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# A Quantile Based Generalized Cross Entropy of Order Statistics

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**Abstract.** In this paper, we propose a generalized cross entropy between the the $i$th order statistic and the parent random variable $X$, defined using the quantile function. This method is more flexible than traditional PDF-based measures, particularly in situations where estimating the underlying density is difficult or unreliable. We investigate the properties of this measure and present examples to illustrate these concepts. Furthermore, we introduce a residual version of the quantile based generalized cross entropy between the the $i$th order statistic and the parent random variable $X$, along with some characterization results. Comparative analyses using simulation and real data indicate that the proposed measure provides improved interpretability and robustness relative to the quantile-based Kerridge inaccuracy measure. This study effectively connects theoretical development with practical application, contributing to the field of statistical analysis.

**Keywords.** Entropy, Cross Entropy, Order Statistics, Quantile Function, Hazard Function.

**MSC:** 62B10, 94A17.

## 1 Introduction

Let $X$ and $Y$ be two non-negative random variables with distribution functions $F$ and $G$, and probability density functions $f$ and $g$, respectively. Kerridge (1961) introduced the inaccuracy measure, or cross entropy, between the distributions of $X$ and $Y$, defined as

$$I(X,Y) = -\int_0^\infty f(x)\log g(x)dx. \tag{1.1}$$

$I(X,Y)$ quantifies the error associated with an experiment when there is insufficient or incorrect information in the experimental results. If $g(x)$ is the same as $f(x)$ in Equation (1.1), it reduces

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to

$$I(X, X) = -\int_0^\infty f(x)\log f(x)dx = H(X),$$

where $H(X)$ is the Shannon differential entropy introduced by Shannon (1948), which is used to measure the uncertainty within a single distribution, implying that uncertainty can be used to measure the error in the experimental results. Furthermore, Nair et al. (2011) proposed the generalized cross entropy between the distributions of $X$ and $Y$ defined as

$$\begin{aligned} I^r(X, Y) &= \frac{1}{r}\int_0^\infty f(x)(1 - g^r(x))dx \\ &= \frac{1}{r}\int_0^\infty f(x)(1 - f^r(x))dx + \frac{1}{r}\int_0^\infty f(x)(f^r(x) - g^r(x))dx \\ &= H^r(X) + K^r(X, Y), \;\; r > -1; r \neq 0. \end{aligned} \tag{1.2}$$

Here, $H^r(X)$ and $K^r(X, Y)$ represent the generalized entropy Khinchin (1957) and the generalized relative entropy Nair et al. (2011), respectively, and $r$ is the entropic index that characterizes the degree of non-additivity. As $r \to 0$, $I^r(X, Y) \to I(X, Y)$. $I^r(X, Y)$ and $H^r(X)$ extend these concepts through the entropic index $r$, which is chosen such that $\phi(t) = \frac{1-t^r}{r}$ is convex with $\phi(1) = 0$, for $r > -1$ and $r \neq 0$.

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed observations, each with distribution function $F(x)$. Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order statistics of $X_1, X_2, \ldots, X_n$, arranged from smallest to largest. The probability density function of the $i$th order statistic is given by

$$f_{i:n}(x) = \frac{(F(x))^{i-1}(1 - F(x))^{n-i} f(x)}{\beta(i, n - i + 1)},$$

where $\beta(i, n - i + 1)$ denotes the beta function with parameters $i$ and $n - i + 1$, Arnold et al. (2008). Order statistics have various applications in different fields, including outlier detection, characterization of probability distributions, quality control, robust statistical estimation, and goodness-of-fit testing. Information-theoretic aspects of order statistics have been studied by Wong and Chen (1990), Park (1995), Ebrahimi et al. (2004), Baratpour et al. (2007, 2008), Thapliyal and Taneja (2013, 2015), and Baratpour and Khammar (2016). In this context, the generalized cross entropy between the distribution of the $i$th order statistic $X_{i:n}$ and the parent distribution is defined as

$$\begin{aligned} I^r(X_{i:n}, X) &= \frac{1}{r}\int_0^\infty f_{i:n}(x)(1 - f_X^r(x))dx \\ &= \frac{1}{r}\int_0^\infty f_{i:n}(x)(1 - f_{i:n}^r(x))dx + \frac{1}{r}\int_0^\infty f_{i:n}(x)(f_{i:n}^r(x) - f_X^r(x))dx \\ &= H^r(X_{i:n}) + K^r(X_{i:n}, X), \quad r > -1; r \neq 0, \end{aligned}$$

where $H^r(X_{i:n})$ and $K^r(X_{i:n}, X)$ represent the generalized entropy of $X_{i:n}$ and the generalized relative entropy between the distributions of $X_{i:n}$ and $X$, respectively. Here $X_{i:n}$ represents the order statistic itself, while $I^r(X_{i:n}, X)$ refers to the distribution between the order statistic and the parent variable, respectively.

Theoretical investigations and applications using information measures typically rely on the distribution function, which may not be suitable when the distribution is not analytically tractable. In such cases, quantile based measures provide a valuable alternative, as they are defined in terms of quantile functions rather than explicit probability densities. This approach is particularly advantageous when estimating the probability density function is difficult, such as with heavy-tailed or skewed distributions, small sample sizes, or incomplete data. Since quantile functions are often more robust in the tails, quantile based entropy measures tend to be computationally more stable than those based on probability density functions. Consequently, an alternative methods for such studies can be expressed in terms of the quantile function,

$$Q(u) = F^{-1}(u) = \inf\{x|F(x) \geq u\}, \quad 0 \leq u \leq 1.$$

Let $Q_1(u)$ and $Q_2(u)$ be the quantile functions corresponding to $F$ and $G$. Using the relation $Q_1(u) = F^{-1}(u)$, we have $G(F^{-1}(u)) = G(Q_1(u))$. Define $Q_3(u) = Q_2^{-1}(Q_1(u))$ as the quantile function of $F(G^{-1})$ with quantile density function $q_3(u)$ where $q_3(u) = \frac{d}{du}(Q_3(u))$. This further implies that $q_1(u)g(Q_1(u)) = q_3(u)$. Kayal et al. (2020) introduced the quantile based cross entropy (quantile based inaccuracy measure), given by

$$\begin{aligned} I_Q(X,Y) &= -\int_0^1 \log g(Q_1(u))du \\ &= -\int_0^1 \log \frac{q_3(u)}{q_1(u)}du. \end{aligned} \tag{1.3}$$

In connection with (1.3), the quantile based generalized cross entropy is defined as

$$I_Q^r(X,Y) = \frac{1}{r}\left(1 - \int_0^1 \left(\frac{q_3(u)}{q_1(u)}\right)^r\right)du, \quad r > -1; r \neq 0. \tag{1.4}$$

Further investigation is required to explore properties of (1.4). As $r \to 0$, the quantile based generalized cross entropy tends to the quantile based cross entropy. The quantile based analogues, $I_Q(X,Y)$ and $I_Q^r(X,Y)$, extend these measures to contexts where quantile functions are more suitable than density functions.

Recently, information measures of order statistics using quantile functions have also emerged in the literature. Sunoj et al. (2017) introduced the quantile-based entropy of order statistics. Kumar et al. (2018) extended this idea by proposing a quantile based Tsallis entropy for order statistics, while Kumar and Singh (2018) developed a quantile based generalized entropy of order statistics. Furthermore, Almanjahie et al. (2021) presented a quantile version of the Mathai–Haubold entropy for order statistics, and Zamani and Madadi (2023) investigated a quantile based entropy function in the context of past lifetime for order statistics and examined its properties. Traditional entropy measures based on probability density functions often become less reliable when data are limited or heavy-tailed, since even small estimation errors can lead to significant distortions. Although quantile based approaches reduce this sensitivity, they do not adequately capture how order statistics relate to each other or to the parent distribution. To enhance the usefulness of information measures for ordered data, we propose a quantile based generalized cross entropy measure that directly evaluates the inaccuracy between the distribution of the $i$th order statistic and that of the parent random variable. This formulation offers a more reliable and interpretable framework for quantifying error and dependence in ordered samples.

This paper is organized as follows: Section 2 examines the quantile based generalized cross entropy between thedistributions of $X_{i:n}$ and $X$ , along with its properties and examples. Section 3 presents the dynamic version of the quantile based generalized cross entropy between the distributions of $X_{i:n}$ and $X$, and discusses its properties. Section 4 reports simulation studies and real data applications to investigate the performance of the quantile based generalized residual cross entropy between the distribution of the specific $i$th order statistics and the parent distribution. Finally, Section 5 provides the conclusions.

## 2 Quantile based generalized cross entropy between distributions of $X_{i:n}$ and $X$

For $F(Q(u)) = u$, the probability density function of the $i$th order statistic is given by

$$f_{i:n}(Q(u)) = \frac{u^{i-1}(1-u)^{n-i}(q(u))^{-1}}{\beta(i, n-i+1)}, \tag{2.1}$$

where $q(u) = \frac{dQ(u)}{du}$ is the quantile density function. Defining the density quantile function by $f(Q(u))$, and the quantile density function by $q(u)$, we have $q(u)f(Q(u)) = 1$. The quantile based generalized entropy for the $i$th order statistic is defined as

$$H_Q^r(X_{i:n}) = \frac{1}{r}\int_0^1 f_{i:n}(Q(u))\left(1 - f_{i:n}^r(Q(u))\right) dQ(u), \tag{2.2}$$

where $r > -1; r \neq 0$. This quantity captures the uncertainty associated with the $i$th order statistic. The quantile based generalized relative entropy between the distributions of the $i$th order statistic $X_{i:n}$ and $X$ is defined as

$$K_Q^r(X_{i:n}, X) = \frac{1}{r}\int_0^1 f_{i:n}(Q(u))(f_{i:n}^r(Q(u)) - f^r(Q(u)))dQ(u), \tag{2.3}$$

where $r > -1, r \neq 0$. The quantity $K_Q^r(X_{i:n}, X)$ measures the dissimilarity between $X_{i:n}$ and $X$. By adding the quantile based generalized entropy of $X_{i:n}$ in (2.2) and the quantile based generalized relative entropy between the distributions of $X_{i:n}$ and $X$ in (2.3), we obtain

$$\begin{aligned} H_Q^r(X_{i:n}) + K_Q^r(X_{i:n}, X) &= \frac{1}{r}\int_0^1 f_{i:n}(Q(u))(1 - f^r(Q(u)))dQ(u) \\ &= I_Q^r(X_{i:n}, X), \end{aligned}$$

where $r > -1, r \neq 0$. This quantity represents the quantile based generalized cross entropy between the distribution of the $i$th order statistic $X_{i:n}$ and the parent distribution.

In reliability engineering, $(n-i+1)$-out-of-$n$ systems are an important type of structure. The lifetime of such a system, represented by the $i$th order statistic, is functional if and only if at least $(n-i+1)$ components out of $n$ function properly. To quantify the discrepancy between the distributions of $X_{i:n}$ and $X$, we use $I_Q^r(X_{i:n}, X)$.

To increase computational feasibility, we introduce a lemma to represent $I_Q^r(X_{i:n}, X)$. In practical situations, this link is essential, as it simplifies the computation of $I_Q^r(X_{i:n}, X)$.

**Lemma 2.1.** *Let $X_1, X_2, \ldots, X_n$ be a random sample with a quantile density function $q(u)$. Then, the quantile based generalized cross entropy between the distribution of the ith order statistic $X_{i:n}$ and the parent distribution can be expressed as*

$$I_Q^r(X_{i:n}, X) = \frac{1}{r}\left(1 - E_{g_i}\Big[(q(W_i))^{-r}\Big]\right), \quad r > -1; r \neq 0, \tag{2.4}$$

*where $E_{g_i}(\cdot)$ denotes expectation with respect to $g_i(u)$, and $g_i(u) = \frac{u^{i-1}(1-u)^{n-i}}{B(i,n-i+1)}, \quad 0 < u < 1$, so that $W_i \sim \mathrm{Beta}(i, n-i+1)$.*

*Proof.* Assume that the parent distribution function $F(x)$ is continuous, strictly increasing on its support, and that $q(u)$ is positive and integrable on $(0, 1)$. By definition,

$$I_Q^r(X_{i:n}, X) = \frac{1}{r}\int_0^1 f_{i:n}(Q(u))\Big(1 - f^r(Q(u))\Big)dQ(u).$$

Using Equation (2.1) along with $f(Q(u)) = 1/q(u)$ and $dQ(u) = q(u)du$, we obtain

$$\begin{aligned} I_Q^r(X_{i:n}, X) &= \frac{1}{r}\int_0^1 \frac{u^{i-1}(1-u)^{n-i}}{\beta(i, n-i+1)}(1 - (q(u)^{-r}))du \\ &= \frac{1}{r}\left(1 - \int_0^1 \frac{u^{i-1}(1-u)^{n-i}}{\beta(i, n-i+1)}(q(u)^{-r})du\right), \\ &= \frac{1}{r}\left(1 - E_{g_i}[(q(W_i))^{-r}]\right), \end{aligned}$$

where $W_i \sim \mathrm{Beta}(i, n-i+1)$ with density $g_i(u) = \frac{u^{i-1}(1-u)^{n-i}}{B(i,n-i+1)}$. □

This lemma provides a useful tool for calculating $I_Q^r(X_{i:n}, X)$. Based on Lemma 2.1, the values of $I_Q^r(X_{i:n}, X)$ for a few standard distributions are presented in Table 1.

Table 1: The quantile based generalized cross entropy between the distributions of $X_{i:n}$ and $X$ for specific lifetime distributions.

| Distribution | Quantile Function | $I_Q^r(X_{i:n}, X)$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}$ | $\frac{1}{r}\left(1 - \lambda^r \frac{\beta(i,n-i+r+1)}{\beta(i,n-i+1)}\right)$ |
| Pareto-I | $b(1-u)^{-\frac{1}{a}}$ | $\frac{1}{r}\left(1 - \left(\frac{b}{a}\right)^{-r} \frac{\beta(i,n-i+\frac{r}{a}+r+1)}{\beta(i,n-i+1)}\right)$ |
| Log-logistic | $\frac{1}{a}\left(\frac{u}{1-u}\right)^{\frac{1}{b}}$ | $\frac{1}{r}\left(1 - \left(\frac{1}{ab}\right)^{-r} \frac{\beta(i-\frac{r}{b}+r,n-i+r+\frac{r}{b}+1)}{\beta(i,n-i+1)}\right)$ |
| Generalized Pareto | $\frac{b}{a}\left((1-u)^{-\frac{a}{a+1}} - 1\right)$ | $\frac{1}{r}\left(1 - \left(\frac{b}{a+1}\right)^{-r} \frac{\beta(i,n-i+r+\frac{ar}{a+1}+1)}{\beta(i,n-i+1)}\right)$ |
| Finite Range | $b\left(1 - (1-u)^{\frac{1}{a}}\right)$ | $\frac{1}{r}\left(1 - \left(\frac{b}{a}\right)^{-r} \frac{\beta(i,n-i+r-\frac{r}{a}+1)}{\beta(i,n-i+1)}\right)$ |
| Power function distribution | $au^{\frac{1}{b}}$ | $\frac{1}{r}\left(1 - \left(\frac{a}{b}\right)^{-r} \frac{\beta(i,n-i+r-\frac{r}{b}+1)}{\beta(i,n-i+1)}\right)$ |

We present the following theorem to establish bounds for the quantile based generalized cross entropy between the distribution of the $i$th order statistic $X_{i:n}$ and the parent distribution. This theorem provides valuable insights into the nature of these bounds and their practical applications.

**Theorem 2.2.** *For any random variable $X$ with quantile based generalized entropy, $H^r_Q(X) = \int_0^1 \frac{1}{r}(1-(q(u))^{-r})du < \infty$, the quantile based generalized cross entropy between the distributions of $X_{i:n}$ and $X$ is bounded as follows:*

*(i) For all $r > -1; r \neq 0$:*

$$I^r_Q(X_{i:n}, X) \geq \frac{1}{r}\left(1 - B_i(1 - rH^r_Q(X))\right), \tag{2.5}$$

*where $B_i = \frac{m_i^{(i-1)}(1-m_i)^{(n-i)}}{\beta(i,n-i+1)}$ is the ith term of the Beta$(i, n-i+1)$ density function evaluated at its mode $m_i = \frac{i-1}{n-1}$. Note that $B_1 = B_n = n$.*

*(ii) Let $M = f(m) < \infty$, where $m$ is the mode of the distribution of $X$. Then, for all $r > -1; r \neq 0$:*

$$I^r_Q(X_{i:n}, X) \geq \frac{1}{r}(1 - M^r). \tag{2.6}$$

*Proof.* (i) Let $g_i$ and $m_i$ denote the probability density function and the mode of the beta distribution with parameters $i$ and $n-i+1$, respectively. From (2.4) and since the mode of this beta distribution is $m_i = \frac{i-1}{n-1}$. Thus $g_i(y) \leq B_i = g_i(m_i) = \frac{m_i^{(i-1)}(1-m_i)^{(n-i)}}{\beta(i,n-i+1)}$ we have, for $r > -1; r \neq 0$,

$$\begin{aligned} -E_{g_i}((q(U))^{-r}) &= -\int_0^1 g_i(u)(q(u))^{-r}du \\ &\geq -\int_0^1 g_i(m_i)(q(u))^{-r}du \\ &= -B_i\int_0^1 (q(u))^{-r}du \\ &= -B_i(1 - rH^r_Q(X)), \end{aligned}$$

where $g_i$ has a beta distribution with parameters $i$ and $n-i+1$. Thus, using (2.4), the result follows.

(ii) For $r > -1; r \neq 0$, we have $f^r(Q(u)) \geq M^r$. By applying definition (2.6), the result follows.

□

This result provides a quantitative measure of the lower bound for the quantile based generalized cross entropy between the distributions of $X_{i:n}$ and $X$. The bounds in Theorem 2.2 highlight the dependence of $I^r_Q(X_{i:n}, X)$ on the mode of the underlying distribution and quantify the relationship between the error associated with $X_{i:n}$ and $X$. Using Theorem 2.2, we now demonstrate the bound of $I^r_Q(X_{i:n}, X)$ for the exponential distribution.

**Example 2.3.** Let $X$ be a random variable following an exponential distribution with parameter $\lambda$. For $i = 1$ in (2.5), we have

$$I_Q^r(X_{1:n}, X) \geq \frac{1}{r}\left(1 - \lambda^r \frac{n}{r+1}\right),$$

while

$$I_Q^r(X_{1:n}, X) = \frac{1}{r}\left(1 - \lambda^r \frac{n}{n+r}\right).$$

The difference between the actual value of $I_Q^r(X_{1:n}, X)$ and the lower bound is strictly increasing in $n$ for $n \geq 1$ when $r > 0$. Therefore, the lower bound is particularly useful for the exponential distribution when $n$ is large. Note that, if multiple lower bounds are available, one may use their maximum.

In many practical situations, the information between the distributions of $X_{i:n}$ and $X$ may vary across quantiles. To capture this variation, we introduce a residual version of the quantile based generalized cross entropy between the distribution of the $i$th order statistic $X_{i:n}$ and the parent distribution in the next section.

## 3 Quantile based generalized residual cross entropy between distributions of $X_{i:n}$ and $X$

The measure given in (1.2) does not adequately describe a system that has survived beyond a certain time, motivating the need for a residual version of the proposed measure. To address this issue, Nair et al. (2011) introduced the generalized cross entropy for the residual lifetimes $X_t = (X - t|X > t)$ and $Y_t = (Y - t|Y > t)$, which is defined as

$$I^r(X, Y; t) = \frac{1}{r}\int_t^{\infty} \frac{f(x)}{\bar{F}(t)}\left(1 - \left(\frac{g(x)}{\bar{G}(t)}\right)^r\right)dx, \; r > -1; r \neq 0.$$

When $t = 0$, $I^r(X, Y; t)$ reduces to $I^r(X, Y)$. The generalized residual cross entropy between the distributions of the $i$th order statistic $X_{i:n}$ and $X$ is defined as

$$I^r(X_{i:n}, X; t) = \frac{1}{r}\int_t^{\infty} \frac{f_{i:n}(x)}{\bar{F}_{i:n}(t)}\left(1 - \left(\frac{f(x)}{\bar{F}(t)}\right)^r\right)dx, \; r > -1; r \neq 0.$$

Here, $\bar{F}_{i:n}(x) = \frac{\bar{\beta}_{F(x)}(i,n-i+1)}{\beta(i,n-i+1)}$ represents the survival function of the $i$th order statistic $X_{i:n}$ and $\bar{\beta}_{F(x)}(i, n-i+1) = \int_{F(x)}^1 t^{i-1}(1-t)^{n-i}dt$ is the incomplete beta function, as detailed in David and Nagaraja (2003). The quantile based generalized residual cross entropy between the distributions of $X_{i:n}$ and $X$ is defined as

$$\begin{aligned} I_Q^r(X_{i:n}, X; u) &= \frac{1}{r}\int_u^1 \frac{f_{i:n}(Q(p))}{\bar{F}_{i:n}(Q(u))}\left(1 - \left(\frac{f(Q(p))}{\bar{F}(Q(u))}\right)^r\right)dQ(p) \\ &= \frac{1}{r}\left(1 - \int_u^1 \frac{p^{i-1}(1-p)^{n-i}}{\bar{\beta}_u(i, n-i+1)}\frac{q(p)^{-r}}{(1-u)^r}dp\right), \; r > -1; r \neq 0, \end{aligned} \tag{3.1}$$

where $q(p)$ is the quantile density function. Differentiating (3.1) with respect to $u$ and simplifying gives

$$\begin{aligned}(q(u))^{-r} &= (1-u)^r + r(1-u)^{r-1-n+i}u^{1-i}\bar{\beta}_u(i,n-i+1)\\ &\quad - rI^r_Q(X_{i:n},X;u)\left((1-u)^r + r(1-u)^{r-1-n+i}u^{1-i}\bar{\beta}_u(i,n-i+1)\right)\\ &\quad + \bar{\beta}_u(i,n-i+1)(1-u)^{r-n+i}u^{1-i}r\frac{d}{du}(I^r_Q(X_{i:n},X;u)).\end{aligned} \tag{3.2}$$

Therefore, (3.2) establishes a direct relationship between the quantile density function $q(u)$ and the quantile based generalized residual cross entropy $I^r_Q(X_{i:n},X;u)$, implying that $I^r_Q(X_{i:n},X;u)$ uniquely determines the quantile function $Q(u)$. For practical computation of $I^r_Q(X_{i:n},X;u)$, the following lemma is useful.

**Lemma 3.1.** *Let $X_1, X_2, \ldots, X_n$ be a random sample with quantile density function $q(u)$. Then the quantile based generalized residual cross entropy between the distributions of $X_{i:n}$ and $X$ can be expressed as*

$$I^r_Q(X_{i:n},X;u) = \frac{1}{r}\left(1 - \frac{1}{(1-u)^r}E_{g_i}\Big[(q(W_i))^{-r} \mid W_i \geq u\Big]\right), \qquad r > -1,\ r \neq 0,$$

*where $g_i$ denotes the density of a* $\text{Beta}(i,n-i+1)$ *distribution and $W_i \sim$* $\text{Beta}(i,n-i+1)$.

*Proof.* Assume that the distribution function $F(x)$ is continuous and strictly increasing on its support, and that $q(u)$ is positive and integrable on $(0,1)$. From equation (3.1) we have

$$I^r_Q(X_{i:n},X;u) = \frac{1}{r}\int_u^1 \frac{f_{i:n}(Q(p))}{\bar{F}_{i:n}(Q(u))}\left(1 - \left(\frac{f(Q(p))}{\bar{F}(Q(u))}\right)^r\right)dQ(p).$$

Since $dQ(p) = q(p)\,dp$ and $q(p) = 1/f(Q(p))$, rearranging gives

$$I^r_Q(X_{i:n},X;u) = \frac{1}{r}\left(1 - \frac{1}{(1-u)^r}\int_u^1 \frac{p^{i-1}(1-p)^{n-i}}{\bar{\beta}_u(i,n-i+1)}\,q(p)^{-r}\,dp\right).$$

Noting that $g_i(p) = \dfrac{p^{i-1}(1-p)^{n-i}}{B(i,n-i+1)}$ is the $\text{Beta}(i,n-i+1)$ density and that

$$\int_u^1 \frac{p^{i-1}(1-p)^{n-i}}{\bar{\beta}_u(i,n-i+1)}\,q(p)^{-r}\,dp = E_{g_i}\Big[q(W_i)^{-r} \mid W_i \geq u\Big],$$

we obtain the stated result. □

This lemma simplifies the computation of $I^r_Q(X_{i:n},X;u)$. Based on lemma 3.1, the values of $I^r_Q(X_{i:n},X;u)$ for some standard distributions are summarized in Table 2.

A particular case of the quantile based generalized residual cross entropy between the distribution of the $i$th order statistic $X_{i:n}$ and the parent distribution is discussed below.

Table 2: The quantile based generalized residual cross entropy between the distributions of $X_{i:n}$ and $X$.

| Distribution | Quantile Function | $I_Q^r(X_{i:n}, X; u)$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}$ | $\frac{1}{r}\left(1-\lambda^r \frac{\bar{\beta}_u(i,n-i+r+1)}{\bar{\beta}_u(i,n-i+1)(1-u)^r}\right)$ |
| Pareto-I | $b(1-u)^{-\frac{1}{a}}$ | $\frac{1}{r}\left(1-\left(\frac{b}{a}\right)^{-r} \frac{\bar{\beta}_u(i,n-i+\frac{r}{a}+r+1)}{\bar{\beta}_u(i,n-i+1)(1-u)^r}\right)$ |
| Log-logistic | $\frac{1}{a}\left(\frac{u}{1-u}\right)^{\frac{1}{b}}$ | $\frac{1}{r}\left(1-\left(\frac{1}{ab}\right)^{-r} \frac{\bar{\beta}_u(i-\frac{r}{b}+r,n-i+r+\frac{r}{b}+1)}{\bar{\beta}_u(i,n-i+1)(1-u)^r}\right)$ |
| Generalized Pareto | $\frac{b}{a}\left((1-u)^{-\frac{a}{a+1}}-1\right)$ | $\frac{1}{r}\left(1-\left(\frac{b}{a+1}\right)^{-r} \frac{\bar{\beta}_u(i,n-i+r+\frac{ar}{a+1}+1)}{\bar{\beta}_u(i,n-i+1)(1-u)^r}\right)$ |
| Finite Range | $b\left(1-(1-u)^{\frac{1}{a}}\right)$ | $\frac{1}{r}\left(1-\left(\frac{b}{a}\right)^{-r} \frac{\bar{\beta}_u(i,n-i+r-\frac{r}{a}+1)}{\bar{\beta}_u(i,n-i+1)(1-u)^r}\right)$ |
| Power function distribution | $au^{\frac{1}{b}}$ | $\frac{1}{r}\left(1-\left(\frac{a}{b}\right)^{-r} \frac{\bar{\beta}_u(i-\frac{r}{b}+r,n-i+1)}{\bar{\beta}_u(i,n-i+1)(1-u)^r}\right)$ |
| Govindarajulu | $ab(b+1)p^{b-1}(1-p)$ | $\frac{1}{r}\left(1-\frac{(ab(b+1))^{-r}\bar{\beta}_u(i,n-i+r-\frac{r}{a}+1)}{\bar{\beta}_u(i,n-i+1)(1-u)^r}\right)$ |

For $i=1$, $X_{1:n}$ denotes the first order statistic with distribution function $F_{1:n}(x) = 1-(\bar{F}(x))^n$, probability density function $f_{1:n}(x) = n(\bar{F}(x))^{n-1}f(x)$, and hazard rate function $h_{1:n}(x) = nh(x)$. In this regard, the quantile based generalized residual cross entropy between the distributions of $X_{1:n}$ and $X$ is defined as

$$\begin{aligned} I_Q^r(X_{1:n}, X; u) &= \frac{1}{r}\left(1-\int_u^1 \frac{(1-p)^{n-1}}{\bar{\beta}_u(1,n)} \frac{q(p)^{-r}}{(1-u)^r}dp\right) \\ &= \frac{1}{r}\left(1-\frac{n}{(1-u)^{n+r}}\int_u^1 (1-p)^{n-1}(q(p))^{-r}dp\right), r>-1, r\neq 0. \end{aligned}$$

To illustrate how $I_Q^r(X_{1:n}, X; u)$ uniquely determines the quantile function, we provide the following example.

**Example 3.2.** Let $X$ follow Govindarajulu distribution with quantile function

$$Q(u) = a\left((b+1)u^b - bu^{b+1}\right), 0 \le u \le 1; a,b>0. \tag{3.3}$$

Then, $I_Q^r(X_{1:n}, X; u)$ for the Govindarajulu distribution is given by

$$I_Q^r(X_{1:n}, X; u) = \frac{1}{r}\left(1-\frac{n(ab(b+1))^{-r}}{(1-u)^{n+r}}\bar{\beta}_u(1-rb+r, n-r)\right). \tag{3.4}$$

Conversely, assume (3.4) holds. Using (3.3), we obtain

$$(q(u))^{-r} = r\frac{d}{du}(I_Q^r(X_{1:n}, X; u))\frac{(1-u)^{r+1}}{n} + (1-u)^r\left(1+\frac{r}{n}\right)(1-rI_Q^r(X_{1:n}, X; u)). \tag{3.5}$$

Also, we have

$$\begin{aligned} \frac{d}{du}(I_Q^r(X_{1:n}, X; u))) = &-\frac{n(ab(b+1))^{-r}}{r}(n+r)\bar{\beta}_u(1-rb+r, n-r)(1-u)^{n+r-1} \\ &+ \frac{n(ab(b+1))^{-r}}{r}u^{-rb+r}(1-u)^{-2r-1}. \end{aligned} \tag{3.6}$$

Substituting (3.4) and (3.6) into (3.5) and simplifying gives

$$q(u) = ab(b+1)(1-u)u^{b-1},$$

where $q(u)$ is the quantile density function of the Govindarajulu distribution given in (3.3).

A function parallel to the hazard rate function, $h(x) = \frac{f(x)}{\bar{F}(x)}$, is the hazard quantile function, which is a highly useful measure in reliability studies. It is denoted by $K(u)$ and is defined as

$$K(u) = h(Q(u)) = \frac{1}{(1-u)q(u)}.$$

For the first order statistic, the hazard quantile function is defined by

$$K_{X_{1:n}}(u) = \frac{n}{(1-u)q(u)},$$

where $q(u)$ is the quantile density funtion. Considering the relationship between $I_Q^r(X_{i:n}, X; u)$ and $K_{X_{i:n}}(u)$ of the first order statistics, we present the following theorem.

**Theorem 3.3.** *Let $X_{1:n}$ denote the first order statistic with hazard quantile function $K_{X_{1:n}}(u)$. Then, the quantile based generalized residual cross entropy between the distributions of $X_{1:n}$ and $X$ is given by*

$$I_Q^r(X_{1:n}, X; u) = \frac{1}{r}\left(1 - c(K_{X_{1:n}}(u))^r\right), r > -1, r \neq 0, \tag{3.7}$$

*if and only if $X$ follows the distributions (i), (ii), and (iii) in Table 3. Here $c$ takes the values $c = \frac{n}{n+r}$, $c = \frac{n(\frac{b}{a})^{-r}}{n+\frac{r}{a}+r}$, and $c = \frac{n(\frac{b}{a})^{-r}}{n-\frac{r}{a}+r}$.*

*Proof.* Assuming (3.7) is valid, we have

$$\frac{1}{r}\left(1 - \frac{n}{(1-u)^{n+r}}\int_u^1 (1-p)^{n-1}(q(p))^{-r}dp\right) = \frac{1}{r}\left(1 - c(K_{X_{1:n}}(u))^r\right).$$

Substituting the expression for $K_{X_{1:n}}(u)$ into the equation and solving for $q(u)$ yields,

$$n\int_u^1 (1-p)^{n-1}(q(p))^{-r}dp = c(1-u)^{(n+r)}(K_{X_{1:n}}(u))^r.$$

Differentiating both sides with respect to $u$ and after some algebraic simplification, we obtain

$$\frac{q'(u)}{q(u)} = \left(\frac{n-nc}{rc}\right)(1-u)^{-1}.$$

This simplifies to

$$q(u) = A(1-u)^{\frac{n(c-1)}{rc}},$$

where $A$ is a constant. Therefore, the underlying distribution is exponential if $c = \frac{n}{n+r}$, Pareto if $c = \frac{n(\frac{b}{a})^{-r}}{n+\frac{r}{a}+r}$, and finite range if $c = \frac{n(\frac{b}{a})^{-r}}{n-\frac{r}{a}+r}$. The "if" part follows directly from Table 3 . □

Table 3: The quantile based generalized residual cross entropy between $X_{1:n}$ and $X$ for specific distributions

| Distribution | Quantile Function | $I_Q^r(X_{i:n}, X; u)$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}$ | $\frac{1}{r}\left(1-\lambda^r \frac{n}{n+r}\right)$ |
| Pareto-I | $\frac{b}{a}\left((1-u)^{-\frac{a}{a+1}}-1\right)$ | $\frac{1}{r}\left(1-\frac{n\left(\frac{b}{a}\right)^r}{n+\frac{r}{a}+r}(1-u)^{\frac{r}{a}}\right)$ |
| Finite Range | $n\left(\frac{b}{a}\right)^{-r}\frac{(1-u)^{\frac{r}{a}}}{n-\frac{r}{a}+r}$ | $\frac{1}{r}\left(1-n\left(\frac{b}{a}\right)^{-r}\frac{(1-u)^{\frac{r}{a}}}{n-\frac{r}{a}+r}\right)$ |
| Generalized Pareto | $n(\frac{b}{a}+1)^{-r}\frac{(1-u)^{\frac{ra}{a+1}}}{n+\frac{ra}{a+1}+r}$ | $\frac{1}{r}\left(1-n(\frac{b}{a}+1)^{-r}\frac{(1-u)^{\frac{ra}{a+1}}}{n+\frac{ra}{a+1}+r}\right)$ |

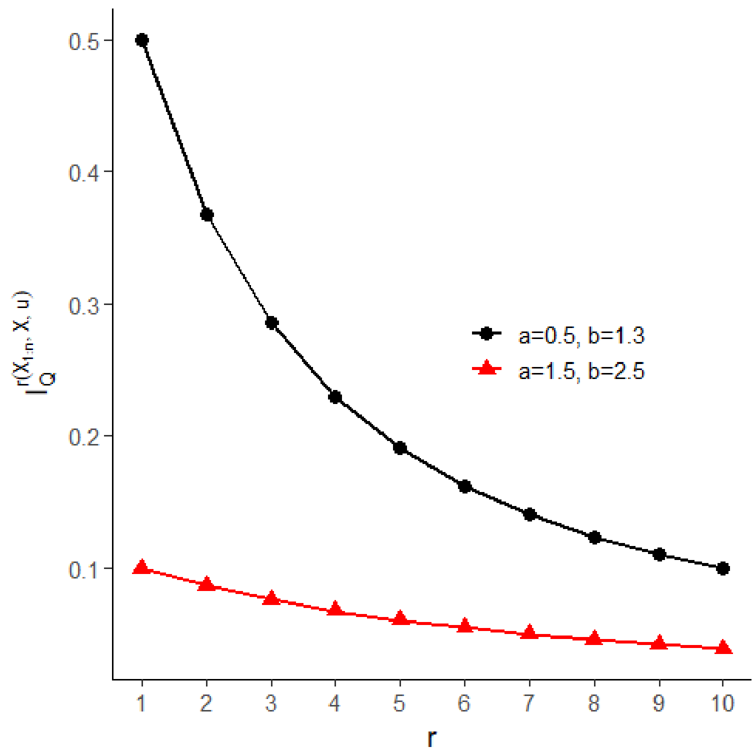



Figure 1: Plot of $I_Q^r(X_{1:n}, X; u)$ for the Pareto-1 distribution under two parameter combinations: $a = 0.5, b = 1.3$, and $a = 1.5, b = 2.5$.

Table 3 provides the quantile based generalized residual cross entropy between $X_{1:n}$ and $X$ for specific distributions. Figure 1 depicts the quantile based generalized residual cross entropy for the first order statistic of the Pareto distribution under two parameter settings: $(a = 0.5, b = 1.3)$ and $(a = 1.5, b = 2.5)$. In both cases, the measure $I_Q^r(X_{1:n}, X; u)$ decreases monotonically as $r$ increases.

# 4 Simulation Study and Application to Real Data

## 4.1 Simulation Study

For the power function distribution, we investigated the performance of the quantile based generalized residual cross entropy between the distribution of the $n$th order statistic and the parent distribution. For the Weibull distribution, the study was conducted with respect to the first order statistic. The parameters of the power function and Weibull distributions were estimated using the method of maximum likelihood. Based on the value of $I^r_Q(X_{n:n}, X; u)$ for the power function distribution as given in Table 2, we computed the estimated values of $I^r_Q(X_{n:n}, X; u)$ for various sample sizes and parameter combinations, using 2000 repetitions. The results of the simulation study, presented in Tables 4 and 5, led to the following conclusions:

(a) The estimated values of $I^r_Q(X_{n:n}, X; u)$ converge to the true value as the sample size increases.

(b) The mean square error (MSE) of $I^r_Q(X_{n:n}, X; u)$ decreases with increasing sample size.

For the Weibull distribution, the quantile based generalized residual cross entropy between the distribution of the first order statistic and the parent distribution is given by $I^r_Q(X_{1:n}, X; u) = \frac{1}{r}\left(1 - \frac{n(a/b)^r}{(1-u)^{(n+r)}} \int_0^{1-u} t^{n+r-1}(-\log t)^{r(1-\frac{1}{a})} dt\right)$. A similar pattern was observed (Table 6 and Figure 3), with the estimates converging to the true values and the RMSE decreasing as the sample size increased. To compare the efficiency of the proposed measure with the existing quantile based Kerridge inaccuracy measure, we also evaluated the latter for the Weibull distribution between the first order statistic and the parent random variable. The quantile based Kerridge inaccuracy measure is given by $I_Q(X_{1:n}, X; u) = \log(\frac{b}{a}) + \frac{1}{n} - \frac{n(a-1)}{a(1-u)^n} \int_0^{1-u} t^{n-1} \log(-\log t) dt$, where $a$ and $b$ denote the shape and scale parameters of the Weibull distribution. The comparative results are presented in Table 6, showing that our proposed measure yields smaller bias and RMSE values, thereby demonstrating improved estimation accuracy. Also, Figure 3b illustrates that the variance of the proposed estimator $I^r_Q(X_{1:n}, X; u)$ decreases rapidly with increasing sample size across all scale parameters b, confirming its consistency and stability.

## 4.2 Application to Real Life Data

The data represent the failure times of 20 oral irrigators reported by Jiang and Murthy (1997). The data values are 1.175, 7.02, 7.58, 9.76, 15.02, 15.57, 17.39, 19.55, 22.47, 23.24, 23.96, 25.05, 32.44, 36.87, 42.76, 43.14, 43.81, 46.95, 56.33, 56.88. The goodness-of-fit of the Weibull and Power function distributions to the failure time data was assessed using the Kolmogorov–Smirnov (K–S) test and the Akaike Information Criterion (AIC). The Power function distribution resulted in a K-S statistic of 0.5169 and an AIC of

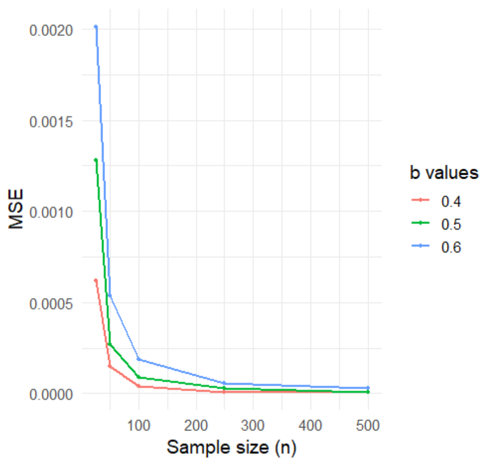
Table 4: Average estimates, bias, and mean squared error (MSE) for $I^r_Q(X_{n:n}, X; u)$ under the power function distribution for different values of $b$, with $a = 1.8, u = 0.01$, and $r = 2$.

| $n$ | Parameter | $E(I^r_Q(X_{n:n}, X; u))$ | Bias | MSE |
|---|---|---|---|---|
| 25 | b=0.4 | 0.45721 | -0.01417 | **0.00062** |
| | b=0.5 | 0.43783 | -0.01938 | 0.00128 |
| | b=0.6 | 0.41668 | -0.02345 | 0.00201 |
| 50 | b=0.4 | 0.46689 | -0.00630 | **0.00015** |
| | b=0.5 | 0.45126 | -0.00774 | 0.00027 |
| | b=0.6 | 0.43092 | -0.01085 | 0.00054 |
| 100 | b=0.4 | 0.47152 | -0.00251 | **0.00004** |
| | b=0.5 | 0.45615 | -0.00369 | 0.00009 |
| | b=0.6 | 0.43771 | -0.00484 | 0.00019 |
| 250 | b=0.4 | 0.47345 | -0.00105 | **0.00001** |
| | b=0.5 | 0.45865 | -0.00167 | 0.00003 |
| | b=0.6 | 0.44126 | -0.00176 | 0.00006 |
| 500 | b=0.4 | 0.47413 | -0.00053 | **0.00001** |
| | b=0.5 | 0.45975 | -0.00072 | 0.00001 |
| | b=0.6 | 0.44219 | -0.00097 | 0.00003 |

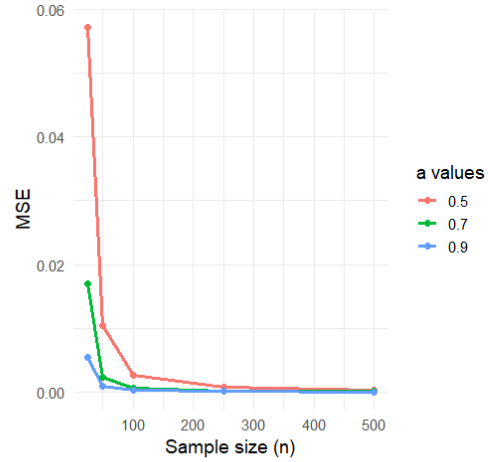
Table 5: Average estimates, bias, and mean squared error (MSE) for $I^r_Q(X_{n:n}, X; u)$ under the power function distribution for different values of $a$ and for value of $b = 0.3, u = 0.01, r = 2$.

| $n$ | Parameter | $E(I^r_Q(X_{n:n}, X; u))$ | Bias | MSE |
|---|---|---|---|---|
| 25 | a=0.5 | 0.13540 | -0.13880 | 0.05717 |
| | a=0.7 | 0.31225 | -0.07255 | 0.01697 |
| | a=0.9 | 0.38831 | -0.04200 | **0.00539** |
| 50 | a=0.5 | 0.24067 | -0.05677 | 0.01045 |
| | a=0.7 | 0.37015 | -0.02650 | 0.00230 |
| | a=0.9 | 0.42034 | -0.01714 | **0.00096** |
| 100 | a=0.5 | 0.28512 | -0.02224 | 0.00266 |
| | a=0.7 | 0.38971 | -0.01200 | 0.00065 |
| | a=0.9 | 0.43321 | -0.00733 | **0.00027** |
| 250 | a=0.5 | 0.30384 | -0.00901 | 0.00073 |
| | a=0.7 | 0.39945 | -0.00507 | 0.00019 |
| | a=0.9 | 0.43966 | -0.00258 | **0.00007** |
| 500 | a=0.5 | 0.31014 | -0.00448 | 0.00032 |
| | a= 0.7 | 0.40316 | -0.00226 | 0.00009 |
| | a=0.9 | 0.44141 | -0.00138 | **0.00003** |

203.22, whereas the Weibull distribution produced a lower K-S statistic of 0.1268 and a smaller AIC of 170.46, indicating that the Weibull distribution provides a better fit to the observed data. The maximum likelihood estimates of the shape and scale parameters of the Weibull distribution are 1.64 and 30.36, respectively. The estimated values of $\hat{I}^r_Q(X_{1:n}, X; u)$ for the Weibull distribution are presented in Table 7. The pro-




(a) MSE against Sample size $n$, under the power function distribution for $a = 1.8$ and varying $b$.



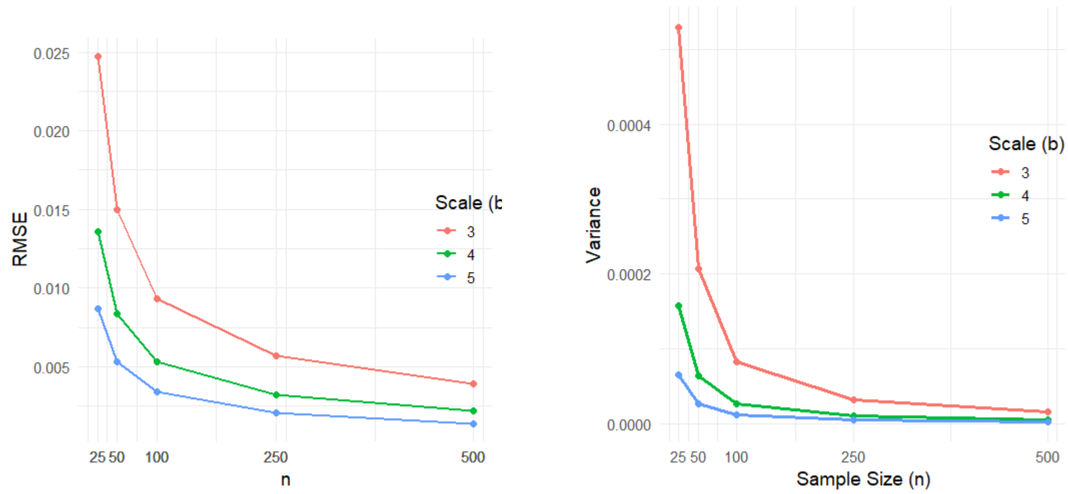

(b) MSE against Sample size $n$, under the power function distribution $b = 0.3$ and varying $a$.

Figure 2: MSE of $I^r_Q(X_{n:n}, X; u)$ plotted against sample size $n$ under the power function distribution. Figure 2a presents the results for $a = 1.8$ and varying $b$, while Figure 2b presents the results for $b = 0.3$ and varying $a$.

Table 6: Average estimates, bias, and root mean squared error (RMSE) for $I^r_Q(X_{1:n}, X; u)$ and $I_Q(X_{1:n}, X; u)$ for the Weibull distribution for different values of $b$ with $a = 2, r = 2$ and $u = 0.3$.

| $n$ | Parameter | $E(I^r_Q(X_{1:n}, X; u))$ | Bias | RMSE | $E(I_Q(X_{1:n}, X; u))$ | Bias | RMSE |
|---|---|---|---|---|---|---|---|
| 25 | b =3 | 0.4190 | -0.0090 | 0.0247 | 0.9101 | -0.0375 | 0.1260 |
| | b =4 | 0.4544 | -0.0052 | 0.0136 | 1.1978 | -0.0399 | 0.1223 |
| | b =5 | 0.4708 | -0.0033 | **0.0087** | 1.4209 | -0.0397 | 0.1247 |
| 50 | b =3 | 0.4197 | -0.0042 | 0.0150 | 0.9143 | -0.0183 | 0.0840 |
| | b = 4 | 0.4548 | -0.0026 | 0.0084 | 1.2020 | -0.0217 | 0.0842 |
| | b = 5 | 0.4711 | -0.0014 | **0.0053** | 1.4251 | -0.0170 | 0.0832 |
| 100 | b = 3 | 0.4202 | -0.0018 | 0.0093 | 0.9173 | -0.0084 | 0.0556 |
| | b = 4 | 0.4551 | -0.0011 | 0.0053 | 1.2050 | -0.0090 | 0.0562 |
| | b = 5 | 0.4713 | -0.0008 | **0.0034** | 1.4281 | -0.0099 | 0.0569 |
| 250 | b =3 | 0.4205 | -0.0006 | 0.0057 | 0.9194 | -0.0028 | 0.0351 |
| | b =4 | 0.4553 | -0.0004 | 0.0032 | 1.2071 | -0.0037 | 0.0355 |
| | b = 5 | 0.4714 | -0.0002 | **0.0021** | 1.4302 | -0.0030 | 0.0355 |
| 500 | b = 3 | 0.4201 | -0.0004 | 0.0039 | 0.9235 | -0.0020 | 0.0243 |
| | b = 4 | 0.4551 | -0.0002 | 0.0022 | 1.2112 | -0.0015 | 0.0246 |
| | b = 5 | 0.4712 | -0.0001 | **0.0014** | 1.4344 | -0.0018 | 0.0237 |

posed quantile based generalized cross entropy serves as a practical tool to evaluate how closely the fitted Weibull model represents the observed data. Higher values of $\hat{I}^r_Q(X_{1:n}, X; u)$ indicate greater error, while the decreasing trend as $r$ increases suggests that higher order measures capture progressively less additional information. This in-




(a) RMSE against sample size for $I^r_Q(X_{i:n}, X; u)$. (b) Variance against sample size for $I^r_Q(X_{i:n}, X; u)$.

Figure 3: Comparison of RMSE and Variance for different parameters of Weibull distribution.

Table 7: Estimated values of $\hat{I}^r_Q(X_{1:n}, X; u)$ for the Weibull distribution.

| $r$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $\hat{I}^r_Q(X_{1:n}, X; u)$ | 0.4993 | 0.3333 | 0.2500 | 0.2000 | 0.1667 | 0.1429 |

sight is useful in reliability analysis, as smaller cross entropy values imply better model performance and higher confidence in lifetime predictions. To further demonstrate the practical relevance of the proposed approach, we employ the quantile based Kerridge inaccuracy measure of order statistics on the same dataset. The calculated value of the quantile based Kerridge inaccuracy measure for the failure times of oral irrigators is 3.322. Compared to our proposed measure, the Kerridge inaccuracy measure yields a higher inaccuracy value, indicating a larger deviation from the expected distribution. The parameter $r$ assigns greater weight to events that are more likely to occur. Smaller values of $r$ approximate the Kerridge inaccuracy measure, while larger values emphasize higher-order deviations, providing a flexible framework for analyzing uncertainty across different reliability scales.

## 5 Conclusion

This study introduced the quantile based generalized cross entropy between the $i$th order statistic and the parent distribution, along with its residual form, and established several key theoretical properties. Numerical investigations through simulations demonstrated the consistency of the proposed measure as the sample size increased, highlighting its practical applicability. In real data applications, the proposed measure exhibited better interpretability compared with the quantile based Kerridge inaccuracy measure. While the present work is limited to univariate order statistics, future studies could extend these measures to multivariate data and other distribution families to enhance their applicability.

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