JIRSS (2025)
Vol. 24, No. 2, pp 107-124
DOI: 10.22034/jirss.2026.2075601.1148

Support Vector Fuzzy Regression with Fuzzy Input-Fuzzy Output and Fuzzy Error

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Received: 23 October 2025, Accepted: 30 January 2026, Published online: 09 April 2026

Abstract. In this paper, we investigate a new approach of fuzzy regression analysis based on support vectors when the available data and error variable are fuzzy quantities. In this approach, based on the concept of the distance between two parallel hyperplanes, we obtain the marginal hyperplanes and then, based on some constraints on the fuzzy data, we present an optimization problem to estimate the parameters of fuzzy regression model. The proposed method is investigated in two cases: with fuzzy fixed error and with fuzzy variable errors. To evaluate the proposed support vector fuzzy regression (SVFR) models, we present two indices of goodness of fit. Based on these indices, the presented SVFR models are compared with some other approaches on the numerical and simulated examples.

Keywords. Fuzzy regression, Fuzzy error, Goodness of fit, Support vector.

MSC: 62J05, 62J86, 03E72.

1 Introduction

A new important approach in regression analysis is the combination of regression models with support vector machines. This concept is known among researchers as support vector regression (SVR) and has been used a lot in recent years. The first study in this field was presented by Vapnik et al. (1997). Later, some extended version of SVR were investigated by Balasundaram and Meena (2019) Basak et al. (2007), Chen et al. (2017), Gu et al. (2015), Yang et al. (2014), and Zhao and Sun (2008, 2010).

In many practical problems, the available information are uncertain, imprecise, and incomplete, and thus the classical regression models are unable to analyze these information.

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A convenient way to solve this problem is to fit fuzzy regression models. This approach is investigated by many authors in recent years. For example, see Arefi and Khammar (2024), Chachi and Roozbeh (2017), Chachi and Taheri (2016), Chachi et al. (2016), D'Urso and Chachi (2022), Khammar et al. (2020, 2021a,b), Taheri et al. (2022), and Zeng et al. (2016).

Some extended approaches of fuzzy regression models based on support vector machines were presented as follows. Asadolahi et al. (2021) presented a robust support vector regression with exact predictors and fuzzy responses. Hao and Chiang (2008) and Hong and Hwang (2003) investigated some methods to fit the fuzzy regression models by support vector learning approaches. Moghadam et al. (2024) studied a new approach for fitting a fuzzy linear regression model based on support vectors, when the response variable, parameters of model, and errors are as fuzzy quantities. An approach to estimate the parameters of a logistic regression model with crisp input-fuzzy output and based on support vectors is investigated by Hesamian and Akbari (2023). For more studies about support vector fuzzy regression, see Luo et al. (2024) and Wieszczy and Grzegorzewski (2016)).

In this paper, we present an approach to fit some fuzzy regression models based on support vectors. In this approach, we assume that the response variable, the explanatory variables, and errors are fuzzy quantities. This approach is a hybrid version of fuzzy regression and support vector machine, in which the marginal hyperplanes are estimated to be the bounds of the response variables and the support vectors are the observed response variables, with at least one bound lying on the hyperplanes. The applications of proposed method are studied on the numerical and simulated examples.

The paper is organized as follows. First at the end of this section, we recall some preliminary concepts about fuzzy sets. The main approaches of paper are expressed in Section 2. In this section, we fit some support vector fuzzy linear/nonlinear regression models with fixed/variable errors on fuzzy dataset (fuzzy explanatory variables and fuzzy response variable). In Section 3, we study the applications of proposed methods on the numerical and simulated examples based on the goodness of fit indices. Finally, some concluding remarks are provided in Section 4.

Let us recall some preliminary concepts about fuzzy sets (for more details, see Zimmermann (2001)).

The fuzzy set $\tilde{A}$ of the universal set X is defined as $\{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, where $\mu_{\tilde{A}} : X \to [0, 1]$ is the membership function of $\tilde{A}$. Briefly, the membership function of $\tilde{A}$ is represented as $\tilde{A}(x) = \mu_{\tilde{A}}(x)$. The $\alpha-$cuts of $\tilde{A}$ is defined as $\tilde{A}[\alpha] = \{x \in X;\ \tilde{A}(x) \geq \alpha\}$ for $0 < \alpha \leq 1$.

Definition 1.1. A fuzzy set $\tilde{N}$ of X is called a fuzzy number, if

- $\tilde{N}(x) = 1$ for some $x \in X$.
- $\tilde{N}[\alpha]$ is a closed bounded interval for $0 < \alpha \leq 1$.

Definition 1.2. A fuzzy number $\tilde{N}$ is called a triangular fuzzy number if its membership function is given as follows:

$$\tilde{N}(x) = \begin{cases} 1 - \frac{n-x}{l_n} & n - l_n < x \leq n, \\ 1 + \frac{n-x}{r_n} & n < x < n + r_n. \end{cases}$$

It is denoted by $\tilde{N} = (n; l_n, r_n)_T$.

Remark 1.3. The fuzzy number $\tilde{N}$ is a symmetric triangular fuzzy number, if $l_n = r_n$. It is denoted by $\tilde{N} = (n; l_n)_T$.

Based on the principle expansion, some arithmetic operations between two triangular fuzzy numbers are obtained as follows.

Proposition 1.4. *Suppose that $\tilde{M} = (m; l_m, r_m)_T$ and $\tilde{N} = (n; l_n, r_n)_T$ are two fuzzy numbers and $\lambda \in R - \{0\}$. Then*

$$\lambda \otimes \tilde{M} = \begin{cases} (\lambda m; \lambda l_m, \lambda r_m)_T & \lambda > 0, \\ (\lambda m; -\lambda r_m, -\lambda l_m)_T & \lambda < 0, \end{cases}$$

and

$$\tilde{M} \oplus \tilde{N} = (m + n; l_m + l_n, r_m + r_n)_T.$$

2 Support vector fuzzy regression with fuzzy error

In this section, we want to fit a support vector fuzzy regression (SVFR) model with fuzzy error when the explanatory variables and the response variable are as fuzzy quantities. In this approach, we present two SVFR models with fuzzy fixed error and fuzzy variable errors.

2.1 Support vector fuzzy regression with fuzzy fixed error

Based on a random sample of size n, we have observed the fuzzy data as $(\tilde{\mathbf{x}}_1, \tilde{y}_1), \ldots, (\tilde{\mathbf{x}}_n, \tilde{y}_n)$ where, $\tilde{\mathbf{x}}_i = (\tilde{x}_{i1}, \ldots, \tilde{x}_{ip})$, $\tilde{x}_{ij} = (x_{ij}; l_{x_{ij}})_T$, and $\tilde{y}_i = (y_i; l_{y_i})_T$, $i = 1, \ldots, n$, $j = 1, \ldots, p$. We want to fit a linear regression model to this data set as follows:

$$\begin{aligned} \tilde{y} &= b \oplus (\boldsymbol{\beta} \otimes \tilde{\mathbf{x}}) \oplus \tilde{\varepsilon} \\ &= b \oplus (\beta_1 \otimes \tilde{x}_1) \oplus (\beta_2 \otimes \tilde{x}_2) \oplus \ldots \oplus (\beta_p \otimes \tilde{x}_p) \oplus \tilde{\varepsilon}, \end{aligned} \tag{2.1}$$

where, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)$ is a vector of the parameters of model. Also, the error component is given by a symmetric triangular fuzzy number $\tilde{\varepsilon} = (0; s)_T$. Hence, we fit the fuzzy regression model as follows:

$$\begin{aligned} \hat{\tilde{y}}_i &= \hat{b} \oplus (\hat{\boldsymbol{\beta}} \otimes \tilde{\mathbf{x}}_i) \oplus \hat{\tilde{\varepsilon}} \\ &= (\hat{b} + \hat{\beta}_1 x_{i1} + \ldots + \hat{\beta}_p x_{ip}; |\hat{\beta}_1| l_{x_{i1}} + \ldots + |\hat{\beta}_p| l_{x_{ip}} + \hat{s})_T. \end{aligned}$$

Now, we want to introduce a method based on support vector regression with fuzzy data. In this method, we suppose that the marginal hyperplanes are the lower and upper bounds as follows:

$$\begin{aligned} \hat{y}^U &= \hat{b} + \hat{\beta}_1 x_1 + |\hat{\beta}_1| l_{x_1} + \ldots + \hat{\beta}_p x_p + |\hat{\beta}_p| l_{x_p} + \hat{s} = \hat{\boldsymbol{\beta}}.\mathbf{x} + |\hat{\boldsymbol{\beta}}|.\mathbf{l_x} + \hat{b} + \hat{s}, \\ \hat{y}^L &= \hat{b} + \hat{\beta}_1 x_1 - |\hat{\beta}_1| l_{x_1} + \ldots + \hat{\beta}_p x_p - |\hat{\beta}_p| l_{x_p} - \hat{s} = \hat{\boldsymbol{\beta}}.\mathbf{x} - |\hat{\boldsymbol{\beta}}|.\mathbf{l_x} + \hat{b} - \hat{s}, \end{aligned} \tag{2.2}$$

where, $\mathbf{x} = (x_1, x_2, \ldots, x_p)'$, $\mathbf{l_x} = (l_{x_1}, l_{x_2}, \ldots, l_{x_p})'$, $\hat{\boldsymbol{\beta}} = (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_p)'$, and $|\hat{\boldsymbol{\beta}}| = (|\hat{\beta}_1|, |\hat{\beta}_2|, \ldots, |\hat{\beta}_p|)'$. By defining the positive auxiliary values $\hat{\beta}_j^+ = \max(0, \hat{\beta}_j)$ and $\hat{\beta}_j^- = -\min(0, \hat{\beta}_j)$ in $\hat{\beta}_j = \hat{\beta}_j^+ - \hat{\beta}_j^-$ and $|\hat{\beta}_j| = \hat{\beta}_j^+ + \hat{\beta}_j^-$, the equations 2.2 are rewritten as follows:

$$\begin{aligned} \hat{y}^U &= \hat{\boldsymbol{\beta}}^+.(\mathbf{x} + \mathbf{l_x}) - \hat{\boldsymbol{\beta}}^-.(\mathbf{x} - \mathbf{l_x}) + \hat{b} + \hat{s}, \\ \hat{y}^L &= \hat{\boldsymbol{\beta}}^+.(\mathbf{x} - \mathbf{l_x}) - \hat{\boldsymbol{\beta}}^-.(\mathbf{x} + \mathbf{l_x}) + \hat{b} - \hat{s}, \end{aligned}$$

where, $\hat{\boldsymbol{\beta}}^+ = (\hat{\beta}_1^+, \hat{\beta}_2^+, \ldots, \hat{\beta}_p^+)'$, and $\hat{\boldsymbol{\beta}}^- = (\hat{\beta}_1^-, \hat{\beta}_2^-, \ldots, \hat{\beta}_p^-)'$. By using the distance between two parallel planes, the distance between two hyperplanes is equal to $\frac{2\hat{s}}{\sqrt{\|\hat{\boldsymbol{\beta}}^+\|^2 + \|\hat{\boldsymbol{\beta}}^-\|^2 + 1}}$ with $\|\hat{\boldsymbol{\beta}}^+\|^2 =$

$\sum_{j=1}^{p}(\hat{\beta}_j^+)^2$ and $\|\hat{\boldsymbol{\beta}}^-\|^2 = \sum_{j=1}^{p}(\hat{\beta}_j^-)^2$. Therefore, by adding a limitation such as including the observed response values by the estimated response values, the programming problem is presented as follows:

$$\min_{\hat{\beta}^+,\hat{\beta}^-,\hat{b},\hat{s}} \frac{4\hat{s}^2}{\|\hat{\boldsymbol{\beta}}^+\|^2 + \|\hat{\boldsymbol{\beta}}^-\|^2 + 1},$$
$$s.t.$$
$$\begin{cases} (\hat{\boldsymbol{\beta}}^+ - \hat{\boldsymbol{\beta}}^-).\mathbf{x_i} - (\hat{\boldsymbol{\beta}}^+ + \hat{\boldsymbol{\beta}}^-).\mathbf{l}_{\mathbf{x_i}} + \hat{b} - \hat{s} \le y_i - l_{y_i}, \\ (\hat{\boldsymbol{\beta}}^+ - \hat{\boldsymbol{\beta}}^-).\mathbf{x_i} + (\hat{\boldsymbol{\beta}}^+ + \hat{\boldsymbol{\beta}}^-).\mathbf{l}_{\mathbf{x_i}} + \hat{b} + \hat{s} \ge y_i + l_{y_i}, \\ \hat{s} \ge l_{y_i}, \hat{\boldsymbol{\beta}}^+ \ge 0, \hat{\boldsymbol{\beta}}^- \ge 0, \ i = 1, 2, \ldots, n. \end{cases}$$

Using the Lagrange coefficients, we have:

$$\begin{aligned} L_p = & \frac{4\hat{s}^2}{\|\hat{\boldsymbol{\beta}}^+\|^2 + \|\hat{\boldsymbol{\beta}}^-\|^2 + 1} \\ & - \sum_{i=1}^{n} \alpha_i \Big[\hat{s} + \hat{b} - y_i - l_{y_i} + (\hat{\boldsymbol{\beta}}^+ - \hat{\boldsymbol{\beta}}^-).\mathbf{x_i} + (\hat{\boldsymbol{\beta}}^+ + \hat{\boldsymbol{\beta}}^-).\mathbf{l}_{\mathbf{x_i}}\Big] \\ & - \sum_{i=1}^{n} \alpha_i^* \Big[\hat{s} - \hat{b} + y_i - l_{y_i} - (\hat{\boldsymbol{\beta}}^+ - \hat{\boldsymbol{\beta}}^-).\mathbf{x_i} + (\hat{\boldsymbol{\beta}}^+ + \hat{\boldsymbol{\beta}}^-).\mathbf{l}_{\mathbf{x_i}}\Big] \\ & - \sum_{i=1}^{n} \eta_i(\hat{s} - l_{y_i}) - \boldsymbol{\gamma}.(\hat{\boldsymbol{\beta}}^+ + \hat{\boldsymbol{\beta}}^-), \end{aligned} \tag{2.3}$$

where, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_p)'$ and $\alpha_i, \alpha_i^*, \eta_i, \gamma_j \ge 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, p$. Now, by taking the derivative with respect to the parameters we have:

$$\begin{cases} \frac{\partial L_P}{\partial \hat{\boldsymbol{\beta}}^+} = \frac{-8\hat{s}^2}{(\|\hat{\boldsymbol{\beta}}^+\|^2+\|\hat{\boldsymbol{\beta}}^-\|^2+1)^2}\hat{\boldsymbol{\beta}}^+ - \sum_{i=1}^{n}(\alpha_i - \alpha_i^*)\mathbf{x_i} - \sum_{i=1}^{n}(\alpha_i + \alpha_i^*)\mathbf{l}_{\mathbf{x_i}} - \boldsymbol{\gamma} = 0, \\ \frac{\partial L_P}{\partial \hat{\boldsymbol{\beta}}^-} = \frac{-8\hat{s}^2}{(\|\hat{\boldsymbol{\beta}}^+\|^2+\|\hat{\boldsymbol{\beta}}^-\|^2+1)^2}\hat{\boldsymbol{\beta}}^- + \sum_{i=1}^{n}(\alpha_i - \alpha_i^*)\mathbf{x_i} - \sum_{i=1}^{n}(\alpha_i + \alpha_i^*)\mathbf{l}_{\mathbf{x_i}} - \boldsymbol{\gamma} = 0, \\ \frac{\partial L_P}{\partial \hat{b}} = \sum_{i=1}^{n}(\alpha_i^* - \alpha_i) = 0, \\ \frac{\partial L_P}{\partial \hat{s}} = \frac{8}{\|\hat{\boldsymbol{\beta}}^+\|^2+\|\hat{\boldsymbol{\beta}}^-\|^2+1}\hat{s} - \sum_{i=1}^{n}(\alpha_i + \alpha_i^* + \eta_i) = 0. \end{cases}$$

So, the parameters of the model are obtained as follows:

$$\begin{cases} \hat{\boldsymbol{\beta}}^+ = \frac{8[-\sum_{i=1}^{n}(\alpha_i-\alpha_i^*)\mathbf{x_i}-\sum_{i=1}^{n}(\alpha_i+\alpha_i^*)\mathbf{l}_{\mathbf{x_i}}-\boldsymbol{\gamma}]}{\Big[\sum_{i=1}^{n}(\alpha_i+\alpha_i^*+\eta_i)\Big]^2}, \\ \hat{\boldsymbol{\beta}}^- = \frac{8[\sum_{i=1}^{n}(\alpha_i-\alpha_i^*)\mathbf{x_i}-\sum_{i=1}^{n}(\alpha_i+\alpha_i^*)\mathbf{l}_{\mathbf{x_i}}-\boldsymbol{\gamma}]}{\Big[\sum_{i=1}^{n}(\alpha_i+\alpha_i^*+\eta_i)\Big]^2}, \\ \sum_{i=1}^{n}(\alpha_i^* - \alpha_i) = 0, \\ \hat{s} = \frac{1}{8}\Big[\sum_{i=1}^{n}(\alpha_i + \alpha_i^* + \eta_i)\Big]\Big[\|\hat{\boldsymbol{\beta}}^+\|^2 + \|\hat{\boldsymbol{\beta}}^-\|^2 + 1\Big]. \end{cases} \tag{2.4}$$

Now, by substituting the values of Equation (2.4) in Equation (2.3), the dual equation is obtained, as:

$$\begin{aligned} L_D = & \frac{16\sum_{i=1}^{n}\sum_{k=1}^{n}(\alpha_i-\alpha_i^*)(\alpha_k-\alpha_k^*)\mathbf{x_i}.\mathbf{x_k}}{\left[\sum_{i=1}^{n}(\alpha_i+\alpha_i^*+\eta_i)\right]^2} \\ & -\frac{1}{16}\Big[\sum_{i=1}^{n}(\alpha_i+\alpha_i^*+\eta_i)\Big]^2+\sum_{i=1}^{n}(\alpha_i+\alpha_i^*+\eta_i)l_{y_i}+\sum_{i=1}^{n}(\alpha_i-\alpha_i^*)y_i. \end{aligned} \tag{2.5}$$

Also, based on conditions KKT, the following limits are obtained as:

$$\sum_{i=1}^{n}(\alpha_i-\alpha_i^*)=0, \qquad \alpha_i,\alpha_i^*,\eta_i,\gamma_j\geq 0.$$

We consider the following matrix forms:

$$\mathbf{W}'=\begin{bmatrix} x_{11} & \dots & x_{n1} & -x_{11} & \dots & -x_{n1} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{1p} & \dots & x_{np} & -x_{1p} & \dots & -x_{np} & 0 & \dots & 0 \end{bmatrix}_{p\times 3n},$$

$$\begin{aligned} \mathbf{T}' &= [-y_1+l_{y_1},\dots,-y_n+l_{y_n},y_1+l_{y_1},\dots,y_n+l_{y_n},l_{y_1},\dots,l_{y_n}]_{1\times 3n}, \\ \mathbf{A}' &= [\alpha_1^*,\dots,\alpha_n^*,\alpha_1,\dots,\alpha_n,\eta_1,\dots,\eta_n] \\ &= [a_1,\dots,a_n,a_{n+1},\dots,a_{2n},a_{2n+1},\dots,a_{3n}]_{1\times 3n}, \\ \mathbf{D}_{3n\times 3n} &= [1,\dots,1,1,\dots,1,1,\dots,1]'[1,\dots,1,1,\dots,1,1,\dots,1], \end{aligned} \tag{2.6}$$

where, $\alpha_i^*=a_i$, $\alpha_i=a_{n+i}$, and $\eta_i=a_{2n+i}$ for $i=1,\dots,n$. Therefore, Equation (2.5) is rewritten as:

$$\min_{\mathbf{A}}(-L_D)=\min_{A}\Big(-\frac{16\mathbf{A}'\mathbf{W}\mathbf{W}'\mathbf{A}}{\mathbf{A}'\mathbf{D}\mathbf{A}}+\frac{1}{16}\mathbf{A}'\mathbf{D}\mathbf{A}-\mathbf{A}'\mathbf{T}\Big).$$

By taking the derivative of the above function, we have:

$$\frac{\partial-L_D}{\partial\mathbf{A}}=-32\Big[\frac{\mathbf{W}\mathbf{W}'\mathbf{A}(\mathbf{A}'\mathbf{D}\mathbf{A})-\mathbf{D}\mathbf{A}(\mathbf{A}'\mathbf{W}\mathbf{W}'\mathbf{A})}{(\mathbf{A}'\mathbf{D}\mathbf{A})^2}\Big]+\frac{1}{8}\mathbf{D}\mathbf{A}-\mathbf{T}.$$

Hence, based on a repetition algorithm, the parameters of the model are estimated as follows:

$$\mathbf{A}^{t+1}=256\mathbf{D}^{-1}\Big[\frac{\mathbf{W}\mathbf{W}'\mathbf{A}^t(\mathbf{A}'^t\mathbf{D}\mathbf{A}^t)-\mathbf{D}\mathbf{A}^t(\mathbf{A}'^t\mathbf{W}\mathbf{W}'\mathbf{A}^t)}{(\mathbf{A}'^t\mathbf{D}\mathbf{A}^t)^2}\Big]+8\mathbf{D}^{-1}\mathbf{T}, \tag{2.7}$$

where, $\mathbf{D}^{-1}=\frac{1}{9n^2}\mathbf{D}$ is the generalized inverse matrix with $\mathbf{D}\mathbf{D}^{-1}\mathbf{D}=\mathbf{D}$ and $\mathbf{D}^{-1}\mathbf{D}\mathbf{D}^{-1}=\mathbf{D}^{-1}$. The set of support vector points is as follows:

$$M=\{i;\quad \alpha_i>0,\alpha_i^*>0\}.$$

The estimation of b based on the set M is obtained as :

$$\hat{b}=\frac{1}{\#M}\sum_{i\in M}\Big[y_i-(\hat{\boldsymbol{\beta}}^+-\hat{\boldsymbol{\beta}}^-).\mathbf{x_i}+sign(\alpha_i^*-\alpha_i)(\hat{s}-l_{y_i}+(\hat{\boldsymbol{\beta}}^++\hat{\boldsymbol{\beta}}^-).\mathbf{l}_{x_i})\Big], \tag{2.8}$$

where $\#M$ is equal to the number of points in the set M.

Remark 2.1. In addition to Algorithm 1, we can also estimate the parameters of model to solve the optimization problem in 2.5 based on Mathematica software.

Algorithm 1 Algorithm 1

1: We obtain **W**, **T** and **D** based on the relations (2.6).
2: In the first step, $t = 0$ is considered.
3: We consider the initial values for the vector $\mathbf{A}^0$ such that $\sum_{i=1}^{n}(\alpha_i - \alpha_i^*) = 0$, $\alpha_i, \alpha_i^*, \eta_i \geq 0$.
4: Based on the Equation (2.7), we repeat this step, when $\max|\mathbf{A}^{t+1} - \mathbf{A}^t| < e$ (e is small value) for $t = 0, 1, 2, \ldots$.
5: We put the obtained Lagrange coefficients in the relation (2.4) and calculate the values of $\hat{\boldsymbol{\beta}}^+$, $\hat{\boldsymbol{\beta}}^-$, and $\hat{s}$.
6: The set of support vector points M is calculated. The estimation of $\hat{b}$ is calculated based on the Equation (2.8).

2.2 Support vector regression with fuzzy variable errors

Now, we assume that the errors are as variable $\tilde{\varepsilon}_i = (0; s_i)_T$. Similar to Tanaka et al. (1982)'s approach, we add the sum of spreads of errors to the proposed objective function. Hence, the optimization problem is rewritten as follows:

$$
\begin{aligned}
&\min_{\hat{\beta}^+, \hat{\beta}^-, \hat{b}, \hat{s}} \frac{4\hat{s}^2}{\|\hat{\boldsymbol{\beta}}^+\|^2 + \|\hat{\boldsymbol{\beta}}^-\|^2 + 1} + C\sum_{i=1}^{n}\hat{s}_i \\
&s.t. \\
&\begin{cases}
(\hat{\boldsymbol{\beta}}^+ - \hat{\boldsymbol{\beta}}^-).\mathbf{x_i} - (\hat{\boldsymbol{\beta}}^+ + \hat{\boldsymbol{\beta}}^-).\mathbf{l}_{\mathbf{x_i}} + \hat{b} - \hat{s}_i \leq y_i - l_{y_i}, \\
(\hat{\boldsymbol{\beta}}^+ - \hat{\boldsymbol{\beta}}^-).\mathbf{x_i} + (\hat{\boldsymbol{\beta}}^+ + \hat{\boldsymbol{\beta}}^-).\mathbf{l}_{\mathbf{x_i}} + \hat{b} + \hat{s}_i \geq y_i + l_{y_i}, \\
\hat{s} \geq \hat{s}_i \geq l_{y_i}, \quad \hat{\boldsymbol{\beta}}^+ \geq 0, \quad \hat{\boldsymbol{\beta}}^- \geq 0, \quad i = 1, 2, \ldots, n,
\end{cases}
\end{aligned}
$$

where, C is the amount of the considered fine. Based on the Lagrange coefficients, the optimization problem is presented as follows:

$$
\begin{aligned}
L_p = &\frac{4\hat{s}^2}{\|\hat{\boldsymbol{\beta}}^+\|^2 + \|\hat{\boldsymbol{\beta}}^-\|^2 + 1} + C\sum_{i=1}^{n}\hat{s}_i \\
&- \sum_{i=1}^{n}\alpha_i\Big[\hat{s}_i + \hat{b} - y_i - l_{y_i} + (\hat{\boldsymbol{\beta}}^+ - \hat{\boldsymbol{\beta}}^-).\mathbf{x_i} + (\hat{\boldsymbol{\beta}}^+ + \hat{\boldsymbol{\beta}}^-).\mathbf{l}_{\mathbf{x_i}}\Big] \\
&- \sum_{i=1}^{n}\alpha_i^*\Big[\hat{s}_i - \hat{b} + y_i - l_{y_i} - (\hat{\boldsymbol{\beta}}^+ - \hat{\boldsymbol{\beta}}^-).\mathbf{x_i} + (\hat{\boldsymbol{\beta}}^+ + \hat{\boldsymbol{\beta}}^-).\mathbf{l}_{\mathbf{x_i}}\Big] \\
&- \sum_{i=1}^{n}\eta_i(\hat{s}_i - l_{y_i}) - \sum_{i=1}^{n}\delta_i(\hat{s} - \hat{s}_i) - \gamma.(\hat{\boldsymbol{\beta}}^+ + \hat{\boldsymbol{\beta}}^-),
\end{aligned}
\tag{2.9}
$$

in which $\gamma = (\gamma_1, \ldots, \gamma_p)'$ and $\alpha_i, \alpha_i^*, \eta_i, \delta_i, \gamma_j \geq 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, p$. In this case, the parameters of the model are obtained as follows:

$$\begin{cases} \hat{\beta}^+ = \dfrac{8[-\sum_{i=1}^n(\alpha_i-\alpha_i^*)\mathbf{x_i}-\sum_{i=1}^n(\alpha_i+\alpha_i^*)\mathbf{l_{x_i}}-\gamma]}{\left[\sum_{i=1}^n \delta_i\right]^2}, \\ \hat{\beta}^- = \dfrac{8[\sum_{i=1}^n(\alpha_i-\alpha_i^*)\mathbf{x_i}-\sum_{i=1}^n(\alpha_i+\alpha_i^*)\mathbf{l_{x_i}}-\gamma]}{\left[\sum_{i=1}^n \delta_i\right]^2}, \\ \sum_{i=1}^n(\alpha_i^* - \alpha_i) = 0, \quad C = \alpha_i + \alpha_i^* + \eta_i - \delta_i \\ \hat{s} = \frac{1}{8}\left[\sum_{i=1}^n \delta_i\right]\left[\|\hat{\beta}^+\|^2 + \|\hat{\beta}^-\|^2 + 1\right]. \end{cases} \tag{2.10}$$

Now, by inserting the obtained values in Equation (2.9) and after simplifying, the double equation is obtained as follows:

$$\begin{aligned} L_D = &\frac{16\sum_{i=1}^n\sum_{k=1}^n(\alpha_i-\alpha_i^*)(\alpha_k-\alpha_k^*)\mathbf{x_i}.\mathbf{x_k}}{\left[\sum_{i=1}^n \delta_i\right]^2} \\ &-\frac{1}{16}\left[\sum_{i=1}^n \delta_i\right]^2 + \sum_{i=1}^n(\alpha_i+\alpha_i^*+\eta_i)l_{y_i} + \sum_{i=1}^n(\alpha_i-\alpha_i^*)y_i. \end{aligned} \tag{2.11}$$

Also, based on the KKT conditions, we have:

$$\sum_{i=1}^n(\alpha_i - \alpha_i^*) = 0, \quad C \geq \alpha_i + \alpha_i^* + \eta_i - \delta_i, \tag{2.12}$$

In this case, we can consider the following matrix forms:

$$\mathbf{W}' = \begin{bmatrix} x_{11} & \ldots & x_{n1} & -x_{11} & \ldots & -x_{n1} & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{1p} & \ldots & x_{np} & -x_{n1} & \ldots & -x_{np} & 0 & \ldots & 0 & 0 & \ldots & 0 \end{bmatrix}_{p\times 4n}$$

$$\begin{aligned} \mathbf{T}' &= [-y_1 + l_{y_1}, \ldots, -y_n + l_{y_n}, y_1 + l_{y_1}, \ldots, y_n + l_{y_n}, l_{y_1}, \ldots, l_{y_n}, 0, \ldots, 0]_{1\times 4n}, \\ \mathbf{A}' &= [\alpha_1^*, \ldots, \alpha_n^*, \alpha_1, \ldots, \alpha_n, \eta_1, \ldots, \eta_n, \delta_1, \ldots, \delta_n] \\ &= [a_1, \ldots, a_n, a_{n+1}, \ldots, a_{2n}, a_{2n+1}, \ldots, a_{3n}, a_{3n+1}, \ldots, a_{4n}]_{1\times 4n}, \\ \mathbf{D}_{4n\times 4n} &= [0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, 1, \ldots, 1]'[0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, 1, \ldots, 1], \end{aligned} \tag{2.13}$$

where, $\alpha_i^* = a_i$, $\alpha_i = a_{i+n}$, $\eta_i = a_{2n+i}$, and $\delta_i = a_{4n+i}$ for $i = 1, \ldots, n$. Therefore, the equation (2.11) can be rewritten in the following matrix form:

$$\min_{\mathbf{A}}(-L_D) = \min_{A}\left(-\frac{16\mathbf{A}'\mathbf{W}\mathbf{W}'\mathbf{A}}{\mathbf{A}'\mathbf{D}\mathbf{A}} + \frac{1}{16}\mathbf{A}'\mathbf{D}\mathbf{A} - \mathbf{A}'\mathbf{T}\right).$$

Based on a repetition algorithm, the parameters of the model can be estimated as follows:

$$\mathbf{A}^{t+1} = 256\mathbf{D}^{-1}\left[\frac{\mathbf{W}\mathbf{W}'\mathbf{A}^t(\mathbf{A}'^t\mathbf{D}\mathbf{A}^t) - \mathbf{D}\mathbf{A}^t(\mathbf{A}'^t\mathbf{W}\mathbf{W}'\mathbf{A}^t)}{(\mathbf{A}'^t\mathbf{D}\mathbf{A}^t)^2}\right] + 8\mathbf{D}^{-1}\mathbf{T}, \tag{2.14}$$

where, $\mathbf{D}^{-1} = \frac{1}{n^2}\mathbf{D}$ is the generalized inverse matrix with $\mathbf{D}\mathbf{D}^{-1}\mathbf{D} = \mathbf{D}$ and $\mathbf{D}^{-1}\mathbf{D}\mathbf{D}^{-1} = \mathbf{D}^{-1}$. Now, according to the KKT condition we have:

$$\begin{cases} \hat{b} = y_i + l_{y_i} - \hat{s}_i - (\hat{\boldsymbol{\beta}}^+ - \hat{\boldsymbol{\beta}}^-).\mathbf{x_i} - (\hat{\boldsymbol{\beta}}^+ + \hat{\boldsymbol{\beta}}^-).\mathbf{l}_{\mathbf{x_i}}, \\ \hat{b} = y_i - l_{y_i} + \hat{s}_i - (\hat{\boldsymbol{\beta}}^+ - \hat{\boldsymbol{\beta}}^-).\mathbf{x_i} + (\hat{\boldsymbol{\beta}}^+ + \hat{\boldsymbol{\beta}}^-).\mathbf{l}_{\mathbf{x_i}}, \\ C - \alpha_i^* + \delta_i \geq \alpha_i > 0, \qquad C - \alpha_i + \delta_i \geq \alpha_i^* > 0, \\ \hat{s} - \hat{s}_i = 0, \qquad \hat{s}_i - l_{y_i} = 0, \qquad \eta_i, \delta_i > 0. \end{cases} \tag{2.15}$$

Therefore, the set of support points is presented as follows:

$$M = \{i; \quad C - \alpha_i^* + \delta_i \geq \alpha_i > 0, \quad C - \alpha_i + \delta_i \geq \alpha_i^* > 0, \quad \eta_i, \delta_i > 0\}.$$

Hence, the estimation of b is as

$$\hat{b} = \frac{1}{\#M}\sum_{i \in M}(y_i - (\hat{\boldsymbol{\beta}}^+ - \hat{\boldsymbol{\beta}}^-).\mathbf{x_i} + sign(\alpha_i^* - \alpha_i)(\hat{s}_i - l_{y_i} + (\hat{\boldsymbol{\beta}}^+ + \hat{\boldsymbol{\beta}}^-).\mathbf{l}_{\mathbf{x_i}})), \tag{2.16}$$

where, $\#M$ is equal to the number of points in the set M. The optimization algorithm for this case is similar to Algorithm 1 with the new matrix forms in relation (2.13) and the initial values based on relations (2.15).

2.3 Goodness of fit measure

Suppose that $\tilde{y}_i = (y_i, l_{y_i})_T$ and $\hat{\tilde{y}}_i = (\hat{y}_i, \hat{l}_{y_i})_T$, $i = 1, 2, \ldots, n$, are the values of the observed and estimated fuzzy responses. Some goodness of fit indices are defined as follows.

i) Based on the distance $d(\tilde{y}_i, \hat{\tilde{y}}_i)$, a goodness of fit index is defined as follows (Khammar et al., 2021b):

$$I = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1 + d(\tilde{y}_i, \hat{\tilde{y}}_i)},$$

where

$$d(\tilde{y}_i, \hat{\tilde{y}}_i) = \frac{1}{3}\Big(|y_i - \hat{y}_i| + |(y_i - l_{y_i}) - (\hat{y}_i - \hat{l}_{y_i})| + |(y_i + l_{y_i}) - (\hat{y}_i + \hat{l}_{y_i})|\Big).$$

ii) The goodness of fit index is defined based on the average of similarity measure as follows (Pappis and Karacapilidis, 1993):

$$MSM = \frac{1}{n}\sum_{i=1}^{n} S_{PK}(\hat{\tilde{y}}_i, \tilde{y}_i),$$

where, $S_{PK}(\hat{\tilde{y}}_i, \tilde{y}_i) = \frac{Card(\hat{\tilde{y}}_i \cap \tilde{y}_i)}{Card(\hat{\tilde{y}}_i \cup \tilde{y}_i)}$, $Card(\tilde{N}) = \int_{\mathbb{R}} \tilde{N}(x)dx$, and also

$$(\hat{\tilde{y}}_i \cap \tilde{y}_i)(x) = \min(\hat{\tilde{y}}_i(x), \tilde{y}_i(x)),$$
$$(\hat{\tilde{y}}_i \cup \tilde{y}_i)(x) = \max(\hat{\tilde{y}}_i(x), \tilde{y}_i(x)).$$

Remark 2.2. The indices I and MSM are on the interval [0,1]. The optimal fuzzy regression model is the model with maximum value of I or MSM.

Remark 2.3. For selecting the optimal value of C in the optimization problem in (2.9), we fit the fuzzy regression models and calculate MSM for the different values of C. The optimal value is C with maximum value of MSM.

Table 1: Data set in Example 3.1.

No.	$(x_i; l_{x_i})$	$(y_i; l_{y_i})_T$
1	$(2; 0.5)_T$	$(4; 0.5)_T$
2	$(3.5; 0.5)_T$	$(5.5; 0.5)_T$
3	$(5.5; 1)_T$	$(7.5; 1)_T$
4	$(7; 0.5)_T$	$(6.5; 0.5)_T$
5	$(8.5; 0.5)_T$	$(8.5; 0.5)_T$
6	$(10.5; 1)_T$	$(8; 1)_T$
7	$(11; 0.5)_T$	$(10.5; 0.5)_T$
8	$(12.5; 0.5)_T$	$(9.5; 0.5)_T$

3 Numerical and simulated examples

In this section, the application of proposed methods are presented by some numerical and simulated examples.

Example 3.1. Consider the data in Table 1 (Sakawa and Yano, 1992). Now, we wish to fit the fuzzy regression models as follows.

A) Based on the method presented in Section 2.1 with fuzzy fixed error, the following fuzzy regression model is fitted to this data (see Figure 1-A):

$$\hat{\tilde{y}}_i = 3.75 \oplus (0.5 \otimes \tilde{x}_i) \oplus (0; 1.5)_T.$$

The values of estimated response variable are obtained in second column of Table 2.

B) Based on the method presented in Section 2.2 with the fuzzy variable errors, we fit the fuzzy regression model with the optimal value $C = 100$ as follows (see Figure 1-B):

$$\hat{\tilde{y}}_i = 3.10 \oplus (0.60 \otimes \tilde{x}_i) \oplus \widehat{\tilde{\varepsilon}_i}.$$

The fuzzy variable errors $\widehat{\tilde{\varepsilon}_i} = (0; \hat{s}_i)_T$, $i = 1, \ldots, 8$, and the values of estimated response variable are obtained in third and 4th columns of Table 2.

C) The other authors fit some fuzzy regression models on this data set. These models are summarized in Table 3.

We compare the proposed fuzzy regression models with other fuzzy regression models based on some goodness of fit indices in Table 3. Based on the given results, the proposed SVFR model with fuzzy variable errors has the performance better than other fuzzy regression models.

Example 3.2. Consider the data set given in Table 4 (Diamond, 1988). Now, we want to fit the different fuzzy regression models as follows:

A) Based on the proposed method in Section 2.1, we fit a SVFR model with fuzzy fixed error as follows (Figure 2-A):

$$\hat{\tilde{y}}_i = 0.8985 \oplus (0.1641 \otimes \tilde{x}_i) \oplus (0; 0.8)_T.$$

Also, the values of estimated response variable are obtained in second column of Table 5.

Table 2: The estimated response variables in Example 3.1.

	Fixed error	Variable errors	
No.	$(\hat{y}_i; \hat{l}_{y_i})_T$	$\widehat{\tilde{\varepsilon}}_i$	$(\hat{y}_i; \hat{l}_{y_i})_T$
1	$(4.75; 1.75)_T$	$(0; 0.5)_T$	$(4.30; 0.80)_T$
2	$(5.50; 1.75)_T$	$(0; 0.5)_T$	$(5.20; 0.80)_T$
3	$(6.50; 2.00)_T$	$(0; 1.5)_T$	$(6.40; 2.10)_T$
4	$(7.25; 1.75)_T$	$(0; 1.0)_T$	$(7.30; 1.30)_T$
5	$(8.00; 1.75)_T$	$(0; 0.5)_T$	$(8.20; 0.80)_T$
6	$(9.00; 2.00)_T$	$(0; 1.8)_T$	$(9.40; 2.40)_T$
7	$(9.25; 1.75)_T$	$(0; 1.0)_T$	$(9.70; 1.30)_T$
8	$(10.00; 1.75)_T$	$(0; 1.3)_T$	$(10.60; 1.60)_T$

Table 3: Goodness of fit indices for different models in Example 3.1.

Approach	Model	I	MSM
SVFR with fuzzy fixed error	$\hat{\tilde{y}}_i = 3.75 \oplus (0.5 \otimes \tilde{x}_i) \oplus (0; 1.5)_T$	0.49	0.24
SVFR with fuzzy variable errors	$\hat{\tilde{y}}_i = 3.10 \oplus (0.60 \otimes \tilde{x}_i) \oplus \widehat{\tilde{\varepsilon}}_i$	0.60	0.28
Sakawa and Yano (1992)	$\hat{\tilde{y}}_i^{SY} = (3.20; 0.17)_T \oplus ((0.57; 0.08)_T \otimes \tilde{x}_i)$	0.39	0.20
Diamond (1988)	$\hat{\tilde{y}}_i^{DM} = (3.56; 0.3)_T \oplus (0.52 \otimes \tilde{x}_i)$	0.60	0.16
Nasrabadi and Nasrabadi (2004)	$\hat{\tilde{y}}_i^{NN} = 3.57 \oplus ((0.54; 0.1)_T \otimes \tilde{x}_i)$	0.59	0.25
Chen and Hsueh (2009)	$\hat{\tilde{y}}_i^{LSE} = (3.57; 0.3)_T \oplus (0.519 \otimes \tilde{x}_i)$	0.60	0.16
Kao and Chyu (2003)	$\hat{\tilde{y}}_i^{KC} = 3.6 \oplus (0.522 \otimes \tilde{x}_i) \oplus (-0.011; 0.94)_T$	0.56	0.23
Wu and Tseng (2002)	$\hat{\tilde{y}}_i^{WT} = (3.57; 0.29)_T \oplus ((0.52; 0.01)_T \otimes \tilde{x}_i)$	0.60	0.18
Hojati et al. (2005)	$\hat{\tilde{y}}_i^{HBS} = (3.41; 0.41)_T \oplus ((0.52; 0.02)_T \otimes \tilde{x}_i)$	0.60	0.21

B) Based on the proposed method in Section 2.2, a SVFR model with fuzzy variable errors with the optimal value $C = 1.7$ is fitted to this data set as follows (Figure 2-B):

$$\hat{\tilde{y}}_i = 1.1666 \oplus (0.1389 \otimes \tilde{x}_i) \oplus \widehat{\tilde{\varepsilon}}_i,$$

where, $\widehat{\tilde{\varepsilon}}_i$, $i = 1, \ldots, 8$, are obtained in third column of Table 5. Also, the values of estimated response variable are obtained in 4th column of Table 5.

C) Some fuzzy regression models on this data set are presented in Table 6.

To compare the proposed fuzzy regression models with other fuzzy regression models based on some goodness of fit indices, the proposed SVFR model with fuzzy variable errors has the performance better than other fuzzy regression models (see Table 6).

Example 3.3. Hong and Hwang (2003) presented an approach for fitting a fuzzy regression model based on support vectors with the fuzzy input-fuzzy output. In this approach, the objective function is similar to the basic idea of SVR in crisp linear regression (i.e. based on the distance between two parallel hyperplanes given by the centers of estimated fuzzy outputs) and some constraints on the centers and bounds of estimated fuzzy outputs. Now, we want to compare the fuzzy regression models proposed in this paper with the model given by Hong and Hwang based on a data set in Table 7 (consider this data set in Hong and Hwang (2003)) as follows:

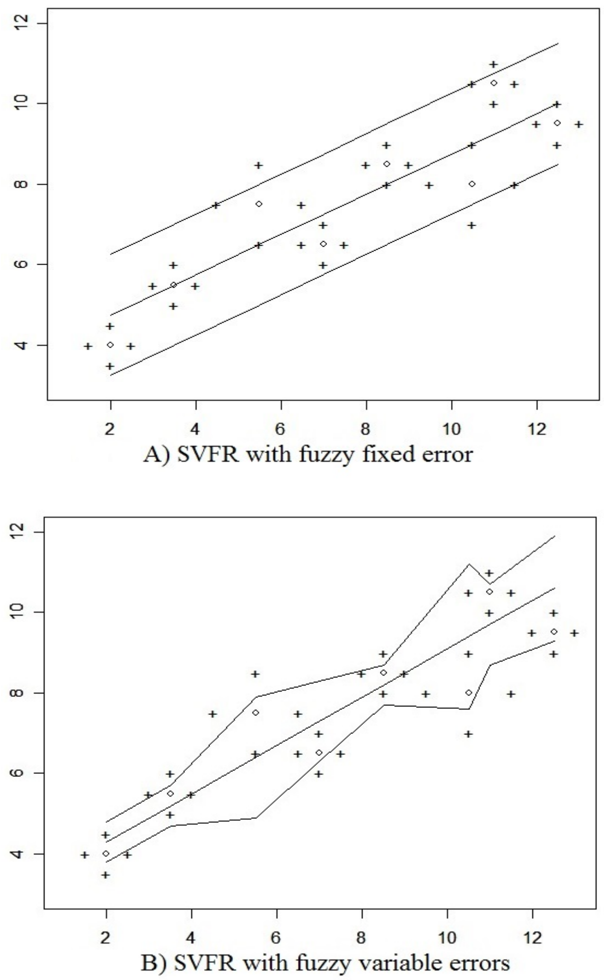

Figure 1: Support vector fuzzy regression models in Example 3.1.

Table 4: Data set in Example 3.2.

No.	$(x_i; l_{x_i})$	$(y_i; l_{y_i})_T$
1	$(21; 2.10)_T$	$(4.0; 0.80)_T$
2	$(15; 2.25)_T$	$(3.0; 0.30)_T$
3	$(15; 2.25)_T$	$(3.5; 0.35)_T$
4	$(9; 1.35)_T$	$(2.0; 0.40)_T$
5	$(12; 1.20)_T$	$(3.0; 0.45)_T$
6	$(18; 1.80)_T$	$(3.5; 0.70)_T$
7	$(6; 1.20)_T$	$(2.5; 0.38)_T$
8	$(12; 2.40)_T$	$(2.5; 0.50)_T$

A) Based on the proposed method in Section 2.1, the SVFR model with fuzzy fixed error is

Table 5: Estimated response variables in Example 3.2.

	Fixed error	Variable errors	
No.	$(\hat{y}_i; \hat{l}_{y_i})_T$	$\widehat{\tilde{\varepsilon}}_i$	$(\hat{y}_i; \hat{l}_{y_i})_T$
1	$(4.3446; 1.1446)_T$	0.7133	$(4.0835; 1.0050)_T$
2	$(3.3600; 1.1692)_T$	0.6292	$(3.2501; 0.9417)_T$
3	$(3.3600; 1.1692)_T$	0.4500	$(3.2501; 0.7625)_T$
4	$(2.3754; 1.0215)_T$	0.5000	$(2.4167; 0.6875)_T$
5	$(2.8677; 0.9969)_T$	0.3000	$(2.8334; 0.4667)_T$
6	$(3.8523; 1.0954)_T$	0.3500	$(3.6668; 0.6000)_T$
7	$(1.8831; 0.9969)_T$	0.7000	$(2.0000; 0.8667)_T$
8	$(2.8677; 1.1938)_T$	0.8000	$(2.8334; 1.1334)_T$

Table 6: Goodness of fit indices for different models in Example 3.2.

Approach	Model	I	MSM
SVFR with fuzzy fixed error	$\hat{\tilde{y}}_i = 0.8985 \oplus (0.1641 \otimes \tilde{x}_i) \oplus (0; 0.8)_T$	0.66	0.35
SVFR with fuzzy variable errors	$\hat{\tilde{y}}_i = 1.1666 \oplus (0.1389 \otimes \tilde{x}_i) \oplus \widehat{\tilde{\varepsilon}}_i$	0.75	0.42
Hong et al. (2001)	$\hat{\tilde{y}}_i^{HSD} = (1.38; 0.43)_T \oplus ((0.12; 0.03)_T \otimes \tilde{x}_i)$	0.69	0.38
Hasanpour et al. (2010)	$\hat{\tilde{y}}_i^{HMY} = 2 \oplus (0.08 \otimes \tilde{x}_i)$	0.73	0.15
Yang and Lin (2002)	$\hat{\tilde{y}}_i^{YL} \simeq (1.37, 1.38)_T \oplus ((0.12, 0.009)_T \otimes \tilde{x}_i)$	0.53	0.27
Arabpour and Tata (2008)	$\hat{\tilde{y}}_i^{AT} \simeq (1.18, 1.37, 1.59)_T \oplus ((0.11, 0.12, 0.123)_T \otimes \tilde{x}_i)$	0.34	0.14
Kelkinnama and Taheri (2012)	$\hat{\tilde{y}}_i^{KT1} = (0.84, 0.39)_T \oplus ((0.14, 0.038)_T \otimes \tilde{x}_i)$	0.64	0.33

as follows (Figure 3-A):

$$\hat{\tilde{y}}_i = -4.8476 \oplus (1.1524 \otimes \tilde{x}_i) \oplus (0; 2.0191)_T.$$

Also, the values of estimated response variable are obtained in second column of Table 8.

B) Based on the proposed method in Section 2.2, a SVFR model with fuzzy variable errors with the optimal value $C = 1.3$ is fitted as follows (Figure 3-B):

$$\hat{\tilde{y}}_i = -4.9458 \oplus (1.1755 \otimes \tilde{x}_i) \oplus \widehat{\tilde{\varepsilon}}_i,$$

where, $\widehat{\tilde{\varepsilon}}_i$, $i = 1, \ldots, 9$, are obtained in third column of Table 8. Also, the values of estimated response variable are obtained in 4th column of Table 8.

C) Based on Hong and Hwang's method, a fuzzy regression model is fitted to this data set as follows (Figure 3-C):

$$\hat{\tilde{y}}_i^{HH} = (-2.4570; 0.0710)_T \oplus (0.8570 \otimes \tilde{x}_i).$$

To compare these models based on goodness of fit indices, the SVFR model with fuzzy variable errors has the performance better than Hong and Hwang's model (see Table 9).

Example 3.4. (Simulation study) In this example, we first simulate 10 data sets with size 150 and then, the performances of the proposed SVFR models are evaluated based on goodness of fit indices. The average of obtained parameters in 10 data sets are as the estimations of parameters of SVFR models. These data sets are as symmetric triangular fuzzy numbers $\tilde{x}_{i1} = (x_{i1}; l_{x_{i1}})_T$,

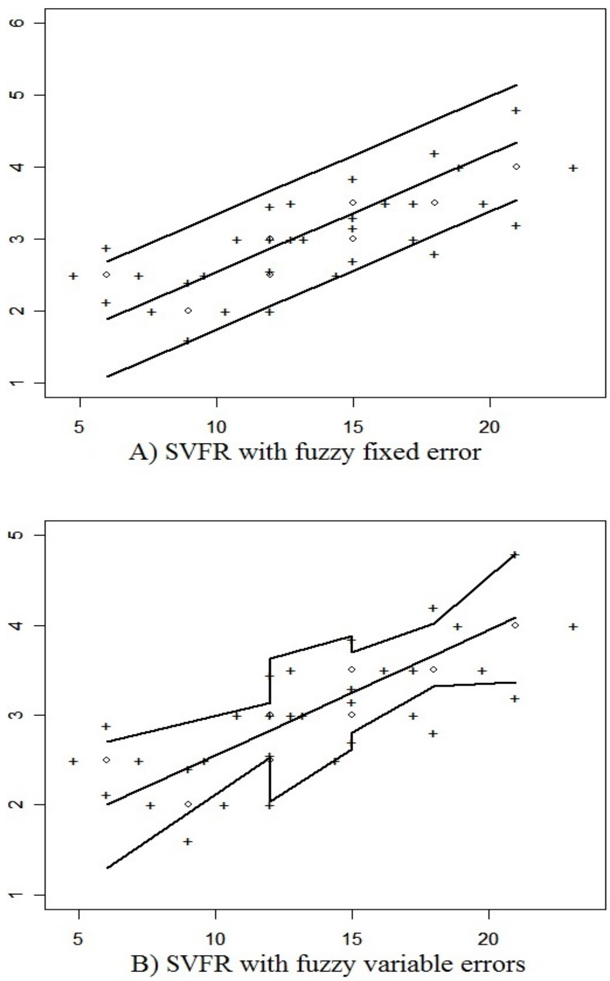

Figure 2: Support vector fuzzy regression models in Example 3.2.

Table 7: Data set in Example 3.3.

No.	$(x_i; l_{x_i})$	$(y_i; l_{y_i})_T$
1	$(1.0; 0.5)_T$	$(-1.6; 0.5)_T$
2	$(3.0; 0.5)_T$	$(-1.8; 0.5)_T$
3	$(4.0; 0.5)_T$	$(-1.0; 0.5)_T$
4	$(5.6; 0.8)_T$	$(1.2; 0.5)_T$
5	$(7.8; 0.8)_T$	$(2.2; 1.0)_T$
6	$(10.2; 0.8)_T$	$(6.8; 1.0)_T$
7	$(11.0; 1.0)_T$	$(10.0; 1.0)_T$
8	$(11.5; 1.0)_T$	$(10.0; 1.0)_T$
9	$(12.7; 1.0)_T$	$(10.0; 1.0)_T$

Table 8: Estimated response variables in Example 3.3.

	Fixed error	Variable errors	
No.	$(\hat{y}_i; \hat{l}_{y_i})_T$	$\widehat{\tilde{\varepsilon}}_i$	$(\hat{y}_i; \hat{l}_{y_i})_T$
1	$(-3.6952; 2.5952)_T$	2.0826	$(-3.7703; 2.6703)_T$
2	$(-1.3905; 2.5952)_T$	0.5000	$(-1.4194; 1.0877)_T$
3	$(-0.2381; 2.5952)_T$	0.6684	$(-0.2439; 1.2561)_T$
4	$(1.6057; 2.9410)_T$	0.5000	$(1.6369; 1.4404)_T$
5	$(4.1409; 2.9410)_T$	2.0826	$(4.2229; 3.0230)_T$
6	$(6.9067; 2.9410)_T$	1.0000	$(7.0441; 1.9404)_T$
7	$(7.8286; 3.1714)_T$	1.8400	$(7.9845; 3.0155)_T$
8	$(8.4047; 3.1714)_T$	1.2523	$(8.5722; 2.4278)_T$
9	$(9.7876; 3.1714)_T$	1.0001	$(9.9828; 2.1756)_T$

Table 9: Goodness of fit indices for different models in Example 3.3.

Approach	Model	I	MSM
SVFR with fuzzy fixed error	$\hat{\tilde{y}}_i = -4.8476 \oplus (1.1524 \otimes \tilde{x}_i) \oplus (0; 2.0191)_T$	0.36	0.19
SVFR with fuzzy variable errors	$\hat{\tilde{y}}_i = -4.9458 \oplus (1.1755 \otimes \tilde{x}_i) \oplus \widehat{\tilde{\varepsilon}}_i$	0.48	0.26
Hong and Hwang (2003)	$\hat{\tilde{y}}_i^{HH} = (1.38; 0.43)_T \oplus ((0.12; 0.03)_T \otimes \tilde{x}_i)$	0.45	0.15

$\tilde{x}_{i2} = (x_{i2}; l_{x_{i2}})_T$, and $\tilde{y}_i = (y_i; l_{y_i})_T$, $i = 1, \ldots, 150$ and are simulated based on the following equation:

$$y_i = 5 + 1.5x_i + e_i, \qquad i = 1, 2, ..., 150,$$

where, $x_i \sim N(1, 4)$, $l_{x_i} \sim U(0, 0.2)$, $l_{y_i} \sim U(0, 0.1)$, and $e_i \sim U(0, 0.5)$.

Based on the method presented in Section 2.1, we can fit the SVFR model on the simulated data sets as follows (see Table 10)

$$\hat{\tilde{y}}_i = 5.4306 \oplus (1.5213 \otimes \tilde{x}_i) \oplus (0; 0.5772)_T.$$

Based on the average of goodness of fit indices in 10 simulated data sets in Table 10, we obtain $I = 0.7471$ and $MSM = 0.3946$. Hence, the performance of the proposed method on simulated data sets is suitable.

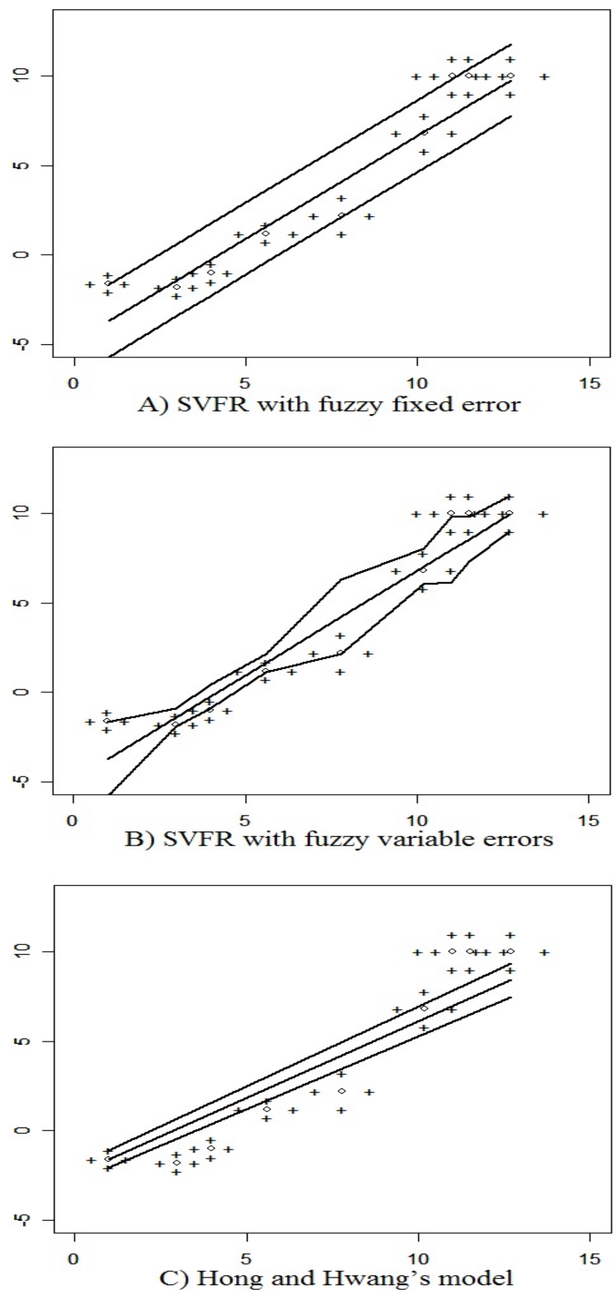

Figure 3: Support vector fuzzy regression models in Example 3.3.

4 Conclusion

A new approach was presented to fit some fuzzy regression models combined with support vectors. This approach has some merits as follows:

1) The approach is a hybrid version of fuzzy regression and support vector machine. We

Table 10: Results of SVFR models in Example 3.4.

	Parameters of model			Goodness of fit	
No.	*b*	*w*	*s*	*I*	*MSM*
Data set 1	5.4650	1.5063	0.5765	0.7514	0.4035
Data set 2	5.4393	1.5169	0.5816	0.7626	0.4340
Data set 3	5.4682	1.5035	0.5637	0.7493	0.3913
Data set 4	5.4218	1.5302	0.5829	0.7411	0.3874
Data set 5	5.4538	1.5235	0.5875	0.7422	0.3983
Data set 6	5.4294	1.5182	0.5817	0.7478	0.3987
Data set 7	5.3951	1.5293	0.5861	0.7438	0.3873
Data set 8	5.4201	1.5185	0.5469	0.7443	0.3729
Data set 9	5.4542	1.5241	0.5911	0.7439	0.3962
Data set 10	5.3589	1.5420	0.5739	0.7440	0.3765
Mean	5.4306	1.5213	0.5772	0.7471	0.3946

called it “support vector fuzzy regression (SVFR)".

2) This approach is based on fuzzy inputs, fuzzy output, crisp parameters, and fuzzy errors.

3) The proposed SVFR models are fitted in two cases: fixed fuzzy errors and variable fuzzy errors

4) By using variable fuzzy errors, we obtain the nonlinear predicted values for fuzzy response variables.

We can investigate some approaches in future researches as follows:

- In addition to fuzzy input-fuzzy output, if the parameters of model are assumed as fuzzy, then the marginal hyperplanes are non-parallel. Hence, we need to introduce a new approach.
- If the fuzzy inputs are complex, we need to map them to the space with higher dimension. Therefore, a new version based on kernel functions needs to be introduced.

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