JIRSS (2025)
Vol. 24, No. 2, pp 50-68
DOI: 10.22034/jirss.2026.2076023.1150

Likelihood Ratio Ordering for Spacings Arising from Multiple-Outlier Models

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Received: 29/10/2025, Accepted: 7/01/2026, Published online: 31/01/2026

Abstract. In this paper, we investigate the stochastic properties of spacings among order statistics derived from a sample of independent, non-negative random variables that are divided into two groups with different distributions. Previous studies have shown that when these distribution functions are exponential distributions with specified hazard rates, the likelihood ratio ordering holds among the spacings under specific conditions. The present work extends these results by considering more general continuous distribution functions. We identify the necessary conditions on the parent distribution functions for preserving the likelihood ratio ordering among spacings in general settings. The comparison results enhance our understanding of stochastic ordering theory and provide valuable insights for applications in reliability, survival analysis, and related fields, aiding in the development of more flexible and accurate statistical models.

Keywords. stochastic order, likelihood ratio order, sample spacings, general multiple-outlier models.

MSC: 62-XX, 62Rxx, 62R10.

1 Introduction

Let $X_{1:n}, \ldots, X_{n:n}$ be the order statistics arising from independent random variables $X_1, \cdots, X_n$, with common distribution function F. Let $D_{i:n} = X_{i:n} - X_{i-1:n}$, $i = 1, \ldots, n$

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denote the ith spacing with $X_{0:n} \equiv 0$. The sample spacings as a function of order statistics have wide applications in many problems such as predictions, tests for outliers, goodness-of-fit tests and stochastic auction theory. For more details on applications of order statistics and of sample spacings, the reader may refer to the books by Balakrishnan and Rao (1998a), Balakrishnan and Rao (1998b). For the case when, X_i's, $i = 1, \ldots, n$ are independent random variables from the exponential distribution with hazard rate λ_i, $i = 1, \ldots, n$, Kochar and Korwar (1996) showed that the survival function of $D_{i:n}$ is given by

$$\overline{F}_{D_{i:n}}(x) = \sum_{r \in A} p_i(r) e^{-\lambda_i(r)x}, \qquad x > 0,$$

where A is a permutation of $(1, \cdots, n)$, $r = (r_1, \cdots, r_n)$ is a member of A, $\lambda_i(r) = \sum_{j=i}^{n} \lambda(r_j)$ and $p_i(r) = \frac{\prod_{i=1}^{n} \lambda_i}{\prod_{i=1}^{n} \sum_{i=j}^{n} \lambda(r_j)}$.

First, we review some notations of stochastic orderings. Let X and Y be two non-negative continuous random variables with density functions f_X and f_Y, distribution functions F_X and F_Y, survival functions $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y$, hazard rate functions $h_X = f_X/\overline{F}_X$ and h_Y, and reversed hazard rate functions $\tilde{h}_X = f_X/F_X$ and $\tilde{h}_Y$, respectively.

(a) X is said to be larger than Y in the *likelihood ratio order* (denoted by $F_X \leq_{lr} F_Y$) if $\frac{f_Y(t)}{f_X(t)}$ is increasing in t.

(b) X is said to be larger than Y in the *hazard rate order* (denoted by $F_X \leq_{hr} F_Y$) if $\frac{\overline{F}_Y(t)}{\overline{F}_X(t)}$ is increasing in t, or, equivalently, $h_X(t) \geq h_Y(t)$ for all t.

(c) X is said to be larger than Y in the *reversed hazard rate order* (denoted by $F_X \leq_{rh} F_Y$) if $\frac{F_Y(t)}{F_X(t)}$ is increasing in t, or, equivalently, $\tilde{h}_X(t) \geq \tilde{h}_Y(t)$ for all t.

(d) X is said to be larger than Y in the usual stochastic order (denoted by $F_X \leq_{st} F_Y$) if $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for all t.

It is well known that

$$X \geq_{lr} Y \Longrightarrow X \geq_{hr[rh]} Y \Longrightarrow X \geq_{st} Y,$$

but neither the reversed hazard rate nor hazard rate orders imply each other. one may refer to Shaked and Shanthikumar (2007) for more details. Kochar and Korwar (1996) conducted stochastic orders between spacings and showed that $D_{2:3} \leq_{hr} D_{3:3}$. Also, in the case when X_i's are independent and identically distribution with log-convex density function, Kochar and Kirmani (1995) showed that $D_{i:n} \leq_{lr} D_{i+1:n}$. Let $X_1, \ldots, X_n$ be a set of non-negative independent random variables where X_i, $i = 1, \ldots, p$, $p \geq 1$, having common distribution function F_1, and X_j, $j = p+1, \ldots, n$, having common distribution function F_2, and two samples are independent. For the case when F_1 is the exponential distribution function with hazard rate λ_1 and F_2 is the exponential distribution function with hazard rate λ_2, Wen et al. (2007) proved that for $\lambda_1 \leq \lambda_2$, $D_{i:n}(p, q) \leq_{lr} D_{i:n}(p-1, q+1)$.

After that, Xu et al. (2007) investigated the likelihood ratio order of m-spacings from the multiple-outlier exponential model. They proved that for $i = 1, \ldots, m-n$, $D^{(m)}_{i:n}(p,q) \leq_{lr} D^{(m)}_{i:n}(p-1,q+1)$, where, $D^{(m)}_{i:n}(p,q) = X_{m+i-1:n}(p,q) - X_{i-1:n}(p,q)$. Motivated by the above analysis, in this framework and in general case, we focus our attention on investigating the stochastic comparisons of the spacings $D^{(1)}_{i:n}(p,q)$ and $D^{(1)}_{i:n}(p-1,q+1)$ with respect to the likelihood ratio order. The study of spacings in the presence of outliers, as deal with in this paper, is motivated by both theoretical advancements in stochastic ordering and practical applications in fields where heterogeneous data arise. Specifically, our work extends the understanding of how outliers, modeled by a distinct distribution F_2 compared to inliers F_1, affect the stochastic properties of spacings (differences between consecutive order statistics) under the likelihood ratio order. This is crucial for analyzing ordered data in heterogeneous populations, which is common in real-world scenarios. The theoretical significance lies in generalizing existing results, such as those in Wen et al. (2007) and Xu et al. (2007), which assume exponential distributions, to arbitrary continuous distributions using permanents. Furthermore, Torrado and Lillo (2015) obtained some new results in the area of stochastic comparisons of simple and normalized spacings from two samples of heterogeneous exponential random variables, Barmalzan et al. (2015) studied some ordering results for the sample spacings arising from the single and multiple-outlier exponential models.

Zhao et al. (2016) extended the analysis to the second sample spacing arising from multiple-outlier exponential models and obtained its stochastic ordering properties in terms of the likelihood ratio order.

This consider open questions about the robustness of stochastic ordering results in non-identical distribution settings, particularly under the strong likelihood ratio order. Practically, our results have direct applications in reliability analysis, where spacings represent inter-failure times in systems with defective components (outliers). For example, understanding how an increased number of outliers affects spacing distributions can inform the design of robust systems in engineering. In risk management, our findings are relevant for modeling extreme events, where outliers represent rare but impactful occurrences, such as financial crashes or environmental anomalies. Additionally, in anomaly detection, the likelihood ratio ordering of spacings can help quantify the impact of outliers on data clustering, aiding in the identification of unusual observations in datasets like sensor networks or quality control processes. To illustrate a practical application, consider a reliability setting with a system consisting of n components where p are standard components (inliers) from distribution F_1 and q are potentially weaker components (outliers) from F_2 with $F_1 \leq_{lr} F_2$ (implying higher failure propensity for outliers). The spacings $D_{i:n}(p,q)$ represent times between successive failures in the combined system. Our main result shows that replacing an inlier with an outlier (shifting to $(p-1, q+1)$) makes the ith spacing stochastically larger in the likelihood ratio sense, meaning longer expected times between failures around that point or increased variability. This insight can guide engineers in assessing the impact of introducing heterogeneous (potentially faulty) components on system reliability and maintenance scheduling.

The main result demonstrates that spacings from a sample with more outliers ($q+1$) are stochastically larger in the likelihood ratio sense than those with fewer outliers (q). This finding has direct practical implications in several areas. Here, we present some application where ordering of spacings can be applied.

- The likelihood ratio ordering of spacings, $D_{i:n}(p,q) \leq_{lr} D_{i:n}(p-1,q+1)$, established in this work, has significant practical implications. In statistical testing, it enhances the robustness of goodness-of-fit and homogeneity tests by quantifying the effect of outliers on spacing distributions, applicable in quality control for detecting defective items.
- In auction theory, the result informs the modeling of bid differences in the presence of outlier bidders, aiding in auction design. Additionally, in reliability analysis, it supports the assessment of inter-failure times in heterogeneous systems, while in anomaly detection, it improves the identification of unusual patterns in data streams, such as in sensor networks.
- In financial modeling, the result contributes to risk management by characterizing intervals between extreme events influenced by outliers. These applications highlight the relevance of our findings in handling heterogeneous data across diverse fields.

In this paper, we establish the likelihood ratio order for the spacings arising from the general multiple-outlier models when the numbers of identically distributed variables are different. In general case, we show that the likelihood ratio order of spacings from $X_1, \ldots, X_p, X_{p+1}, \ldots, X_n$ is a function of the likelihood ratio order between inlier and outlier random variables. Specifically, this work establishes the likelihood ratio ordering of spacings, $D_{i:n}(p,q) \leq_{lr} D_{i:n}(p-1,q+1)$, for general continuous distributions F_1 and F_2 satisfying $F_1 \leq_{lr} F_2$. Unlike previous studies, such as Wen et al. (2007) and Xu et al. (2007), which focused on exponential distributions, our approach uses the theory of permanents to derive the joint density of order statistics, enabling a general framework. The main result demonstrates that spacings from samples with more outliers are stochastically larger, providing a robust tool for analyzing heterogeneous data.

2 Main result

The key tool used to prove the main result is the theory of permanents. We first recall the definition of permanents, and introduce some useful notation. It is useful to represent the joint density functions of order statistics by using the theory of permanents when the underlying random variables are not identical. If $\mathbf{A} = \{a\}_{i,j}$ is an $n \times n$ matrix, then the permanent of $\mathbf{A}$ is defined as

$$perm\{A\} = \sum_{\sigma} \prod_{i=1}^{n} \{a\}_{i,\sigma(i)},$$

where the summation is over all permutations $\sigma = (\sigma(1), \ldots, \sigma(n))$ of $\{1, \ldots, n\}$. If $\mathbf{v}_1, \mathbf{v}_2, \ldots$ are column random vectors on $\mathbb{R}^n$, then the permanent

$$\Bigg[\underbrace{\mathbf{v}_1}_{r_1}, \underbrace{\mathbf{v}_1}_{r_2}, \ldots\Bigg],$$

is obtained by taking r_1 copies of $\mathbf{v}_1$, r_2 copies of $\mathbf{v}_1$ and so on.

Before stating our main result, let us first introduce the following two lemmas, which are also needed to prove the main result.

Lemma 1. *(?). Let $\boldsymbol{a}_1, ..., \boldsymbol{a}_{n-1}$ be non-negative vectors in $\mathbb{R}^n$, and let $\boldsymbol{b}$ be any vector in $\mathbb{R}^n$, where $n \geq 2$. Then*

$$[\boldsymbol{a}_1, ..., \boldsymbol{a}_{n-1}, \boldsymbol{b}]^2 \geq [\boldsymbol{a}_1, ..., \boldsymbol{a}_{n-1}, \boldsymbol{a}_{n-1}]\,[\boldsymbol{a}_1, ..., \boldsymbol{a}_{n-2}, \boldsymbol{b}, \boldsymbol{b}]\,.$$

Lemma 2. *(Misra and van der Meulen (2003)). Let Θ be a subset of real line and U be a non-negative random variable having a cdf belonging to a stochastically ordered family $P = \{H(\cdot|\theta), \theta \in \Theta\}$, that is, for $\theta_1, \theta_2 \in \Theta$, $H(\cdot|\theta_1) \leq_{st} (\geq_{st}) H(\cdot|\theta_2)$ whenever $\theta_1 \leq \theta_2$. Suppose a real function $\psi(u, \theta)$ on $\mathbb{R} \times \Theta$ is measurable in u for each θ such that $E_\theta[\psi(U, \theta)]$ exists. Then*

(i) $E_\theta[\psi(U, \theta)]$ is increasing in θ if $\psi(u, \theta)$ increasing in θ and increasing(decreasing) in u.

(ii) $E_\theta[\psi(U, \theta)]$ is decreasing in θ if $\psi(u, \theta)$ decreasing in θ and decreasing(increasing) in u.

Theorem 3. *Let $X_1, \ldots, X_n$ be independent random variables follow the multiple-outlier model with X_i's, $i = 1, \ldots, p$ having the distribution function F_1, for $i = 1, \ldots, p$ and X_j's, $j = p+1, \ldots, n$ having the common distribution function F_2, where two samples are independent. If $F_1 \leq_{lr} F_2$, then for $i = 1, \ldots, n$,*

$$D_{i:n}(p, q) \leq_{lr} D_{i:n}(p-1, q+1),$$

where $q = n - p$.

Proof. The proof proceeds in three main stages: (1) expressing the joint densities of the relevant order statistics using permanents of structured matrices involving f_1, f_2, and F_1, F_2; (2) deriving closed-form expressions for the densities of the spacings $D_{i:n}(p, q)$ and $D_{i:n}(p-1, q+1)$ by integrating these joint densities; (3) showing that the ratio of these densities is increasing in x by establishing the monotonicity of a key ratio involving permanents, leveraging the likelihood ratio order $F_1 \leq_{lr} F_2$ to ensure the required inequalities hold through sign-preserving properties and matrix factorizations.

Let us define for each pair (p, q),

$$[\mathbf{F}(x)]_{p,q} = \begin{bmatrix} F(x)\mathbf{1}_p \\ F(x)\mathbf{1}_q \end{bmatrix}_{p,q}, [\overline{F}(x)]_{p,q} = \begin{bmatrix} \overline{F}(x)\mathbf{1}_p \\ \overline{F}(x)\mathbf{1}_q \end{bmatrix}_{p,q},$$

$$[\mathbf{f}(x)]_{p,q} = \begin{bmatrix} f(x)\mathbf{1}_p \\ f(x)\mathbf{1}_q \end{bmatrix}_{p,q},$$

where $p \geq 1$, $p + q = n \geq 2$ and the entries of both $\mathbf{1}_p$ and $\mathbf{1}_q$ are all ones. Note that each matrix is an $n \times n$ matrix with structured repetitions based on p and q.

The joint density function of $X_{i:n}(p,q)$ and $X_{i-1:n}(p,q)$ is given by

$$\begin{aligned} f_{X_{i-1:n}(p,q),X_{i:n}(p,q)}(x,y) &= \frac{1}{(i-1)!(n-i)!}\left(pf_1(x)\left[\underbrace{\mathbf{F}(x)}_{i-2}, \mathbf{f}(y), \underbrace{\overline{F}(y)}_{n-i}\right]_{p-1,q}\right. \\ &\quad \left. + qf_2(x)\left[\underbrace{\mathbf{F}(x)}_{i-2}, \mathbf{f}(y), \underbrace{\overline{F}(y)}_{n-i}\right]_{p,q-1}\right) \\ &= \frac{1}{(i-1)!(n-i)!}\left[\underbrace{\mathbf{F}(x)}_{i-2}, \mathbf{f}(x), \mathbf{f}(y), \underbrace{\overline{F}(y)}_{n-i}\right]_{p,q}. \end{aligned} \tag{2.1}$$

From (2.1) the density function of $D_{i:n}(p-1,q+1)$ is given by

$$f_{D_{i:n}(p-1,q+1)}(x) = \frac{1}{(i-1)!(n-i)!}\int_0^\infty \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(t+x), \underbrace{\overline{F}(t+x)}_{n-i}\right]_{p-1,q+1} dt.$$

Similarly, we have

$$f_{D_{i:n}(p,q)}(x) = \frac{1}{(i-1)!(n-i)!}\int_0^\infty \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(t+x), \underbrace{\overline{F}(t+x)}_{n-i}\right]_{p,q} dt.$$

To reach the desired result, it suffices to prove that

$$
\begin{aligned}
\frac{f_{D_{i:n}(p-1,q+1)}(x)}{f_{D_{i:n}(p,q)}(x)} &= \frac{\int_0^\infty \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(t+x), \underbrace{\overline{F}(t+x)}_{n-i}\right]_{p-1,q+1} dt}{\int_0^\infty \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(t+x), \underbrace{\overline{F}(t+x)}_{n-i}\right]_{p,q} dt} \\
&= \int_0^\infty \frac{\left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(t+x), \underbrace{\overline{F}(t+x)}_{n-i}\right]_{p-1,q+1}}{\left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(t+x), \underbrace{\overline{F}(t+x)}_{n-i}\right]_{p,q}} \frac{\left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(t+x), \underbrace{\overline{F}(t+x)}_{n-i}\right]_{p-1,q+1}}{\int_0^\infty \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(t+x), \underbrace{\overline{F}(t+x)}_{n-i}\right]_{p,q} dt} dt \\
&= \int_0^\infty \Psi(t, t+x)\mathbf{h}(t|x)dt \\
&= E_x[\Psi(T, T+x)], \qquad (2.2)
\end{aligned}
$$

is increasing in x, where, the distribution function of the random variable T belongs to a family of conditional distributions parameterized by x, $P = \{H(\cdot|x), x \in \mathbb{R}^+\}$, with density function

$$
\mathbf{h}(t|x) = a(x) \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(t+x), \underbrace{\overline{F}(t+x)}_{n-i}\right]_{p,q}
$$

and a normalizing constant

$$
a(x) = \frac{1}{\int_0^\infty \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(t+x), \underbrace{\overline{F}(t+x)}_{n-i}\right]_{p,q} dt},
$$

such that $\int_0^\infty \mathbf{h}(t|x)dt = 1$ and

$$\Psi(t, t+x) = \frac{\left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(t+x), \underbrace{\overline{F}(t+x)}_{n-i}\right]_{p-1,q+1}}{\left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(t+x), \underbrace{\overline{F}(t+x)}_{n-i}\right]_{p,q}}.$$

For convenience, we define that

$$\Psi(t, x) = \frac{\left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i}\right]_{p-1,q+1}}{\left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i}\right]_{p,q}}. \tag{2.3}$$

We use Lemma 2 to prove that (2.2) is increasing in x. Therefore, it is enough to verify that $\Psi(t, x)$ is increasing in t and x and for any $x_1 \le x_2$, $H(\cdot|x_1) \le_{st} H(\cdot|x_2)$. In the following parts (a) and (b), we will prove that $\Psi(t, x)$ is increasing in t and x, respectively.

Part (a): For convenience, we introduce the following notations:

$$\mathbf{A}_{p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i}\right]_{p-1,q+1}, \quad \mathbf{A}_{1,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-3}, \mathbf{f}(t), \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i}\right]_{p-2,q+1},$$

$$\mathbf{A}_{2,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-3}, \mathbf{f}(t), \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i}\right]_{p-1,q}, \quad \mathbf{A}_{3,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i}\right]_{p-2,q+1},$$

$$\mathbf{A}_{4,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i}\right]_{p-1,q}, \quad \mathbf{B}_{p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i}\right]_{p,q}$$

$$\mathbf{B}_{1,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-3}, \mathbf{f}(t), \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i}\right]_{p-1,q}, \quad \mathbf{B}_{2,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-3}, \mathbf{f}(t), \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i}\right]_{p,q-1}$$

$$\mathbf{B}_{3,p,q}(t,x)=\left[\underbrace{\mathbf{F}(t)}_{i-2},\mathbf{f}(x),\underbrace{\overline{F}(x)}_{n-i}\right]_{p-1,q},\quad \mathbf{B}_{4,p,q}(t,x)=\left[\underbrace{\mathbf{F}(t)}_{i-2},\mathbf{f}(x),\underbrace{\overline{F}(x)}_{n-i}\right]_{p,q-1}.$$

For a clearer illustration of the structure and dimensions of the matrices $\mathbf{A}_{p,q}(t,x)$, $\mathbf{B}_{p,q}(t,x)$, and $\mathbf{C}_{p,q}(t,x)$ used in the permanent expressions, explicit examples for small values of n, p, and q are provided in the Appendix.

According to the Laplace expansion, it follows that

$$pF_1(x)\mathbf{A}_{1,p,q}(t,x)+qF_2(x)\mathbf{A}_{2,p,q}(t,x)=\mathbf{A}_{p,q}(t,x)=pf_1(x)\mathbf{A}_{3,p,q}(t,x)+qf_2(x)\mathbf{A}_{4,p,q}(t,x)$$

and

$$pF_1(x)\mathbf{B}_{1,p,q}(t,x)+qF_2(x)\mathbf{B}_{2,p,q}(t,x)=\mathbf{B}_{p,q}(t,x)=pf_1(x)\mathbf{B}_{3,p,q}(t,x)+qf_2(x)\mathbf{B}_{4,p,q}(t,x).$$

Upon using the Laplace expansion, the derivative $\Psi(t,x)$ with respect to t follows that

$$\begin{aligned}\frac{\partial}{\partial t}\Psi(t,x) &\overset{sgn}{=} \mathbf{B}_{p,q}(t,x)\{(i-2)(p-1)f_1(t)\mathbf{A}_{1,p,q}(t,x)+(i-2)(q+1)f_2(t)\mathbf{A}_{2,p,q}(t,x)\\ &\quad+(p-1)f_1'(t)\mathbf{A}_{3,p,q}(t,x)+(q+1)f_2'(t)\mathbf{A}_{4,p,q}(t,x)\}\\ &\quad-\mathbf{A}_{p,q}(t,x)\{(i-2)pf_1(t)\mathbf{B}_{1,p,q}(t,x)+(i-2)qf_2(t)\mathbf{B}_{2,p,q}(t,x)\\ &\quad+pf_1'(t)\mathbf{B}_{3,p,q}(t,x)+qf_2'(t)\mathbf{B}_{4,p,q}(t,x)\}\\ &= \Upsilon_1(t,x)+\Upsilon_2(t,x),\end{aligned}$$

where $\overset{sgn}{=}$ means that two sides of the equality have the same sign and

$$\begin{aligned}\Upsilon_1(t,x) &= (i-2)^2[p(q+1)\mathbf{A}_{2,p,q}(t,x)\mathbf{B}_{1,p,q}(t,x)-(p-1)q\mathbf{A}_{1,p,q}(t,x)\mathbf{B}_{2,p,q}(t,x)]\\ &\quad\times[f_2(t)F_1(t)-f_1(t)F_2(t)]\\ &= (i-2)^2I_1\times I_2\end{aligned}$$

and

$$\begin{aligned}\Upsilon_2(t,x) &= [p(q+1)\mathbf{A}_{4,p,q}(t,x)\mathbf{B}_{3,p,q}(t,x)-(p-1)q\mathbf{A}_{3,p,q}(t,x)\mathbf{B}_{4,p,q}(t,x)]\\ &\quad\times[f_2'(t)f_1(t)-f_1'(t)f_2(t)]\\ &= J_1\times J_2.\end{aligned}$$

We know that $\mathbf{A}_{2,p,q}(t,x)=\mathbf{B}_{1,p,q}(t,x)$, $\mathbf{A}_{4,p,q}(t,x)=\mathbf{B}_{3,p,q}(t,x)$ and $p+q>1$. Therefore, from Lemma 1, we have

$$\mathbf{A}^2_{2,p,q}(t,x)\geq\mathbf{A}_{1,p,q}(t,x)\mathbf{B}_{2,p,q}(t,x)$$

and

$$\mathbf{A}^2_{4,p,q}(t,x)\geq\mathbf{A}_{3,p,q}(t,x)\mathbf{B}_{4,p,q}(t,x).$$

Hence, $I_1\geq 0$ and $J_1\geq 0$. Next, one can see that

$$I_2\overset{sgn}{=}\frac{f_2(t)}{F_2(t)}-\frac{f_1(t)}{F_1(t)}=\tilde{h}_{F_2}(t)-\tilde{h}_{F_1}(t)\geq 0, \tag{2.4}$$

where $\tilde{h}$ is the reverse hazard rate function. Due to the fact that the likelihood ratio order implies the reversed hazard rate order, the inequality in (2.4) follows from the assumption that $F_1 \leq_{lr} F_2$. Likewise,

$$J_2 \overset{sgn}{=} \frac{f_2'(t)}{f_2(t)} - \frac{f_1'(t)}{f_1(t)} \geq 0,$$

where the inequality follows from the assumption of the theorem that $F_1 \leq_{lr} F_2$. Therefore , from these observations we have $\frac{\partial}{\partial t}\Psi(t,x) \geq 0$.

Part (b): Similarly, before taking the derivative $\Psi(t,x)$ with respect to x, we introduce the following notations:

$$\mathbf{C}_{1,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \underbrace{\overline{F}(x)}_{n-i}\right]_{p-2,q+1}, \quad \mathbf{C}_{2,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \underbrace{\overline{F}(x)}_{n-i}\right]_{p-1,q},$$

$$\mathbf{C}_{3,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i-1}\right]_{p-2,q+1}, \quad \mathbf{C}_{4,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i-1}\right]_{p-1,q},$$

$$\mathbf{D}_{1,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \underbrace{\overline{F}(x)}_{n-i}\right]_{p-1,q}, \quad \mathbf{D}_{2,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \underbrace{\overline{F}(x)}_{n-i}\right]_{p,q-1},$$

$$\mathbf{D}_{3,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i-1}\right]_{p-1,q}, \quad \mathbf{D}_{4,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(x), \underbrace{\overline{F}(x)}_{n-i-1}\right]_{p,q-1}.$$

According to the Laplace expansion, we have

$$p\overline{F}_1(x)\mathbf{C}_{3,p,q}(t,x) + q\overline{F}_2(x)\mathbf{C}_{4,p,q}(t,x) = \mathbf{C}_{p,q}(t,x) = pf_1(x)\mathbf{C}_{1,p,q}(t,x) + qf_2(x)\mathbf{C}_{2,p,q}(t,x)$$

and

$$p\overline{F}_1(x)\mathbf{D}_{3,p,q}(t,x) + q\overline{F}_2(x)\mathbf{D}_{4,p,q}(t,x) = \mathbf{D}_{p,q}(t,x) = pf_1(x)\mathbf{D}_{1,p,q}(t,x) + qf_2(x)\mathbf{D}_{2,p,q}(t,x).$$

Again, upon using the Laplace expansion, it follows that

$$\begin{aligned}
\frac{\partial}{\partial x}\Psi(t,x) \overset{sgn}{=}\; & \mathbf{B}_{p,q}(t,x)\{(p-1)f_1'(x)\mathbf{C}_{1,p,q}(t,x) + (q+1)f_2'(x)\mathbf{C}_{2,p,q}(t,x) \\
& -(n-i)(p-1)f_1(x)\mathbf{C}_{3,p,q}(t,x) - (n-i)(q+1)f_2(x)\mathbf{C}_{4,p,q}(t,x)\} \\
& -\mathbf{A}_{p,q}(t,x)\{pf_1'(x)\mathbf{D}_{1,p,q}(t,x) + qf_2'(x)\mathbf{D}_{2,p,q}(t,x) \\
& -(n-i)pf_1(x)\mathbf{D}_{3,p,q}(t,x) - (n-i)qf_2(x)\mathbf{D}_{4,p,q}(t,x)\} \\
=\; & \Lambda_1(x) + \Lambda_2(x),
\end{aligned}$$

where

$$\begin{aligned}\Lambda_1(x) &= (n-i)^2[p(q+1)\mathbf{C}_{2,p,q}(t,x)\mathbf{D}_{1,p,q}(t,x)-(p-1)q\mathbf{C}_{1,p,q}(t,x)\mathbf{D}_{2,p,q}(t,x)]\\ &\quad\times[f_1(x)\overline{F}_2(x)-f_2(x)\overline{F}_1(x)]\\ &= (n-i)^2 I_1\times I_2\end{aligned}$$

and

$$\begin{aligned}\Lambda_2(x) &= [p(q+1)\mathbf{C}_{4,p,q}(t,x)\mathbf{D}_{3,p,q}(t,x)-(p-1)q\mathbf{C}_{3,p,q}(t,x)\mathbf{D}_{4,p,q}(t,x)]\\ &\quad\times[f_2'(x)f_1(x)-f_1'(x)f_2(x)]\\ &= J_1\times J_2.\end{aligned}$$

We know that $\mathbf{C}_{2,p,q}(t,x)=\mathbf{D}_{1,p,q}(t,x)$ and $\mathbf{C}_{4,p,q}(t,x)=\mathbf{D}_{3,p,q}(t,x)$. Therefore, from Lemma 1, it follows that

$$\mathbf{C}^2_{2,p,q}(t,x)\geq \mathbf{C}_{1,p,q}(t,x)\mathbf{D}_{2,p,q}(t,x)$$

and

$$\mathbf{C}^2_{4,p,q}(t,x)\geq \mathbf{C}_{3,p,q}(t,x)\mathbf{D}_{4,p,q}(t,x),$$

which is reveals that $I_1\geq 0$ and $J_1\geq 0$. Next, we have

$$I_2 \overset{sgn}{=} \frac{f_1(x)}{\overline{F}_1(x)}-\frac{f_2(x)}{\overline{F}_2(x)} = h_{F_1}(t)-h_{F_2}(t)\geq 0, \tag{2.5}$$

where h is the hazard rate function. Due to the fact that the likelihood ratio order implies the hazard rate order, the inequality in (2.5) follows from the assumption that $F_1\leq_{lr} F_2$. Also, we have

$$J_2 \overset{sgn}{=} \frac{f_2'(x)}{f_2(x)}-\frac{f_1'(x)}{f_1(x)}\geq 0.$$

Therefore, from these observations we have $\frac{\partial}{\partial x}\Psi(t,x)\geq 0$.

Finally, according to Lemma 1, it is enough to show that $H(\cdot|x_1)\leq_{st} H(\cdot|x_2)$, $x_1\leq x_2$. Due to the fact that the likelihood ratio order implies the usual stochastic order, it is enough to show that

$$\Phi(t)=\frac{\mathbf{h}(t|x_2)}{\mathbf{h}(t|x_1)} \;=\; \frac{\mathbf{A}^*_{p,q}(t,x_2)}{\mathbf{B}^*_{p,q}(t,x_1)};\quad \text{whenever } x_1\leq x_2,$$

is increasing in t. In the following two steps, we will prove that $\Phi'(t)\geq 0$.

Step 1: Using the same argument used to prove **P**art (a), in this step we will show that the derivative of $\Phi'(t)$ along the first $i-2$ columns and the second column is non-negative. Before to do that likewise to previous parts, for convenient we have the following notations.

$$\mathbf{A}^*_{p,q}(t,x_2)=\left[\underbrace{\mathbf{F}(t)}_{i-2},\mathbf{f}(t),\mathbf{f}(t+x_2),\underbrace{\overline{\mathbf{F}}(t+x_2)}_{n-i}\right]_{p,q}$$

$$\mathbf{A}^*_{1,p,q}(t,x_2)=\left[\underbrace{\mathbf{F}(t)}_{i-3},\mathbf{f}(t),\mathbf{f}(t+x_2),\underbrace{\overline{\mathbf{F}}(t+x_2)}_{n-i}\right]_{p-1,q},$$

$$\mathbf{A}^*_{2,p,q}(t,x_2)=\left[\underbrace{\mathbf{F}(t)}_{i-3},\mathbf{f}(t),\mathbf{f}(t+x_2),\underbrace{\overline{\mathbf{F}}(t+x_2)}_{n-i}\right]_{p,q-1},$$

$$\mathbf{A}^*_{3,p,q}(t,x_2)=\left[\underbrace{\mathbf{F}(t)}_{i-2},\mathbf{f}(t+x_2),\underbrace{\overline{\mathbf{F}}(t+x_2)}_{n-i}\right]_{p-1,q},$$

$$\mathbf{A}^*_{4,p,q}(t,x_2)=\left[\underbrace{\mathbf{F}(t)}_{i-2},\mathbf{f}(t+x_2),\underbrace{\overline{\mathbf{F}}(t+x_2)}_{n-i}\right]_{p,q-1},$$

$$\mathbf{B}^*_{p,q}(t,x_1)=\left[\underbrace{\mathbf{F}(t)}_{i-2},\mathbf{f}(t),\mathbf{f}(t+x_1),\underbrace{\overline{\mathbf{F}}(t+x_1)}_{n-i}\right]_{p,q}$$

$$\mathbf{B}^*_{1,p,q}(t,x_1)=\left[\underbrace{\mathbf{F}(t)}_{i-3},\mathbf{f}(t),\mathbf{f}(t+x_1),\underbrace{\overline{\mathbf{F}}(t+x_1)}_{n-i}\right]_{p-1,q},$$

$$\mathbf{B}^*_{2,p,q}(t,x_1)=\left[\underbrace{\mathbf{F}(t)}_{i-3},\mathbf{f}(t),f(t+x_1),\underbrace{\overline{\mathbf{F}}(t+x_1)}_{n-i}\right]_{p,q-1},$$

$$\mathbf{B}^*_{3,p,q}(t,x)=\left[\underbrace{\mathbf{F}(t)}_{i-2},\mathbf{f}(t+x_1),\underbrace{\overline{\mathbf{F}}(t+x_1)}_{n-i}\right]_{p-1,q},$$

$$\mathbf{B}^*_{4,p,q}(t,x_1)=\left[\underbrace{\mathbf{F}(t)}_{i-2},\mathbf{f}(t+x_1),\underbrace{\overline{\mathbf{F}}(t+x_1)}_{n-i}\right]_{p,q-1}.$$

Then, it holds that

$$\Phi'(t) \overset{sgn}{=} \Upsilon_1^*(t) + \Upsilon_2^*(t),$$

where

$$\begin{aligned}\Upsilon_1^*(t) &= (i-2)^2 pq[\mathbf{A}^*_{2,p,q}(t,x_2)\mathbf{B}^*_{1,p,q}(t,x_1) - \mathbf{A}^*_{1,p,q}(t,x_2)\mathbf{B}^*_{2,p,q}(t,x_1)] \\ &\quad \times[f_2(t)F_1(t) - f_1(t)F_2(t)] \\ &= pq(i-2)^2 I_1^* \times I_2\end{aligned}$$

and

$$\begin{aligned}\Upsilon_2^*(t) &= pq[\mathbf{A}^*_{4,p,q}(t,x_2)\mathbf{B}^*_{3,p,q}(t,x_1) - \mathbf{A}^*_{3,p,q}(t,x_2)\mathbf{B}^*_{4,p,q}(t,x_1)] \\ &\quad \times[f_2'(t)f_1(t) - f_1'(t)f_2(t)] \\ &= pqJ_1^* \times J_2. \end{aligned} \tag{2.6}$$

Using the same argument in **P**art (a), it reveals that $I_2 \geq 0$ and $J_2 \geq 0$. Next, our task is to show that I_1^* and J_1^* are also non-negative. In the following, we firstly show the non-negativity of I_1^*. This is equivalent to show that

$$\frac{\mathbf{A}^*_{2,p,q}(t,x_2)}{\mathbf{A}^*_{1,p,q}(t,x_2)} \geq \frac{\mathbf{B}^*_{2,p,q}(t,x_1)}{\mathbf{B}^*_{1,p,q}(t,x_1)}.$$

Set $t + x_2 = u$ and $t + x_1 = v$, we have $u \geq v$, therefore, from which we only need to show that

$$\Lambda_1(u) = \frac{\mathbf{A}^*_{2,p,q}(t,u)}{\mathbf{A}^*_{1,p,q}(t,u)},$$

is increasing in u. Let us define the following notations

$$\mathbf{C}^*_{1,p,q}(t,u) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \underbrace{\overline{F}(u)}_{n-i}\right]_{p-2,q}, \quad \mathbf{C}^*_{2,p,q}(t,u) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \underbrace{\overline{F}(u)}_{n-i}\right]_{p-1,q-1},$$

$$\mathbf{C}^*_{3,p,q}(t,u) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(u), \underbrace{\overline{F}(x)}_{n-i-1}\right]_{p-2,q}, \quad \mathbf{C}^*_{4,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(u), \underbrace{\overline{F}(u)}_{n-i-1}\right]_{p-1,q-1},$$

$$\mathbf{D}^*_{1,p,q}(t,u) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \underbrace{\overline{F}(u)}_{n-i}\right]_{p-1,q-1}, \quad \mathbf{D}^*_{2,p,q}(t,u) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \underbrace{\overline{F}(u)}_{n-i}\right]_{p,q-2},$$

$$\mathbf{D}^*_{3,p,q}(t,u) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(u), \underbrace{\overline{F}(u)}_{n-i-1} \right]_{p-1,q-1}, \quad \mathbf{D}^*_{4,p,q}(t,x) = \left[\underbrace{\mathbf{F}(t)}_{i-2}, \mathbf{f}(t), \mathbf{f}(u), \underbrace{\overline{F}(u)}_{n-i-1} \right]_{p,q-2}.$$

Using proof of **P**art (b), the sing of derivative $\Lambda_1(u)$ with respect to u can be evaluated as the following

$$\Lambda_1'(u) \stackrel{sgn}{=} \xi_1^*(u) + \xi_2^*(u),$$

where

$$\begin{aligned} \xi_1^*(u) &= (n-i)^2[pq\mathbf{C}^*_{2,p,q}(t,u)\mathbf{D}^*_{1,p,q}(t,u) - (p-1)(q-1)\mathbf{C}^*_{1,p,q}(t,u)\mathbf{D}^*_{2,p,q}(t,u)] \\ &\quad \times[f_1(u)\overline{F}_2(u) - f_2(u)\overline{F}_1(u)] \\ &= (n-i)^2 A_1 \times I_2 \end{aligned}$$

and

$$\begin{aligned} \xi_2^*(u) &= [pq\mathbf{C}^*_{4,p,q}(t,u)\mathbf{D}^*_{3,p,q}(t,u) - (p-1)(q-1)\mathbf{C}^*_{3,p,q}(t,u)\mathbf{D}^*_{4,p,q}(t,u)] \\ &\quad \times[f_2'(x)f_1(u) - f_1'(u)f_2(u)] \\ &= B_1 \times J_2. \end{aligned}$$

We know that $\mathbf{C}^*_{2,p,q}(t,u) = \mathbf{D}^*_{1,p,q}(t,u)$ and also $\mathbf{C}^*_{4,p,q}(t,u) = \mathbf{D}^*_{3,p,q}(t,u)$. Therefore, using the similar way in proof of the previous parts, one can be proved that A_1, I_2, B_1 and J_2 are non-negative, which implies that $\xi_1^*(u)$ and $\xi_2^*(u)$ are non-negative. Hence, we obtain that $\Lambda_1(u)$ is an increasing function in u for the first $i-2$ columns and the second column. Hence, Υ_1^* is non-negative. Finally, to show that Υ_2^* is also non-negative, it suffices to show that J_1^* is non-negative, or equivalently,

$$\frac{\mathbf{A}^*_{4,p,q}(t,x_2)}{\mathbf{A}^*_{3,p,q}(t,x_2)} \geq \frac{\mathbf{B}^*_{4,p,q}(t,x_1)}{\mathbf{B}^*_{3,p,q}(t,x_1)},$$

where $x_1 \leq x_2$. This is equivalent to saying that

$$\Lambda_2(u) = \frac{\mathbf{A}^*_{4,p,q}(t,u)}{\mathbf{A}^*_{3,p,q}(t,u)},$$

is increasing in u. Using the same argument to prove that $\Lambda_1(u)$ is increasing in u, it is easy to be proved. Hence, Υ_2^* is also non-negative. Therefore, $\Phi'(t) \geq 0$, which this reveals that $\Phi(t)$ is increasing in t along the first $i-2$ columns and the second column.

Step 2: In this step, our task is to show that $\Phi(t)$ is also increasing in t along the third columns and the forth $n-2$ column. Using the same argument in **S**tep 1, the proof is validated. □ □

Note that the condition $F_1 \leq_{lr} F_2$ is a sufficient condition for the likelihood ratio ordering between spacings and applies to all continuous distributions satisfying this stochastic order. This includes many common parametric families, such as exponential, Weibull (with common shape parameter), gamma (with common shape parameter), and normal (with common variance) distributions, where the order between their parameters implies the likelihood ratio order. While the assumption is strong, it provides a general and verifiable condition under which the result holds uniformly across a wide class of distributions.

As be mentioned in the introduction, Xu et al. (2007) established the likelihood ratio order between m-spacings in multiple-outlier exponential samples. Therefore, it should be mentioned here that, by comparing the results stated in Xu et al. (2007) and the present result, we see that the conditions stated in the present work is a quite general. In Xu et al. (2007) and in Appendix, they considered the conditions (14)-(17) to prove the main results. To be more specific, in the following we want to show that the conditions given in Theorem 3 satisfied in conditions (14)-(17) in Xu et al. (2007). Therefore, the condition that $F_1 \leq_{lr} F_2$ is equivalent to $\frac{\bar{F}_1(u)}{\bar{F}_2(u)}$ is increasing in u, which is implied that

$$\frac{\bar{F}_1(u)}{\bar{F}_2(u)} \leq \frac{\bar{F}_1(u+x)}{\bar{F}_2(u+x)} \qquad for\ all\ x, u > 0.$$

Hence, the inequality (14) in Xu et al. (2007) is given. From the fact that $F_1 \leq_{lr} F_2$ implies $F_1 \leq_{rh} F_2$, it follows that

$$\frac{f_1(u)}{F_1(u)} \leq \frac{f_2(u)}{F_2(u)},$$

which reveals that inequality (15) in Xu et al. (2007) is also satisfied. On the other hands, $F_1 \leq_{lr} F_2$ implies $F_1 \leq_{st} F_2$ and then we have

$$\bar{F}_1(x+u) \leq \bar{F}_2(x+u) \qquad and \qquad \bar{F}_1(u) \leq \bar{F}_2(u)$$

or equivalently,

$$\frac{1}{\bar{F}_1(x+u)} \geq \frac{1}{\bar{F}_2(x+u)} \qquad for\ all\ x, u > 0.$$

Hence, the inequality (16) in Xu et al. (2007) is also satisfied. Notice that only the inequality (17) in Xu et al. (2007) may be does not hold for the general case. We can consider the inequality (17) as a assumption in the statement of theorem, because there are some family of distributions that are satisfied in (17). Therefore, in the following we verify the inequality (17) for Weibull distribution as an illustration.

Remark 4. *Let X be a random variable from Weibull distribution with shape and scale parameters, α and λ, respectively. It is known that the distribution function of X is $F(x) = 1 - e^{-(\lambda x)^{\alpha}}$,*

$x > 0$. Now, let us take $\lambda = \lambda_1$, $\lambda^* = \lambda_2$ and $\lambda_2 \geq \lambda_1$. By replacing Weibull above observations in the following inequality

$$\frac{f(x+u)}{F(x+u)-F(u)} \geq \frac{f^*(x+u)}{F^*(x+u)-F^*(u)},$$

it observed that

$$\frac{\lambda_1^\alpha}{e^{-\lambda_1^\alpha(u^\alpha-(x+u)^\alpha)}-1} \geq \frac{\lambda_2^\alpha}{e^{-\lambda_2^\alpha(u^\alpha-(x+u)^\alpha)}-1}.$$

Hence,

$$\frac{\lambda_1^\alpha}{e^{-\lambda_1^\alpha t}-1} \geq \frac{\lambda_2^\alpha}{e^{-\lambda_2^\alpha t}-1},$$

where $u^\alpha - (x+u)^\alpha = t$. Let $\lambda_1^\alpha = u_1$, $\lambda_2^\alpha = u_2$. Thus, one has

$$\frac{tu_1}{e^{-u_1 t}-1} \geq \frac{tu_2}{e^{-u_2 t}-1}, \qquad u_1 < u_2,$$

and this is because from Lemma 2.1 of Khaledi and Kochar (2000), one concludes that $\phi(x) = \frac{x}{e^{-x}-1}$ is decreasing in $x > 0$.

Therefore,

$$\frac{\lambda_1^\alpha}{e^{-\lambda_1^\alpha t}-1} \geq \frac{\lambda_2^\alpha}{e^{-\lambda_2^\alpha t}-1}$$

is true for $\lambda_2 > \lambda_1$. Note that if X_1 and X_2 be non-negative random variables from Weibull distribution with common shape parameter α and the scale parameters λ_1 and λ_2, respectively, then it is easy to see that $F_2 \leq_{lr} F_1$ if, and only if, $\lambda_2 > \lambda_1$ and hence the inequalities (14)-(17) in Xu et al. (2007) are satisfied.

Example 5. *To illustrate the main result, Theorem 4.1 (Section 4, page 4), which establishes the likelihood ratio ordering $D_{i:n}(p,q) \leq_{lr} D_{i:n}(p-1,q+1)$, we provide a numerical example using Weibull distributions. Consider a sample of size $n = 5$, with spacing index $i = 2$, $p = 3$ inliers following distribution F_1, and $q = 2$ outliers following distribution F_2. Let $F_1 \sim Weibull(2,1)$ with density $f_1(x) = 2xe^{-x^2}$,for $x \geq 0$, and $F_2 \sim Weibull(2,1.5)$ with density $f_2(x) = \frac{2x}{2.25}e^{-(x/1.5)^2} = \frac{2x}{2.25}e^{-x^2/2.25}$,for $x \geq 0$. These distributions satisfy $F_1 \leq_{lr} F_2$, and the likelihood ratio is given by*

$$\frac{f_1(x)}{f_2(x)} = \frac{2xe^{-x^2}}{\frac{2x}{2.25}e^{-x^2/2.25}} = 2.25e^{-x^2(1-\frac{1}{2.25})} = 2.25e^{-x^2(1-\frac{4}{9})} = 2.25e^{-\frac{5x^2}{9}},$$

which is non-increasing for $x \geq 0$ since the exponent $-\frac{5x^2}{9}$ decreases as x increases. We compute the density functions of the spacings $D_{2:5}(3,2) = X_{3:5}(3,2) - X_{2:5}(3,2)$ and $D_{2:5}(2,3) = X_{3:5}(2,3) - X_{2:5}(2,3)$, where $X_{i:5}(p,q)$ denotes the i-th order statistic from a sample with p

variables from F_1 *and* q *variables from* F_2*. For Weibull distributions, the density of spacings is complex due to the non-exponential nature, but it can be numerically approximated using the permanent-based formulas. To visually confirm the likelihood ratio ordering, we numerically evaluate the density functions* $f_{D_{2:5}(3,2)}(t)$ *and* $f_{D_{2:5}(2,3)}(t)$ *for* $t \in [0, 3]$*, and compute the likelihood ratio* $\frac{f_{D_{2:5}(3,2)}(t)}{f_{D_{2:5}(2,3)}(t)}$*. Figure 1 plots the likelihood ratio as a function of* t*, demonstrating its non-increasing behavior, which validates* $D_{2:5}(3, 2) \leq_{lr} D_{2:5}(2, 3)$*. The plot in Figure 1 provides a*

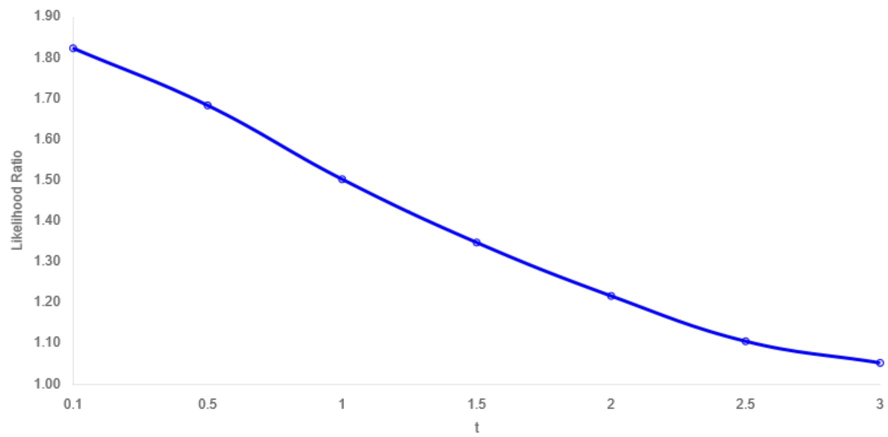

Figure 1: The likelihood ratio $\frac{f_{D_{2:5}(3,2)}(t)}{f_{D_{2:5}(2,3)}(t)}$.

clear visual representation of the likelihood ratio ordering. The decreasing trend indicates that the spacing $D_{2:5}(3, 2)$*, from a sample with fewer outliers (*$q = 2$*), is stochastically smaller than* $D_{2:5}(2, 3)$*, from a sample with more outliers (*$q = 3$*), in the likelihood ratio sense. This behavior reflects the influence of additional outliers, which tend to increase the magnitude of spacings due to the heavier tail of* F_2 *(Weibull with larger scale parameter* $\lambda = 1.5$*).*

To ensure reproducibility of the numerical results presented in Figure 1, we note that the density functions of the spacings and their likelihood ratio were computed exactly using the closed-form expressions for the permanents derived in Theorem 2.3. These computations were performed in Mathematica without employing any numerical approximations or Monte Carlo simulations. The explicit permanent formulas allow direct evaluation for the chosen parameters, guaranteeing precise and fully reproducible results.

3 Conclusion

This work presents the new result to compare the spacings from the multiple-outlier statistical model in the general case where inlier and outlier random variables follow arbitrary continuous distributions. For the case when X_i, $i = 1, \ldots, p$, having common distribution function F_1, and X_j, $j = p+1, \dots, n$, having common distribution function F_2, it was verified that if $F_1 \leq_{lr} F_2$, then $D_{i:n}(p,q) \leq_{lr} D_{i:n}(p-1, q+1)$, $q = n-p$. Motivate by this, it will be interesting to examine whether under the assumption of Theorem 3, $D_{i:n}(p,q) \leq_{lr} D_{i+1:n}(p,q)$, $i = 1, \dots, n$ and $D_{i:n}(p,q) \leq_{lr} D_{i+1:n+1}(p+1,q)$, $i = 1, \dots, n$. These comparisons would provide deeper insights into the stochastic behavior of spacings across different indices i and sample sizes n. However, establishing these orderings presents significant theoretical challenges due to the complex structure of the joint density functions of order statistics, which are derived using permanents. The dependencies introduced by the mixture of distributions F_1 and F_2, require advanced techniques beyond the scope of the current framework. For instance, comparing $D_{i:n}(p;q)$ and $D_{i+1:n}(p;q)$ involves analyzing the conditional distributions of consecutive spacings, which necessitates handling higher-order permanents and ensuring the monotonicity of the likelihood ratio under general continuous distributions. Similarly, comparing $D_{i:n}(p;q)$ with $D_{i:n+1}(p;q)$ requires addressing the effect of increasing sample size on the spacing distributions, further complicating the permanent-based density computations. We have previously considered these orderings during the development of this work, recognizing their potential to complement the main result, $D_{i:n}(p,q) \leq_{lr} D_{i:n}(p-1, q+1)$, established in Section 2. However, the complexity of the proofs, particularly in maintaining the likelihood ratio order's strict requirements, has led us to defer these investigations to future research. We are currently working on this problem and hope to report these findings in a future paper.

Acknowledgements

The authors are grateful to the Editor and the anonymous referees for their careful reading of the manuscript and their constructive comments and suggestions, which have significantly improved the quality and presentation of the paper.

References

Balakrishnan N, Rao CR. Handbook of statistics. v. 16: Order statistics: theory and methods. 1998;.

Balakrishnan N, Rao C. Handbook of statistics. v. 17: Order statistics: applications. 1998;.

Barmalzan G, Haidari A, Payandeh Najafabadi AT. Stochastic comparisons of sample spacings in single-and multiple-outlier exponential models. Statistics. 2015;49(4):815–830.

Khaledi BE, Kochar S. Some new results on stochastic comparisons of parallel systems. Journal of Applied Probability. 2000;37(4):1123–1128.

Kochar SC, Kirmani S. Some results on normalized spacings from restricted families of distributions. Journal of Statistical Planning and inference. 1995;46(1):47–57.

Kochar SC, Korwar R. Stochastic orders for spacings of heterogeneous exponential random variables. Journal of Multivariate Analysis. 1996;57(1):69–83.

Misra N, van der Meulen EC. On stochastic properties of m-spacings. Journal of Statistical Planning and Inference. 2003;115(2):683–697.

Shaked M, Shanthikumar JG. Stochastic orders. Springer; 2007.

Torrado N, Lillo RE. Likelihood ratio comparisons among spacings related to both one or two samples. Statistics. 2015;49(4):831–841.

Wen S, Lu Q, Hu T. Likelihood ratio orderings of spacings of heterogeneous exponential random variables. Journal of Multivariate Analysis. 2007;98(4):743–756.

Xu M, Li X, Zhao P, Li Z. Likelihood ratio order of m-spacings in multiple-outlier models. Communications in Statistics Theory and Methods. 2007;36(8):1507–1525.

Zhao P, Qiao J, Balakrishnan N. Likelihood ratio order of the second spacing in multiple-outlier exponential models. Statistics. 2016;50(1):206–218.

Appendix

Consider $n = 3$, with $p = 2$ inliers following F_1 and $q = 2$ outliers following F_2. Take $i = 2$. Then, the expression for $\mathbf{B}_{p,q}(t, x)$ involves the permanent of the matrix with rows corresponding to the functions at different points and columns repeated according to the following types

$$\begin{pmatrix} f_1(t) & f_1(t) & f_2(t) \\ f_1(t+x) & f_1(t+x) & f_2(t+x) \\ F_1(t+x) & F_1(t+x) & F_2(t+x) \end{pmatrix}$$

Its permanent is $2\left[f_1(t)f_1(t+x)F_2(t+x) + f_1(t)f_2(t+x)F_1(t+x) + f_2(t)f_1(t+x)F_1(t+x)\right]$. Similarly, for $\mathbf{A}_{p,q}(t, x)$ (for $p - 1 = 1$, $q + 1 = 2$), we have

$$\begin{pmatrix} f_1(t) & f_2(t) & f_2(t) \\ f_1(t+x) & f_2(t+x) & f_2(t+x) \\ F_1(t+x) & F_2(t+x) & F_2(t+x) \end{pmatrix}$$

Its permanent is $2\left[f_1(t)f_2(t+x)F_2(t+x) + f_2(t)f_1(t+x)F_2(t+x) + f_2(t)f_2(t+x)F_1(t+x)\right]$. This explicit representation illustrates the structure of the permanents used in the proof. Similarly, explicit forms for $\mathbf{C}_{p,q}(t, x)$ is provided, along with a step-by-step explanation of how the Laplace expansion is applied in the proof.