b}} (1 - \tfrac{z}{z_{h,b}})\, \frac{(1-z)\, z^{h-b}\, p_b(z)}{p_{b+3}(z) + z^{h-b}\, p_b(z)}$$

and $R_{h,b}(z)$ is a regular rational function for $|z| \leq z_{h,b}$. Applying D'Hospital's rule and using the relation $z_{h,b}^{h-b} = -\frac{p_{b+3}(z_{h,b})}{p_b(z_{h,b})}$, we find $\widehat{K}_{h,b} \equiv K_{h,b}$, where $K_{h,b}$ is the quantity defined in the theorem. Thus, $\langle z^n; G_{h,b}(z)\rangle = K_{h,b}\, z_{h,b}^{-n} + \langle z^n; R_{h,b}(z)\rangle$, where the coefficient $\langle z^n; R_{h,b}(z)\rangle$ involves all the zeros of $P_{h,b}(z) = p_{b+3}(z) + z^{h-b}\, p_b(z)$, $b \geq 2$, with the exception of $z_{h,b}$. Unfortunately, we have only insufficient information about these zeros at hand. It seems to be true that all roots of the polynomial $P_{h,b}(z)$ are simple but the author is unable to prove this conjecture. However, there is a general theorem established in [13, Theorem 9.2] which can be successively applied in order to determine the asymptotic growth of $\langle z^n; R_{h,b}(z)\rangle$. For this purpose we consider the rational function $\widehat{G}_{h,b}(z) := z^{-2}G_{h,b}(z) = \frac{\widehat{N}_{h,b}(z)}{P_{h,b}(z)}$, where $\widehat{N}_{h,b}(z) := (1-z)\, z^{h-b-2}\, p_b(z)$, and $P_{h,b}(z) = p_{b+3}(z) + z^{h-b}\, p_b(z)$. Note that $\langle z^n; \widehat{G}_{h,b}(z)\rangle = \langle z^{n+2}; G_{h,b}(z)\rangle$, that the degree of the polynomial $\widehat{N}_{h,b}(z)$ is less than the degree of $P_{h,b}(z)$ and that $P_{h,b}(0) = p_{b+3}(0) \neq 0$. Translating the result established in [13, Theorem 9.2] into our case (set $k := 1$ and $\rho_1 := z_{h,b}$), we obtain

$$\underbrace{\Big|\langle z^n; \widehat{G}_{h,b}(z)\rangle + \underbrace{\frac{\widehat{N}_{h,b}(z_{h,b})}{P'_{h,b}(z_{h,b})}\, z_{h,b}^{-n-1}}_{=-K_{h,b}\, z_{h,b}^{-n-2}}\Big|}_{=\langle z^{n+2}; R_{h,b}(z)\rangle} \leq W\, R^{-n} + \delta^{-1}\, R^{-n}\, M,$$

where the following conditions have to hold:

(i) $P_{h,b}(z)$ has only one simple root in $|z| < R$;
(ii) $R - z_{h,b} > \delta$ for some $\delta > 0$;
(iii) $\max_{|z|=R} |\widehat{G}_{h,b}(z)| \leq W$;
(iv) $M = \Big|\frac{\widehat{N}_{h,b}(z_{h,b})}{P'_{h,b}(z_{h,b})}\Big|$.

Now, let $R := r_b$ and $\delta := r_b - \widehat{r}_b$, with $\widehat{r}_b := [4\cos(\frac{2\pi}{2b+5})]^{-1}$. The condition (i) is clearly satisfied by Theorem 3.1(a), (b). By Theorem 3.1(c) and (d), we have $R - z_{h,b} = r_b - z_{h,b} > r_b - z_{b+3,b} > r_b - \widehat{r}_b = \delta > 0$, and the condition (ii) holds, too. Moreover, it is easily verified that δ^{-1} is monotonically increasing with growing $b \geq 2$, and that $\delta^{-1} \leq \frac{1}{2}\cos^2(\frac{2\pi}{9})(2\cos^2(\frac{2\pi}{9}) - 1)^{-1}\, b^3 = \mathcal{O}(b^3)$, for $b \geq 2$.

Next, the following computation shows that the upper bound W introduced in condition (iii) can be chosen as a constant, i.e. $W = \mathcal{O}(1)$. Indeed, we have for $|z| = r_b$, $2 \leq b \leq h-3$,

$$\begin{aligned} |\widehat{G}_{h,b}(z)| &= \frac{|z|^{h-b-2}|\,|1-z|\,|p_b(z)|}{|p_{b+3}(z) + z^{h-b}\, p_b(z)|} \leq \frac{|z|^{h-b-2}(1+|z|)\,|p_b(z)|}{\underbrace{|p_{b+3}(z)| - |z|^{h-b}|p_b(z)|}_{\overset{(17)}{>}\,0}} \\ &= \frac{r_b^{h-b-2}(1+r_b)}{\frac{|p_{b+3}(z)|}{|p_b(z)|} - \underbrace{r_b^{h-b}}_{\leq r_b^3}} \overset{Lem. 3.4}{\leq} \frac{r_b^{h-b-2}(1+r_b)}{\Upsilon\, r_b^2} \leq \underbrace{\frac{1+r_b}{\Upsilon\, r_b}}_{:=u(b)} = \mathcal{O}(1). \end{aligned}$$

Note that the sequence $(u(b))_{b\geq 2}$ is positive and monotonically increasing with the limit $10\,(4+3\sqrt{2}\,) \approx 82.426\,406\ldots$.

Finally, let us turn to the quantity M appearing in condition (iv). Since $r_{b+1} < z_{h,b} < r_b$ by Theorem 3.1(b), $p_2(z) = 1$, and $p_b(z_{h,b}) < p_b(r_{b+1}) = (1-r_{b+1})\, r_{b+1}^{\frac{1}{2}(b-3)}$, $b \geq 3$, by Lemma 3.1(d) and (e), we obtain, for $2 \leq b \leq h-3$,

$$\begin{aligned} |\widehat{N}_{h,b}(z_{h,b})| &= |z_{h,b}|^{h-b-2}\, \underbrace{|1-z_{h,b}|}_{\leq 1+|z_{h,b}|}\, |p_b(z_{h,b})| \\ &\leq r_b^{h-b-2}(1+r_b)(1-r_{b+1})\, r_{b+1}^{\frac{1}{2}(b-3)} \leq r_b\,(\,1+r_b)\, r_{b+1}^{\frac{1}{2}(b-3)}. \end{aligned}$$

Hence, by Lemma 3.2(a)

$$M \leq \frac{r_b\,(1+r_b)(4\,r_{b+1}-1)\, r_{b+1}^{\frac{1}{2}(b-3)}}{(b-1)\, r_{b+1}\, r_b^{\frac{1}{2}b}} \leq \underbrace{\frac{r_b\,(1+r_b)(4\,r_{b+1}-1)}{(b-1)\, r_{b+1}^{\frac{5}{2}}}}_{:=v(b)} = \mathcal{O}(1),$$

because $r_{b+1} < r_b$. Note that the sequence $(v(b))_{b\geq 2}$ is positive and monotonically decreasing with $v(b) \leq v(2) = \frac{3}{8}(5+3\sqrt{5}) = 4.390\,576\ldots$.

Thus, we have verified the relation $|\langle z^{n+2}; R_{h,b}(z)\rangle| = \mathcal{O}(b^3\, r_b^{-n})$, $2 \leq b \leq h-3$. This completes the proof because $r_b^2 \in [\frac{1}{16}, \frac{1}{4}[$, $b \geq 2$. $\square$

Remark 3.2. **(a)** Using Lemma 3.1(d), (e) and Lemma 3.2(a), it is not hard to verify that the quantity $K_{h,b}$ introduced in the preceding theorem is bounded, i.e. $K_{h,b} = \mathcal{O}(1)$, $1 \leq b < h$, $h \geq 3$.

(b) Evaluating the explicit expression for $K_{h,b}$ stated in the previous theorem by means of (18) and using the explicit expressions presented in Lemma 3.1(d), we find the following approximation, for $b = \circ(h)$ and large h, by a lengthy computation:

$$\begin{aligned} K_{h,b} &= k_b^{(1)}\, r_{b+1}^{h-b-3} + k_{h,b}^{(2)}\, r_{b+1}^{2(h-b)-6} + k_{h,b}^{(3)}\, r_{b+1}^{3(h-b)-9} \\ &\quad + \mathcal{O}((h-b)^3\, r_{b+1}^{4(h-b)}), \end{aligned} \tag{20}$$

where

$$k_b^{(1)} = \tfrac{1}{b+3}\,(1-r_{b+1})^2(4\,r_{b+1}-1);$$

$$k^{(2)}_{h,b} = \tfrac{1}{(b+3)^2}(1-r_{b+1})^2(4\,r_{b+1}-1)\,[\,e^{(1)}_b(h-b)+e^{(0)}_b\,],$$
$$\text{with } e^{(0)}_b := 6\,r^2_{b+1}+3(b-4)\,r_{b+1}-b+3,$$
$$\text{and } e^{(1)}_b := 2(1-r_{b+1})(4\,r_{b+1}-1);$$
$$k^{(3)}_{h,b} = \tfrac{1}{6\,(b+3)^3}(1-r_{b+1})^2(4\,r_{b+1}-1)$$
$$[\,f^{(2)}_b(h-b)^2+f^{(1)}_b(h-b)+f^{(0)}_b\,],$$
$$\begin{aligned}\text{with } f^{(0)}_b := {} & 24(b^2+b+28)\,r^4_{b+1}-2(b^3-b^2-289\,b+1\,003)\,r^3_{b+1}\\ & +3(6\,b^3+7\,b^2-361\,b+636)\,r^2_{b+1}\\ & -3(4\,b^3+7\,b^2-164\,b+216)\,r_{b+1}\\ & +2\,b^3+4\,b^2-65\,b+74,\end{aligned}$$
$$\begin{aligned}\text{and } f^{(1)}_b := {} & 3(1-r_{b+1})(4\,r_{b+1}-1)\\ & \times[\,22(b+5)\,r^2_{b+1}+(9\,b^2+16\,b-169)\,r_{b+1}\\ & \quad -3\,b^2-8\,b+41],\end{aligned}$$
$$\text{and } f^{(2)}_b := 9(b+5)\,(1-r_{b+1})^2(4\,r_{b+1}-1)^2.$$

The exact and approximate values for $K_{h,b}$, $1 \le b < h$, $3 \le h \le 12$, are presented in Table 3. Even for small h, the derived approximation yields very good values for the quantity $K_{h,b}$. $\diamond$

4 Enumeration and Distribution Results

In this section we shall compute asymptotic equivalents to the number of all b-balanced ordered trees with n nodes and of all such trees with height h. Furthermore, assuming that all b-balanced ordered trees with n nodes are equally likely, we shall determine the exact asymptotic behaviour of the average height of such a tree together with the variance.

Theorem 4.1. *Let* $r_w := [4\cos^2\left(\frac{\pi}{w+2}\right)]^{-1}$, $w \in \mathbb{N}$.

(a) The number $\ell_{h,b}(n)$ *of all* b*-balanced ordered trees of height* h *with* n *nodes is given by* $\ell_{1,0}(n) = \delta_{n,1}$, *by* $\ell_{2,0}(n) = 1-\delta_{n,1}$, *by* $\ell_{h,h-1}(n) = 0$, $h \ge 2$, *and for* $0 \le b \le h-2$, $h \ge 3$ *by*

$$\ell_{h,b}(n) = K_{h+1,b+1}\,z^{-n}_{h+1,b+1} + \mathcal{O}((b+1)^3 r^{-n}_{b+1}),$$

$h \setminus b$	1	2	3	4	5	6	7	8	9	10
3	$K_{3,1}$	$K_{3,2}$								
4	$\vert$ .170 820 *.148 437*	$K_{4,2}$	$K_{4,3}$							
5	$\vert$ .054 036 *.056 396*	$\vert$ .060 660 *.066 727*	$K_{5,3}$	$K_{5,4}$						
6	$\vert$ .021 283 *.022 308*	$\vert$ .018 638 *.019 420*	$\vert$ .029 841 *.031 651*	$K_{6,4}$	$K_{6,5}$					
7	$\vert$ .009 317 *.009 552*	$\vert$ .006 425 *.006 494*	$\vert$ .008 959 *.009 091*	$\vert$ .017 153 *.017 458*	$K_{7,5}$	$K_{7,6}$				
8	$\vert$ .004 318 *.004 363*	$\vert$ .002 339 *.002 345*	$\vert$ .002 843 *.002 851*	$\vert$ .005 077 *.005 097*	$\vert$ .010 845 *.010 787*	$K_{8,6}$	$K_{8,7}$			
9	$\vert$ .002 067 *.002 075*	$\vert$ .000 873 *.000 874*	$\vert$ .000 927 *.000 928*	$\vert$ .001 529 *.001 530*	.003 182 *.003 182*	$\vert$ .007 320 *.007 223*	$K_{9,7}$	$K_{9,8}$		
10	$\vert$ .001 008 *.001 009*	.000 330 *.000 330*	.000 306 *.000 306*	.000 466 *.000 466*	$\vert$ .000 925 *.000 926*	$\vert$ .002 136 *.002 134*	$\vert$ .005 184 *.005 126*	$K_{10,8}$	$K_{10,9}$	
11	.000 497 *.000 497*	.000 125 *.000 125*	.000 101 *.000 101*	.000 143 *.000 143*	.000 270 *.000 270*	$\vert$ .000 607 *.000 606*	$\vert$ .001 508 *.001 506*	$\vert$ .003 811 *.003 800*	$K_{11,9}$	$K_{11,10}$

Table 3: The quantity $K_{h,b}$ for $3 \leq b < h$, $3 \leq h \leq 11$. The values of $K_{h,h-2}$ and $K_{h,h-1}$, $h \geq 3$, are explicitly given by $\frac{1}{h}\tan^2(\frac{1}{h})$. The upper (resp. lower) number appearing in the remaining entries corresponds to the exact (resp. approximative) value of $K_{h,b}$. The values are not rounded to the sixth decimal place. The exact and the approximate value coincide up to the sixth decimal place regarding the entries not marked with $|$.

where $z_{h,b}$ (resp. $K_{h,b}$) is the dominant singularity of the enumerator $G_{h,b}(z)$ (resp. the quantity) introduced in Theorem 3.1 (resp. in Theorem 3.2) with the asymptotic expansion established in (18) (resp. (20)).

(b) The number $t_b(n)$ of all b-balanced ordered trees with n nodes is for fixed $b \in \mathbb{N}_0$ and all $\delta > 0$ given by

$$t_b(n) = -\tfrac{1-r_{b+2}}{\ln(r_{b+2})}\, n^{-1}\, r_{b+2}^{-n}\,[\,1 + \eta_{0,b}(n)\,] + \mathcal{O}(\tfrac{\ln(n)}{n^{2-\delta}}\, r_{b+2}^{-n}),$$

where $\eta_{s,b}(n)$, $s \geq 0$, is the oscillating function

$$\eta_{s,b}(n) = 2\sum_{k\geq 1} \Re\Big(\Gamma^{(s)}(1 - \tfrac{2\pi ik}{\ln(r_{b+2})})\, e^{2\pi ik\,\frac{\ln\left(\frac{n}{b+4}(1-r_{b+2})(4r_{b+2}-1)\right)}{\ln(r_{b+2})}}\Big)$$

with $\eta_{s,b}(n) = \eta_{s,b}(r_{b+2}\, n)$ and $|\eta_{0,b}(n)| < 8.646\,828\ldots \times 10^{-3}$. Here, $i^2 = -1$ and $\Gamma^{(s)}$ denotes the s-th derivative of the complete gamma function.

Proof: **(a)** Inserting the explicit expression for $G_{h,b}(z)$ presented in (6) into (8), we obtain $L_{1,0}(z) = z$, $L_{2,0}(z) = \frac{z^2}{1-z}$ and $L_{h,h-1}(z) = 0$, $h \geq 2$. This proves the result for $\ell_{h,b}(n) = \langle z^n; L_{h,b}(z)\rangle$, $(h,b) \in \{(1,0),(2,0),(h,h-1)\}$, $h \geq 2$.

With respect to the general case $\ell_{h,b}(n)$, $0 \leq b \leq h-2$, $h \geq 3$, formula (8) and Theorem 3.2 tell us

$$\ell_{h,0}(n) = K_{h+1,1}\, z_{h+1,1}^{-n} + \mathcal{O}(r_1^{-n}),\ \ h \geq 3,$$

and additionally for $b \geq 1$ by means of Theorem 3.1(b) and Remark 3.2(a)

$$\begin{aligned}
\ell_{h,b}(n) &= [K_{h+1,b+1}\, z_{h+1,b+1}^{-n} + \mathcal{O}((b+1)^3\, r_{b+1}^{-n})] \\
&- \ [K_{h+1,b}\, z_{h+1,b}^{-n} + \mathcal{O}(b^3\, r_b^{-n})] \\
&- \ [K_{h,b}\, z_{h,b}^{-n} + \mathcal{O}(b^3\, r_b^{-n})] \\
&+ \ [K_{h,b-1}\, z_{h,b-1}^{-n} + \mathcal{O}((b-1)^3\, r_{b-1}^{-n})] \\
&= \ K_{h+1,b+1}\, z_{h+1,b+1}^{-n} + \mathcal{O}((b+1)^3\, r_{b+1}^{-n}),\quad 1 \leq b \leq h-2,\ h \geq 3.
\end{aligned}$$

(b) We obtain by the first three alternatives for $\ell_{h,b}(n)$ given in part (a)

$$t_b(n) = \sum_{b<h\leq n} \ell_{h,b}(n) = \delta_{b,0} + \sum_{\max\{3,b+2\}\leq h\leq n} \ell_{h,b}(n), \tag{21}$$

and further by the fourth alternative and by $(K_{3,1}, z_{3,1}) = (1,1)$ (cf. Theorem 3.2 and Remark 3.1(a))

$$r_{b+2}^{n}\, t_b(n) = S_b(n) + \mathcal{O}(n\,(b+1)^3\, r_{b+1}^{-n}\, r_{b+2}^{n}),\ \ b \geq 0, \tag{22}$$

where

$$S_b(n) := \sum_{b+2 \leq h \leq n} K_{h+1,b+1}\,(r_{b+2}^{-1}\, z_{h+1,b+1})^{-n}.$$

In order to compute an asymptotic equivalent to the sum appearing on the right-hand side, we first approximate the quantity

$$(r_{b+2}^{-1}\, z_{h+1,b+1})^{-n}.$$

Using the asymptotic expansion established in (18) and the expansions of $\ln(1+x)$ and e^x, the standard "exp/log"-technique ([11, pp. 174-175 and p. 190]) yields, for all fixed $\epsilon > 0$,

$$\begin{aligned}&(r_{b+2}^{-1}\, z_{h+1,b+1})^{-n} \\ &\quad = \begin{cases} \zeta_{h,b}(n)\, e^{-c_{b+1}^{(1)}\, n\, r_{b+2}^{h-b-3}}, & \text{if } h \geq b+3+\mathsf{N}_{\epsilon,b}(n); \\ \mathcal{O}(e^{-n^{\epsilon}}), & \text{if } h < b+3+\mathsf{N}_{\epsilon,b}(n), \end{cases}\end{aligned} \tag{23}$$

where

$$\mathsf{N}_{\epsilon,b}(n) := \frac{\ln(c_{b+1}^{(1)}\, n^{1-\epsilon})}{\ln(r_{b+2}^{-1})} = (1-\epsilon)\log_4(n) - 3\log_4(b) + \mathcal{O}(1),\ \ n \to \infty,$$

and

$$\begin{aligned}\zeta_{h,b}(n) &= 1 + \kappa_{h,b}^{(1)}\, r_{b+2}^{2(h-b-3)}\, n + \kappa_{h,b}^{(2)}\, r_{b+2}^{3(h-b-3)}\, n + \kappa_{h,b}^{(3)}\, r_{b+2}^{4(h-b-3)}\, n^2 \\ &\quad + \mathcal{O}\big(\tfrac{[\ln(n)]^3}{n^{3-\epsilon}}\big),\end{aligned}$$

with

$$\begin{aligned}\kappa_{h,b}^{(1)} &:= \tfrac{1}{2}\,(\,[\,c_{b+1}^{(1)}\,]^2 - 2\,c_{h+1,b+1}^{(2)}) = -\tfrac{\zeta_b^{(0)}}{4\,(b+4)^2}\,[\,\zeta_b^{(0)}\,(h-b) + \zeta_b^{(1)}\,], \\ \kappa_{h,b}^{(2)} &:= -\tfrac{1}{3}\,(\,[\,c_{b+1}^{(1)}\,]^3 - 3\,c_{b+1}^{(1)}\,c_{h+1,b+1}^{(2)} + 3\,c_{h+1,b+1}^{(3)}) \\ &= -\tfrac{\zeta_b^{(0)}}{48\,(b+4)^3}\,[\,9\,[\,\zeta_b^{(0)}\,]^2\,(h-b)^2 + 18\,\zeta_b^{(0)}\,\zeta_b^{(1)}\,(h-b) + 4\,\zeta_b^{(2)}\,], \\ \kappa_{h,b}^{(3)} &:= \tfrac{1}{8}\,(\,[c_{b+1}^{(1)}\,]^2 - 2\,c_{h+1,b+1}^{(2)})^2 = \tfrac{1}{2}\,[\,\kappa_{h,b}^{(1)}\,]^2.\end{aligned}$$

Here, $c_b^{(1)}$, $c_{h,b}^{(2)}$ and $c_{h,b}^{(3)}$ are the quantities defined in Remark 3.1(b) and

$$\begin{aligned}
\zeta_b^{(0)} &:= 2(1-r_{b+2})(4\,r_{b+2}-1),\\
\zeta_b^{(1)} &:= 10\,r_{b+2}^2+(3\,b-10)\,r_{b+2}-b+2,\\
\zeta_b^{(2)} &:= 224\,r_{b+2}^4-2\,(b^2-73\,b+214)r_{b+2}^3+6\,(3\,b^2-35\,b+45)\,r_{b+2}^2\\
&\quad -(12\,b^2-87\,b+65)\,r_{b+2}+(2\,b-1)(b-5).
\end{aligned}$$

Now, taking the approximation (23) with $\zeta_{h,b}(n)=1+\mathcal{O}(\frac{\ln(n)}{n^{1-\epsilon}})$, we find by means of the asymptotic expansion for $K_{h,b}$ stated in (20) that

$$S_b(n)=S_{h,b}^{(1)}(n)+S_{h,b}^{(2)}(n)-S_{h,b}^{(3)}(n),$$

where

$$\begin{aligned}
S_{h,b}^{(1)}(n) &= \sum_{b+2\le h<b+3+\mathsf{N}_{b,\epsilon}(n)} K_{h+1,b+1}\,(r_{b+2}^{-1}\,z_{h+1,b+1})^{-n}\\
&= \mathcal{O}(e^{-n^\epsilon})\underbrace{\sum_{b+2\le h<b+3+\mathsf{N}_{b,\epsilon}(n)}[\,k_{b+1}^{(1)}\,r_{b+2}^{h-b-3}+\mathcal{O}((h-b)\,r_{b+2}^{2(h-b)})\,]}_{=\mathcal{O}(1)\text{ since } r_{b+2}\le r_2=\frac{1}{2}}\\
&= \mathcal{O}(e^{-n^\epsilon}),
\end{aligned}$$

and

$$\begin{aligned}
S_{h,b}^{(2)}(n) &= \sum_{h\ge b+3+\mathsf{N}_{b,\epsilon}(n)} K_{h+1,b+1}\,(r_{b+2}^{-1}\,z_{h+1,b+1})^{-n}\\
&= [\,1+\mathcal{O}(\tfrac{\ln(n)}{n^{1-\epsilon}})\,]\sum_{h\ge b+3+\mathsf{N}_{b,\epsilon}(n)}[\,k_{b+1}^{(1)}\,r_{b+2}^{h-b-3}+\mathcal{O}((h-b)\,r_{b+2}^{2(h-b)})\,]\\
&\qquad\times\, e^{-c_{b+1}^{(1)}\,n\,r_{b+2}^{h-b-3}}\\
&= \Big[\,k_{b+1}^{(1)}\sum_{h\ge b+3+\mathsf{N}_{b,\epsilon}(n)} r_{b+2}^{h-b-3}e^{-c_{b+1}^{(1)}\,n\,r_{b+2}^{h-b-3}}\Big]+\mathcal{O}(\tfrac{\ln(n)}{n^{2-2\,\epsilon}})\\
&= k_{b+1}^{(1)}\Big[\sum_{h\ge 1} r_{b+2}^{h-b-3}\,e^{-c_{b+1}^{(1)}\,n\,r_{b+2}^{h-b-3}}\\
&\qquad-\sum_{1\le h<b+3+\mathsf{N}_{b,\epsilon}(n)} r_{b+2}^{h-b-3}\,\underbrace{e^{-c_{b+1}^{(1)}\,n\,r_{b+2}^{h-b-3}}}_{=\mathcal{O}(e^{-n^\epsilon})}\Big]+\mathcal{O}(\tfrac{\ln(n)}{n^{2-2\,\epsilon}})
\end{aligned}$$

$$= \quad k_{b+1}^{(1)} \sum_{h\geq 1} r_{b+2}^{h-b-3}\, e^{-c_{b+1}^{(1)}\, n\, r_{b+2}^{h-b-3}} + \mathcal{O}(\tfrac{\ln(n)}{n^{2-2\,\epsilon}}),$$

and

$$\begin{aligned} S_{h,b}^{(3)}(n) &= \sum_{h>n} K_{h+1,b+1}\,(r_{b+2}^{-1}\, z_{h+1,b+1})^{-n} \\ &= [\,1+\mathcal{O}(\tfrac{\ln(n)}{n^{1-\epsilon}})\,] \sum_{h>n} [\, k_{b+1}^{(1)}\, r_{b+2}^{h-b-3} + \mathcal{O}((h-b)\, r_{b+2}^{2(h-b)})\,] \\ &\qquad \times \underbrace{e^{-c_{b+1}^{(1)}\, n\, r_{b+2}^{h-b-3}}}_{\leq 1} \\ &= \mathcal{O}(r_{b+2}^{-n}). \end{aligned}$$

Thus, introducing the series

$$U_{s,q,y}(n) := \sum_{h\geq 1} h^s\, y^h\, e^{-n\,q\,y^h}, \;\; s\geq 0,\;\; q>0,\;\; y<1,\;\; s,q,y \text{ fixed} \quad (24)$$

we have shown

$$S_b(n) = k_{b+1}^{(1)}\, r_{b+2}^{-b-3}\, U_{0,c_{b+1}^{(1)}\, r_{b+2}^{-b-3}, r_{b+2}}(n) + \Omega,$$

where $\Omega := \mathcal{O}(e^{-n^{\epsilon}}) + \mathcal{O}(\frac{\ln(n)}{n^{2-2\,\epsilon}}) + \mathcal{O}(r_{b+2}^{-n}) = \mathcal{O}(\frac{\ln(n)}{n^{2-2\,\epsilon}})$. Therefore by (22)

$$\begin{aligned} r_{b+2}^{n}\, t_b(n) &= k_{b+1}^{(1)}\, r_{b+2}^{-b-3}\, U_{0,c_{b+1}^{(1)}\, r_{b+2}^{-b-3}, r_{b+2}}(n) \\ &\quad + \underbrace{\mathcal{O}(\tfrac{\ln(n)}{n^{2-2\,\epsilon}}) + \mathcal{O}(n\,(b+1)^3\,[\,\underbrace{r_{b+1}^{-1}\, r_{b+2}}_{\leq\alpha<1}\,]^n)}_{\mathcal{O}(\frac{\ln(n)}{n^{2-2\,\epsilon}})}. \quad (25) \end{aligned}$$

It remains to derive an asymptotic equivalent to the series $U_{s,q,y}(n)$ established in (24). This can be done by the MELLIN-transform technique (cf. e.g. [3]) yielding a complex integral which can be computed by an application of the residue theorem. In this way, an asymptotic equivalent to a function very similar to $U_{s,q,y}(n)$ has already been computed in [5, Lemma 2]. Almost the same computation carried out there yields in the present case

$$U_{s,q,y}(n) \;=\; -\tfrac{[\,-\ln(n\,q)\,]^s}{q\,[\,\ln(y)\,]^{s+1}}\, n^{-1}\,[\,\Gamma(1) + \theta_{0,q,y}(n)\,]$$

$$\begin{aligned} &- \; s \, \tfrac{[-\ln(n\,q)\,]^{s-1}}{q\,[\ln(y)\,]^{s+1}} \, n^{-1} \, [\, \Gamma'(1) + \theta_{1,q,y}(n) \,] \\ &- \; \tbinom{s}{2} \, \tfrac{[-\ln(n\,q)\,]^{s-2}}{q\,[\ln(y)\,]^{s+1}} \, n^{-1} \, [\, \Gamma''(1) + \theta_{2,q,y}(n) \,] \\ &+ \; \mathcal{O}(\tfrac{[\ln(n)^{s-3}\,]}{n}), \end{aligned} \tag{26}$$

where $\theta_{m,q,y}$ is the oscillating function

$$\theta_{m,q,y}(n) := 2 \sum_{k\geq 1} \Re\Big(\Gamma^{(m)}(1 - \tfrac{2\pi ik}{\ln(y)})\, e^{2\pi ik \frac{\ln(n\,q)}{\ln(y)}}\Big),\ m \geq 0,$$

with $\theta_{m,q,y}(n) = \theta_{m,q,y}(y\,n)$. Note that $\theta_{s,c_{b+1}^{(1)}\,r_{b+2}^{-b-3},r_{b+2}}$ is identical with the function $\eta_{s,b}$ defined in the theorem. Now, inserting the asymptotic expansion given in (26) with

$$(m,q,y) := (0, c_{b+1}^{(1)}\, r_{b+2}^{-b-3}, r_{b+2})$$

into (25), we obtain the asymptotic result stated in part (b) of our theorem by means of the explicit expressions for $c_b^{(1)}$ and $k_b^{(1)}$ presented in (18) and Remark 3.2(b), respectively.

Finally, since the identity $|\Gamma(1+i\,t)|^2 = \frac{\pi t}{\sinh(\pi t)}$ is valid, we find the upper bound

$$\begin{aligned} |\eta_{0,b}(n)| \;&\leq\; 2\sum_{k\geq 1} |\Gamma(1 - \tfrac{2\pi ik}{\ln(r_{b+2})})| \\ &=\; \overline{\eta}_{0,b} := 2\sum_{k\geq 1} \Big[\, \frac{2k\pi^2}{\ln(r_{b+2})\sinh(\frac{2k\pi^2}{\ln(r_{b+2})})} \,\Big]^{\frac{1}{2}} \\ &\overset{(\star)}{\leq}\; \overline{\eta}_{0,\frac{1}{4}} = 8.646\,828\ldots \times 10^{-3}. \end{aligned}$$

The estimate $(\star)$ is valid because $r_{b+2} > \frac{1}{4}$ and $\frac{x}{\sinh(x)}$ is a strictly monotonically increasing function for $x < 0$. □

Remark 4.1. (a) The first few numerical values of the upper bound $\overline{\eta}_{0,b}$ for $|\eta_{0,b}(n)|$ introduced at the end of the proof of the preceding theorem are summarized in Table 4.

(b) By the previous theorem we have, for fixed b and large n,

$$t_b(n) \sim \alpha_b\, \xi_b^n\, n^{-1}\, [\,1 + \eta_{0,b}],\ |\eta_{0,b}(n)| < \overline{\eta}_{0,b},\ \eta_{0,b}(n) = \eta_{0,b}(r_{b+2}\,n),$$

with $\alpha_0 = \frac{1}{2\,\ln(2)} \geq \alpha_b = -\frac{1-r_{b+2}}{\ln(r_{b+2})} = \frac{3}{8\,\ln(2)} + \frac{\pi^2(3-2\,\ln(2))}{16\,[\ln(2)]^2}\, b^{-2} + \mathcal{O}(b^{-3})$, $b \to \infty$, and $\xi_0 = 2 \leq \xi_b = r_{b+2}^{-1} = 4 - 4\,\pi^2\, b^{-2} + \mathcal{O}(b^{-3})$, $b \to \infty$. In

b	0	1	2	3	4	5
$\overline{\eta}_{0,b}$	$9.884\,450 \times 10^{-6}$	$4.506\,932 \times 10^{-4}$	$1.504\,155 \times 10^{-3}$	$2.656\,699 \times 10^{-3}$	$3.665\,929 \times 10^{-3}$	$4.489\,543 \times 10^{-3}$

b	10	20	30	40	50	100	∞
$\overline{\eta}_{0,b}$	$6.718\,478 \times 10^{-3}$	$7.956\,383 \times 10^{-3}$	$8.298\,338 \times 10^{-3}$	$8.437\,674 \times 10^{-3}$	$8.507\,612 \times 10^{-3}$	$8.609\,158 \times 10^{-3}$	$8.646\,828 \times 10^{-3}$

Table 4: The numerical values of the upper bound $\overline{\eta}_{0,b}$. The values are not rounded to the sixth decimal place.

particular, we find for $b \in \{0,1,2,3,4\}$ and for large n:

$$\begin{aligned}
t_0(n) &\sim \tfrac{1}{\ln(2)\,n}\, 2^{n-1}\,[\,1+\eta_{0,0}(n)],\ |\eta_{0,0}(n)| < \overline{\eta}_{0,0},\ \eta_{0,0}(n) = \eta_{0,0}(\tfrac{1}{2}\,n);\\
t_1(n) &\sim \tfrac{1}{2\,\ln(\phi)\,n}\,\phi^{2n-1}\,[\,1+\eta_{0,1}(n)],\ |\eta_{0,1}(n)| < \overline{\eta}_{0,1},\ \eta_{0,1}(n)\\
&\quad = \eta_{0,1}(\phi^{-2}\,n);\quad \text{where } \phi \text{ is the `golden ratio'};\\
t_2(n) &\sim \tfrac{2}{\ln(3)\,n}\,3^{n-1}\,[\,1+\eta_{0,2}(n)], |\eta_{0,2}(n)| < \overline{\eta}_{0,2}, \eta_{0,2}(n) = \eta_{0,2}(\tfrac{1}{3}\,n);\\
t_3(n) &\sim \tfrac{c-1}{\ln(c)\,n}\,c^{n-1}\,[\,1+\eta_{0,3}(n)],\ |\eta_{0,3}(n)| < \overline{\eta}_{0,3},\ \eta_{0,3}(n)\\
&\quad = \eta_{0,3}(c^{-1}\,n)\ \text{ with } c := 4\,\cos^2(\tfrac{\pi}{7});\\
t_4(n) &\sim \tfrac{c-1}{\ln(c)\,n}\,c^{n-1}\,[\,1+\eta_{0,4}(n)],\ |\eta_{0,4}(n)| < \overline{\eta}_{0,4},\ \eta_{0,4}(n)\\
&\quad = \eta_{0,4}(c^{-1}\,n);\ \text{with } c := 2+\sqrt{2}.
\end{aligned}$$

Note that the result with respect to $t_0(n)$ has already been proved in [5]. The first few exact and asymptotic values for $t_b(n)$ are summarized in Table 5.

(c) The oscillating functions $\eta_{s,b}(n)$ introduced in the previous theorem are bounded for fixed $s \geq 1$, too. We obtain $|\eta_{s,b}(n)| < \overline{\eta}_{s,b} := 2\sum_{k\geq 1} |\Gamma^{(s)}(1-\frac{2\pi i k}{\ln(r_{b+2})})|$. The first few numerical values of the upper bound $\overline{\eta}_{s,b}$, $s \in \{1,2\}$, are presented in Table 6. ◇

Assuming that all b-balanced ordered trees with n nodes are equally likely, the quotient $\omega_{h,b}(n) := \frac{\ell_{h,b}(n)}{t_b(n)}$ is the probability that such a tree has the height h. The s-th moment about the origin $\mathbb{E}[H_b^s(n)]$ of the random variable $H_b(n)$ taking the value h with the probability

b	0		1		2		3		10	
n	*ex.*	*as.*	*ex.*	*as.*	*ex.*	*as.*	*ex.*	*as.*	*ex.*	*as.*
1	1	1.442 7								
2	1	1.442 7								
3	2	1.923 5								
4	3	2.885 4	2	7.538 9						
5	5	4.616 5	7	1.579 8 $\times 10^{1}$	2	2.946 9 $\times 10^{1}$				
6	8	7.694 3	21	3.447 7 $\times 10^{1}$	11	7.378 3 $\times 10^{1}$	2	1.145 4 $\times 10^{2}$		
7	14	1.319 0 $\times 10^{1}$	55	7.735 4 $\times 10^{1}$	46	1.898 7 $\times 10^{2}$	15	3.194 0 $\times 10^{2}$		
8	24	2.308 3 $\times 10^{1}$	141	1.771 3 $\times 10^{2}$	165	4.982 9 $\times 10^{2}$	78	9.090 4 $\times 10^{2}$		
9	43	4.103 6 $\times 10^{1}$	351	4.121 1 $\times 10^{2}$	552	1.327 7 $\times 10^{3}$	341	2.625 8 $\times 10^{3}$		
10	77	7.386 5 $\times 10^{1}$	868	9.709 6 $\times 10^{2}$	1 763	3.581 7 $\times 10^{3}$	1 359	7.672 8 $\times 10^{3}$		
20	3.867 4 $\times 10^{4}$	3.781 9 $\times 10^{4}$	7.402 7 $\times 10^{6}$	7.347 3 $\times 10^{6}$	9.327 6 $\times 10^{7}$	1.059 3 $\times 10^{8}$	3.001 3 $\times 10^{8}$	4.976 4 $\times 10^{8}$	8.585 1 $\times 10^{6}$	1.096 7 $\times 10^{10}$
40	2.005 8 $\times 10^{10}$	1.982 8 $\times 10^{10}$	8.503 2 $\times 10^{14}$	8.409 8 $\times 10^{14}$	1.844 9 $\times 10^{17}$	1.841 8 $\times 10^{17}$	3.891 1 $\times 10^{18}$	4.229 9 $\times 10^{18}$	6.823 7 $\times 10^{19}$	2.193 0 $\times 10^{21}$
100	9.185 7 $\times 10^{27}$	9.144 1 $\times 10^{27}$	4.050 9 $\times 10^{39}$	4.030 3 $\times 10^{39}$	3.131 7 $\times 10^{45}$	3.123 7 $\times 10^{45}$	8.314 6 $\times 10^{48}$	8.282 8 $\times 10^{48}$	1.464 7 $\times 10^{55}$	5.487 6 $\times 10^{55}$
200	5.808 9 $\times 10^{57}$	5.795 8 $\times 10^{57}$	1.266 0 $\times 10^{81}$	1.263 4 $\times 10^{81}$	8.092 4 $\times 10^{92}$	8.071 0 $\times 10^{92}$	5.805 5 $\times 10^{99}$	5.794 4 $\times 10^{99}$	1.753 3 $\times 10^{113}$	2.783 9 $\times 10^{113}$
400	4.661 9 $\times 10^{117}$	4.656 7 $\times 10^{117}$	2.488 4 $\times 10^{164}$	2.486 4 $\times 10^{164}$	1.070 4 $\times 10^{188}$	1.069 3 $\times 10^{188}$	5.749 9 $\times 10^{201}$	5.744 9 $\times 10^{201}$	1.280 9 $\times 10^{229}$	1.376 8 $\times 10^{229}$
500	4.726 7 $\times 10^{147}$	4.722 5 $\times 10^{147}$	1.248 7 $\times 10^{206}$	1.247 8 $\times 10^{206}$	4.422 3 $\times 10^{235}$	4.417 0 $\times 10^{235}$	6.445 0 $\times 10^{252}$	6.442 6 $\times 10^{252}$	1.073 5 $\times 10^{287}$	1.107 8 $\times 10^{287}$
1000	7.732 8 $\times 10^{297}$	7.729 3 $\times 10^{297}$	6.067 4 $\times 10^{414}$	6.065 3 $\times 10^{414}$	8.012 2 $\times 10^{473}$	8.010 6 $\times 10^{473}$	1.778 4 $\times 10^{508}$	1.777 2 $\times 10^{508}$	5.585 1 $\times 10^{576}$	5.588 6 $\times 10^{576}$
2000	4.141 9 $\times 10^{598}$	4.141 0 $\times 10^{598}$	2.867 3 $\times 10^{832}$	2.866 6 $\times 10^{832}$	5.310 4 $\times 10^{950}$	5.309 1 $\times 10^{950}$	2.667 5 $\times 10^{1019}$	2.667 3 $\times 10^{1019}$	2.822 8 $\times 10^{1156}$	2.822 3 $\times 10^{1156}$

Table 5: The exact [ex.] and asymptotical [as.] values of the number $t_b(n)$ of b-balanced ordered trees with n nodes. The values are not rounded to the fourth decimal place.

b	0	1	2	3	4	5
$\overline{\eta}_{1,b}$	2.645 078 $\times 10^{-5}$	1.081 682 $\times 10^{-3}$	3.446 692 $\times 10^{-3}$	5.938 005 $\times 10^{-3}$	8.070 653 $\times 10^{-3}$	9.785 936 $\times 10^{-3}$
$\overline{\eta}_{2,b}$	6.978 935 $\times 10^{-5}$	2.530 362 $\times 10^{-3}$	7.643 819 $\times 10^{-3}$	1.278 774 $\times 10^{-2}$	1.706 847 $\times 10^{-2}$	2.044 857 $\times 10^{-2}$

b	10	20	30	40	50	100	∞
$\overline{\eta}_{1,b}$	1.434 160 $\times 10^{-2}$	1.682 802 $\times 10^{-2}$	1.751 019 $\times 10^{-2}$	1.778 761 $\times 10^{-2}$	1.792 674 $\times 10^{-2}$	1.812 861 $\times 10^{-2}$	1.820 346 $\times 10^{-2}$
$\overline{\eta}_{2,b}$	2.921 189 $\times 10^{-2}$	3.388 670 $\times 10^{-2}$	3.515 772 $\times 10^{-2}$	3.567 326 $\times 10^{-2}$	3.593 152 $\times 10^{-2}$	3.630 591 $\times 10^{-2}$	3.644 462 $\times 10^{-2}$

Table 6: The numerical values of the upper bound $\overline{\eta}_{s,b}$, $s \in \{1,2\}$. The values are not rounded to the sixth decimal place.

$\omega_{h,b}(n)$ is given by

$$\mathbb{E}[H_b^s(n)] := \sum_{b<h\leq n} h^s\,\omega_{h,b}(n) = \frac{1}{t_b(n)} \sum_{b<h\leq n} h^s\,\ell_{h,b}(n),\ \ s \geq 1. \quad (27)$$

The following lemma gives us information on the s-th moment $\mathbb{E}[H_b^s(n)]$.

Lemma 4.1. *Let* $r_w := [4\cos^2\left(\frac{\pi}{w+2}\right)]^{-1}$, $w \in \mathbb{N}$, *and* $\beta_b(n) := \ln(\frac{n}{b+4}\,(1-r_{b+2})(4\,r_{b+2}-1)\,r_{b+2}^{-b-3}\,)$. *We have for fixed* b *and large* n

$$\begin{aligned}\mathbb{E}[H_b^s(n)] &= \tfrac{(-1)^s}{[\ln(r_{b+2})]^s}\,[\,\beta_b(n)\,]^s \\ &- s\,\tfrac{(-1)^s}{[\ln(r_{b+2})]^s}\,[\,\beta_b(n)\,]^{s-1}\,\tfrac{\Gamma'(1)+\eta_{1,b}(n)}{1+\eta_{0,b}(n)} \\ &+ \tbinom{s}{2}\,\tfrac{(-1)^s}{[\ln(r_{b+2})]^s}\,[\,\beta_b(n)\,]^{s-2}\,\tfrac{\Gamma''(1)+\eta_{2,b}(n)}{1+\eta_{0,b}(n)} + \mathcal{O}([\,\ln(n)\,]^{s-3}).\end{aligned}$$

Here, $\eta_{s,b}(n)$ *is the oscillating function introduced in Theorem 4.1(b).*

Proof: Using the definition of $U_{s,q,y}(n)$ given in (24), the result presented in Theorem 4.1(a) and the approximation stated in (23) together with the asymptotic expansion for $K_{h,b}$ presented in (20), we find by a similar lengthy computation as in the proof of Theorem 4.1(b)

$$\begin{aligned}r_{b+2}^n \sum_{b<h\leq n} h^s\,\ell_{h,b}(n) \;=\; r_{b+2}^n\Big[&\delta_{b,0}\,(\delta_{n,1}+2^s[\,1-\delta_{n,1}]\,) \\ &+ \sum_{\max\{3,b+2\}\leq h\leq n} h^s\,\ell_{h,b}(n)\Big]\end{aligned}$$

$$= \sum_{b+2\le h\le n} h^s\, K_{h+1,b+1}\,(r_{b+2}^{-1}\, z_{h+1,b+1})^{-n}$$

$$+\,\mathcal{O}(n^{s+1}\,(b+1)^3\, r_{b+1}^{-n}\, r_{b+2}^{n})$$

$$= k_{b+1}^{(1)}\, r_{b+2}^{-b-3}\, U_{s,c_{b+1}^{(1)} r_{b+2}^{-b-3}, r_{b+2}}(n) + \mathcal{O}(\tfrac{[\ln(n)]^{s+1}}{n^{2-2\epsilon}}).$$

Hence, by (27)

$$\mathbb{E}[H_b^s(n)] = \tfrac{1}{t_b(n)}\, k_{b+1}^{(1)}\, r_{b+2}^{-n-b-3}\, U_{s,c_{b+1}^{(1)} r_{b+2}^{-b-3}, r_{b+2}}(n) + \mathcal{O}(\tfrac{[\ln(n)]^{s+1}}{r_{b+2}^{n}\, t_b(n)\, n^{2-2\epsilon}}).$$

Using Theorem 4.1(b) and the asymptotic expression for $U_{s,q,y}(n)$ given in (26), a straightforward computation yields the stated result. □

Following the classical paper [1], the average height of an ordered tree with n nodes is asymptotically given by $\sim \sqrt{\pi\, n} - \frac{1}{2}$, $n \to \infty$, provided that all n-node ordered trees are equally likely; the variance is asymptotically given by $\sim \frac{1}{3}\,\pi\,(\pi-3)\,n + \frac{1}{12}\,(\pi+1)$, $n\to\infty$ (see [4]).

The following theorem presents analogous asymptotic equivalents to the average height $\underline{h}_b(n) := \mathbb{E}[H_b(n)]$ of a b-balanced ordered tree with n nodes and to the variance $\sigma_b(n)^2 := \mathbb{E}[H_b^2(n)] - (\mathbb{E}[H_b(n)])^2$.

Theorem 4.2. *Assume that all b-balanced ordered trees with n nodes are equally likely. The average height $\underline{h}_b(n)$ of such a tree is asymptotically given by*

$$\underline{h}_b(n) \sim -\tfrac{1}{\ln(r_{b+2})}\,\ln(n) + \beta_b + \chi_b(n),\ \ b \textit{ fixed},\ \ n\to\infty.$$

Here, β_b denotes the quantity $\beta_b := -\frac{1}{\ln(r_{b+2})}\,\ln(e^{\gamma}\,\frac{(1-r_{b+2})(4r_{b+2}-1)}{(b+4)\,r_{b+2}^{b+3}})$, where γ is Euler*'s constant, and $\chi_{h,b}(n)$ is a bounded oscillating function with $\chi_b(n) = \chi_b(r_{b+2}\,n)$ and $|\chi_b(n)| < \mathsf{C}_1 := 3.375\,453\ldots \times 10^{-2}$.*

The variance is asymptotically given by $\sigma_b^2(n) \sim \frac{\pi^2}{6\,[\ln(r_{b+2})]^2} + \varphi_b(n)$, where $\varphi_b(n)$ is a bounded oscillating function with $\varphi_b(n) = \varphi_b(r_{b+2}\,n)$ and $|\varphi_b(n)| < \mathsf{C}_2 := 1.595\,432\ldots \times 10^{-1}$.

Proof: Since $\Gamma'(1) = -\gamma$, the preceding lemma, with $s=1$, tells us that

$$\underline{h}_b(n) \sim -\tfrac{1}{\ln(r_{b+2})}\,\beta_b(n) + \tfrac{1}{\ln(r_{b+2})}\,\tfrac{\eta_{1,b}(n)-\gamma}{\eta_{0,b}(n)+1}.$$

Rearranging the terms appearing on the right-hand side, we find the asymptotic equivalent to $\underline{h}_b(n)$ stated in the theorem, where

$$\chi_b(n) := \frac{1}{\ln(r_{b+2})} \frac{\eta_{1,b}(n) + \gamma\, \eta_{0,b}(n)}{1 + \eta_{0,b}(n)}.$$

Since $\eta_{s,b}(n) = \eta_{s,b}(r_{b+2}\, n)$, we also have $\chi_b(n) = \chi_b(r_{b+2}\, n)$. Moreover, we find by Remark 4.1(a), (c)

$$\begin{aligned} |\chi_b(n)| \le \overline{\chi}_b \quad &:= \tfrac{1}{|\ln(r_{b+2})|\,(1-|\eta_{0,b}(n)|)}\, (|\eta_{1,b}(n)| + \gamma\, |\eta_{0,b}(n)|) \\ &< \tfrac{1}{|\ln(r_{b+2})|\,(1-\overline{\eta}_{0,b})}\, (\overline{\eta}_{1,b} + \gamma\, \overline{\eta}_{0,b}) \\ &< \tfrac{1}{\ln(2)\,(1-\overline{\eta}_{0,\infty})}\, (\overline{\eta}_{1,\infty} + \gamma\, \overline{\eta}_{0,\infty}) = \mathsf{C}_1. \end{aligned}$$

With respect to the variance, the preceding Lemma 4.1 with $s \in \{1, 2\}$ yields

$$\begin{aligned} \sigma_b^2(n) \quad &\sim \quad \frac{1}{[\ln(r_{b+2})\,]^2} \Big[\frac{\Gamma''(1) + \eta_{0,b}(n)}{1 + \eta_{0,b}(n)} - \Big(\frac{\Gamma'(1) + \eta_{1,b}(n)}{1 + \eta_{0,b}(n)}\Big)^2\Big] \\ &= \quad \frac{\pi^2}{6\,[\ln(r_{b+2})\,]^2} + \varphi_b(n), \end{aligned}$$

where $\varphi_b(n) := \frac{[\,\eta_{2,b}(n) - \frac{\pi^2}{6}\,\eta_{0,b}(n)\,]\,[\,1+\eta_{0,b}(n)\,] + \gamma^2\,\eta_{0,b}(n) + \eta_{1,b}(n)\,[\,2\,\gamma - \eta_{1,b}(n)\,]}{[\ln(r_{b+2})\,]^2\,[\,1+\eta_{0,b}(n)]^2}$. Here, we have used the equality $\Gamma''(1) = \gamma^2 + \frac{\pi^2}{6}$. Again, the relation $\varphi_b(n) = \varphi_b(r_{b+2}\, n)$ is fulfilled and we obtain by Remark 4.1(a), (c)

$$\begin{aligned} |\varphi_b(n)| \le \overline{\varphi}_b \quad &:= \tfrac{1}{[\ln(r_{b+2})\,]^2\,[\,1-|\eta_{0,b}(n)|]^2} \\ &\qquad \times [(|\eta_{2,b}(n)| + \tfrac{\pi^2}{6}\,|\eta_{0,b}(n)|)(1 + |\eta_{0,b}(n)|) \\ &\qquad\quad + \gamma^2\,|\eta_{0,b}(n)| + |\eta_{1,b}(n)|\,(2\,\gamma + |\eta_{1,b}(n)|)] \\ &< \tfrac{1}{[\ln(2)\,]^2\,[\,1-\overline{\eta}_{0,\infty}]^2}\, [(\overline{\eta}_{2,\infty} + \tfrac{\pi^2}{6}\,\overline{\eta}_{0,\infty})(1 + \overline{\eta}_{0,\infty}) \\ &\qquad\quad + \gamma^2\,\overline{\eta}_{0,\infty} + \overline{\eta}_{1,\infty}\,(2\gamma + \overline{\eta}_{1,\infty})] = \mathsf{C}_2. \end{aligned}$$

This completes the proof. □

Remark 4.2. **(a)** The first few numerical values of the quantity β_b appearing in the preceding theorem are summarized in Table 7. It is easily verified that $\beta_b = b - \frac{3}{\ln(4)}\,\ln(b) + \frac{\gamma + \ln(48\,\pi^2)}{\ln(4)} - \frac{6}{\ln(2)}\,b^{-1} + \mathcal{O}(\frac{\ln(b)}{b^2})$, $b \to \infty$.

b	0	1	2	3	4	5
β_b	.832 746	1.763 614	2.525 404	3.284 408	4.058 874	4.850 579

b	10	20	30	40	50	100
β_b	9.015 213	17.924 081	27.195 403	36.649 051	56.212 250	94.804 514

Table 7: The numerical values for β_b. The values are not rounded to the sixth decimal place.

b	0	1	2	3	4	5
$\overline{\chi}_b$	4.639 211 $\times 10^{-5}$	1.394 848 $\times 10^{-3}$	3.933 521 $\times 10^{-3}$	6.360 903 $\times 10^{-3}$	8.326 226 $\times 10^{-3}$	9.852 836 $\times 10^{-3}$
$\overline{\varphi}_b$	2.495 175 $\times 10^{-4}$	5.049 852 $\times 10^{-3}$	1.215 409 $\times 10^{-2}$	1.810 486 $\times 10^{-2}$	2.257 247 $\times 10^{-2}$	2.588 387 $\times 10^{-2}$

b	10	20	30	40	50	100	∞
$\overline{\chi}_b$	1.373 470 $\times 10^{-2}$	1.577 109 $\times 10^{-2}$	1.632 140 $\times 10^{-2}$	1.654 424 $\times 10^{-2}$	1.665 579 $\times 10^{-2}$	1.681 741 $\times 10^{-2}$	1.687 726 $\times 10^{-2}$
$\overline{\varphi}_b$	3.383 205 $\times 10^{-2}$	3.778 771 $\times 10^{-2}$	3.883 567 $\times 10^{-2}$	3.925 765 $\times 10^{-2}$	3.946 839 $\times 10^{-2}$	3.977 313 $\times 10^{-2}$	3.988 581 $\times 10^{-2}$

Table 8: The numerical values of the amplitudes $\overline{\chi}_b$ and $\overline{\varphi}_b$. The values are not rounded to the sixth decimal place.

(b) The first few numerical values for the upper bound $\overline{\chi}_b$ (resp. $\overline{\varphi}_b$) of the amplitude $|\chi_b(n)|$ (resp. $|\varphi_b(n)|$) of the oscillating functions appearing in the proof of the previous theorem are given in Table 8.
(c) The first few exact and asymptotic values of the average height $\underline{h}_b(n)$ and of the variance $\sigma_b^2(n)$ are summarized in Table 9.

5 Concluding Remarks

In this paper we have presented a detailed average case analysis of b-balanced ordered n-node trees for fixed $b \geq 0$ and large n. However, the computations and results presented raise some questions:

- What is the average behaviour of b-balanced trees when b is not assumed to be fixed?
- For the expected height of b-balanced ordered trees with $n \geq$

b	0		1		2		3		10	
n	*ex.*	*as.*	*ex.*	*as.*	*ex.*	*as.*	*ex.*	*as.*	*ex.*	*as.*
1	1.000 0 *0.000 0*	0.832 7 *3.423 8*								
2	2.000 0 *0.000 0*	1.832 7 *3.432 8*								
3	2.500 0 *0.250 0*	2.417 7 *3.423 5*								
4	3.000 0 *0.666 6*	2.832 7 *3.423 8*	3.000 0 *0.000 0*	3.205 3 *1.773 4*						
5	3.400 0 *1.040 0*	3.154 7 *3.423 6*	3.285 7 *0.204 0*	3.436 1 *1.772 9*	4.000 0 *0.000 0*	3.993 7 *1.354 6*				
6	3.750 0 *1.437 5*	3.417 7 *3.423 5*	3.523 8 *0.439 9*	3.624 2 *1.776 7*	4.181 8 *0.148 7*	4.156 5 *1.356 5*	5.000 0 *0.000 0*	4.811 7 *1.175 5*		
7	4.000 0 *1.714 2*	3.640 0 *3.423 7*	3.727 2 *0.671 0*	3.784 2 *1.779 3*	4.326 0 *0.306 7*	4.293 8 *1.363 3*	5.133 3 *0.115 5*	4.939 9 *1.173 5*		
8	4.208 3 *1.998 2*	3.832 7 *3.423 8*	3.907 8 *0.863 8*	3.923 8 *1.779 0*	4.442 4 *0.452 7*	4.414 4 *1.369 1*	5.243 5 *0.235 5*	5.049 1 *1.178 2*		
9	4.372 0 *2.187 1*	4.002 6 *3.423 8*	4.062 6 *1.021 7*	4.047 2 *1.776 7*	4.545 2 *0.581 2*	4.522 4 *1.371 4*	5.331 3 *0.344 7*	5.145 8 *1.185 5*		
10	4.506 4 *2.353 8*	4.154 7 *3.423 6*	4.201 6 *1.144 8*	4.157 3 *1.774 3*	4.636 4 *0.690 8*	4.620 2 *1.370 3*	5.405 4 *0.441 2*	5.223 6 *1.192 3*		
20	5.388 6 *2.857 5*	5.154 7 *3.423 6*	5.000 5 *1.504 8*	4.875 5 *1.779 4*	5.286 3 *1.174 6*	5.250 2 *1.360 9*	5.867 0 *0.931 6*	5.833 8 *1.174 6*	12.249 7 *0.280 0*	11.254 1 *0.943 3*
40	6.300 5 *3.056 1*	6.154 7 *3.423 6*	5.669 2 *1.642 9*	5.595 5 *1.776 1*	5.936 3 *1.257 8*	5.886 8 *1.358 0*	6.438 8 *1.132 3*	6.412 7 *1.198 5*	12.506 7 *0.604 5*	11.780 0 *0.902 4*
100	7.551 9 *3.228 1*	7.476 6 *3.423 5*	6.585 2 *1.702 9*	6.547 9 *1.775 0*	6.748 9 *1.316 1*	6.718 4 *1.366 6*	7.202 6 *1.169 7*	7.189 3 *1.189 0*	12.744 6 *0.814 7*	12.474 1 *0.919 5*
200	8.521 1 *3.305 9*	8.476 6 *3.423 5*	7.294 1 *1.723 6*	7.270 0 *1.772 5*	7.358 9 *1.342 1*	7.344 7 *1.366 0*	7.800 5 *1.157 1*	7.789 2 *1.176 7*	13.029 3 *0.945 7*	12.971 0 *0.927 4*
400	9.502 3 *3.354 5*	9.476 6 *3.423 5*	8.003 2 *1.750 8*	7.989 1 *1.777 9*	7.992 6 *1.336 1*	7.982 5 *1.354 8*	8.372 8 *1.189 5*	8.366 6 *1.198 2*	13.479 0 *0.978 6*	13.513 6 *0.914 1*
500	9.819 9 *3.366 2*	9.798 4 *3.423 8*	8.234 0 *1.749 2*	8.222 2 *1.773 1*	8.189 0 *1.343 8*	8.181 7 *1.357 5*	8.568 6 *1.183 6*	8.562 4 *1.192 9*	13.643 5 *0.966 0*	13.678 7 *0.899 5*
1000	10.810 6 *3.390 9*	10.798 4 *3.423 8*	8.947 3 *1.766 2*	8.940 8 *1.778 7*	8.821 2 *1.352 7*	8.816 1 *1.361 5*	9.148 1 *1.181 0*	9.145 5 *1.185 5*	14.163 4 *0.965 1*	14.178 5 *0.944 9*
2000	11.805 2 *3.405 3*	11.798 4 *3.423 8*	9.663 4 *1.770 7*	9.660 0 *1.777 5*	9.443 6 *1.346 8*	9.440 3 *1.369 9*	9.746 3 *1.175 8*	9.744 1 *1.179 5*	14.704 4 *0.917 1*	14.714 5 *0.899 3*
		≀ *3.423 71* $+\varphi_0(n)$		≀ *1.775 88* $+\varphi_1(n)$		≀ *1.362 88* $+\varphi_2(n)$		≀ *1.185 93* $+\varphi_3(n)$		≀ *0.922 26* $+\varphi_{10}(n)$

Table 9: The exact [ex.] and asymptotical [as.] values of the average height $\underline{h}_b(n)$ and of the variance $\sigma_b^2(n)$. In each entry the upper number and lower number (in italics) indicate $\underline{h}_b(n)$ and $\sigma_b^2(n)$, respectively. The upper bounds $\overline{\varphi}_b$ of the amplitudes $|\varphi_b(n)|$ are given in Table 8. The values are not rounded to the fourth decimal place. ◇

3 nodes the parameter $b \in \mathbb{N}_0$ controls the transition from all totally balanced n-node trees ($b = 0$) to all ordered trees with n nodes ($b \leq n - 3$). In the course of this transition, the expected height changes from $\Theta(\mathrm{ld}(n))$ (see [5]) to $\Theta(\sqrt{n})$ (see [1]). How could we characterize this transition with respect to b as a function in n?

- Another interesting problem is the computation of the average value $b(n)$ such that an ordered tree with n nodes is $b(n)$-balanced assuming a uniform distribution of all ordered n-node trees. [It is easy to show that for uniformly distributed ordered trees with n nodes and fixed height h a random tree is asymptotically b_h-balanced with $b_h \sim h - 2$, $n \to \infty$; the variance is asymptotically $o(1)$, $n \to \infty$.]

- The distribution of the heights in ordered trees obeys a limiting theta distribution (see [2], [4]). What is the limit law for the height of b-balanced ordered trees?

- It remains an open problem whether the polynomials $P_{h,k}(z)$ introduced in Lemma 3.2 only have simple roots. There is numerical evidence that this is indeed the case but so far the author was not able to prove this conjecture.

- The considerations presented in this paper can be extended to other classes of ordered trees, at least in a formal sense. Following [10], the generating function $Y(z) := \sum_{n\geq 1} t(n)\, z^n$ of the number $t(n)$ of all trees with n nodes appearing in a *simply generated family of trees* satisfies the functional equation $Y(z) := z\,\phi(Y(z))$, where $\phi(y) := \sum_{\lambda\geq 0} c_\lambda\, y^\lambda$ with $c_0 = 1$, $c_\lambda \in \mathbb{N}_0$ for $\lambda \in \mathbb{N}$, and $c_\lambda \in \mathbb{N}$ for some $\lambda \in \mathbb{N}\backslash\{1\}$. This definition includes all families of unlabelled trees defined by restrictions on the set of the allowed node degrees such as *t-ary trees* ($\phi(y) := 1 + y^t$, $t \in \mathbb{N}\backslash\{1\}$), *extended binary trees* ($\phi(y) := 1 + y^2$), *binary trees* ($\phi(y) := (1 + y)^2$), *unary-binary trees* ($\phi(y) := 1 + y + y^2$), *unbalanced 2-3-trees* ($\phi(y) := 1 + y^2 + y^3$) or *ordered trees* ($\phi(y) := (1 - y)^{-1}$) considered in this paper.

It is not hard to see that the generating function $A_h(z,y)$ of the number of all 0-balanced simply generated trees with n nodes, m leaves and height h (cf. formula (1)) is recursively given by $A_{h+1}(z,y) = z\,(\,\phi(A_h(z,y)) - 1\,)$ with the initial condition $A_1(z,y) = z\,y$. Since the generating function $F_k(z)$ of the number of all simply generated trees with n nodes and height less than or equal to k (cf. formula (3)) satisfies the recurrence (see [2]) $F_{k+1}(z) = z\,\phi(F_k(z))$ with the initial condition $F_0(z) = 0$, the generating functions $T_{h,b}(z)$, $G_{h,b}(z)$ and therefore $L_{h,b}(z)$ introduced in the formulae (2), (6) and (8) are defined in a recursive way for b-balanced simply generated trees.

For ordered trees, the recurrences for $A_h(z,y)$ and $F_k(z)$ have the explicit solutions presented in (1) and (3), respectively; for t-ary trees, the recurrence for $A_h(z,y)$ has the solution $A_h(z,y) = z^{\frac{t^h-1}{t-1}}\, y^{t^{h-1}}$, but there does not exist an explicit expression for $F_k(z)$. Generally, explicit solutions for $A_h(z,y)$ and $F_k(z)$ are not available and we only have recursive formulae. Thus, in order to obtain enumeration and distribution results in the case of b-balanced simply generated trees, more powerful and refined methods must be developed.

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