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Generalized Skewed Linnik Distribution and its Application in Time Series

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Abstract. Heavy-tailed distributions have recently gained prominence in science, economics, and industry as robust alternatives to the Gaussian distribution, particularly for modeling data with extreme variability or outliers. Several studies in the literature have introduced and examined the Pakes generalized Linnik distribution and its related distributions due to their flexibility in capturing heavy-tailed behavior. However, most existing works focus exclusively on the special case of symmetric random variables—a significant limitation, given that real-world data often exhibit skewness. To address this gap, this paper proposes a new class of generalized skewed Linnik distributions, extending previous symmetric models to accommodate asymmetric data structures. We investigate their theoretical properties, including moments, tail behavior, and stability under linear transformations. Furthermore, we develop an autoregressive (AR) model based on this framework, enabling time-series analysis with skewed, heavy-tailed innovations. Additionally, we introduce a novel class of geometric skewed Linnik distributions, which arise as the limit of random sums and exhibit unique dependence structures. The practical utility of these models is demonstrated through theoretical derivations and potential applications in finance, risk assessment, and signal processing. Our results broaden the scope of Linnik-based models, offering more accurate tools for skewed, heavy-tailed data analysis.

Keywords. Skewed Linnink Distribution, Characteristic Function, Geometric Distribution, Autoregressive Process.

MSC: 60E5, 62E10, 62M10.

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1 Introduction

In recent years, the Linnik distribution $L(\alpha, \gamma)$, introduced by Linnik (1962), has attracted significant attention across various scientific disciplines. Notably, it has been applied to model financial phenomena, such as the S&P index (Anderson & Arnold, 1993; Kozubowski, 1999, 2001). Further theoretical investigations by Devroye (1990), Kotz & Ostrovskii (1996), Lin (1998), and Pakes (1998) have explored its probability density function (PDF), characterized by the characteristic function (CHF):

$$\phi(\lambda) = (1 + |\gamma\lambda|^{\alpha})^{-1}, \tag{1.1}$$

where $\gamma > 0$ is the scale parameter,$\lambda \in R$, and $0 < \alpha \leq 2$. For the special case $L(\alpha, 1)$ the PDF and cumulative distribution function (CDF) are given by:

$$f(x) = \frac{\sin(\pi\alpha/2)}{\pi} \int_0^{\infty} \frac{y^{\alpha} \exp(-xy)}{y^{2\alpha} + 2y^{\alpha} \cos(\pi\alpha/2) + 1} dy,$$

and

$$F(x) = 1 - \frac{\sin(\pi\alpha/2)}{\pi} \int_0^{\infty} \frac{y^{\alpha-1} \exp(-xy)}{y^{2\alpha} + 2y^{\alpha} \cos(\pi\alpha/2) + 1} dy, \quad x > 0,$$

(Dexter (2012)). Kozubowski (2001) established a structural representation of $L(\alpha, \gamma)$:

$$L \stackrel{d}{=} \gamma D R^{1/\alpha}, \tag{1.2}$$

where D follows a standard Laplace distribution (location 0, scale 1), and R has the density:

$$f_R(r) = \frac{\sin(\pi\rho)}{\rho\pi[r^2 + 2r\cos(\rho\pi) + 1]}, \quad 0 < \rho < 1, \quad r > 0.$$

The $L(\alpha, \gamma)$ is geometrically stable. Recently, Pillai & Satheesh (2024) examined its stochastic representations and self-decomposability, proving that a Lévy process $\{X(t), t \geq 0\}$ with symmetric $X(1)$ is Linnik-distributed if and only if the CHF $h(u)$ of the stochastic integral $\int_0^1 t^{\lambda} dx(v(t))$ is given by:

$$Ln[h(u)] = 1 - (1 + |u|^{-\alpha}) Ln[1 + |u|^{\alpha}], \tag{1.3}$$

where $v(t) = t^{\alpha\lambda}, 0 < \alpha \leq 2, \lambda > 0$.

The normalized Linnik distribution (Linnik, 1962), obtained by setting $\gamma = 1$ in (1.1), has the CHF:

$$\varphi(t) = \frac{1}{1 + |t|^{\alpha}}, \quad t \in R, 0 < \alpha \leq 2. \tag{1.4}$$

This distribution is also known as the α-Laplace distribution, reducing to the symmetric Laplace distribution when $\alpha = 2$. Pillai (1985) later generalized this to the semi-α-Laplace distribution, while Sabu and Pillai (1987) derived its multivariate extension. Further generalizations of Laplace-type distributions were investigated by Kotz et al.

(2001). Anderson (1992) examined the properties of Linnik distributions and developed Linnik processes for modeling stock price returns. Geometric variants, including the geometric Laplace and geometric α-Laplace distributions, were introduced by Seetha Lekshmi and Jose (2003, 2004). Pakes (1998) extended this framework by proposing the generalized Linnik distribution with CHF:

$$\varphi(t) = \frac{1}{(1+|t|^{\alpha})^{\nu}}, \quad 0 < \alpha \leq 2, \quad \nu > 0,$$

This distribution is referred to as the Pakes generalized Linnik distribution. Two important special cases emerge:

- when $\nu = 1$, it reduces to the classical Linnik distribution.
- when $\alpha = 2$, it yields the generalized Laplacian distribution introduced by Mathai (1993a).

Seetha Lekshmi and Jose (2004) made significant contributions by:

- Developing first-order autoregressive processes based on the Pakes generalized Linnik distribution
- Introducing the geometric Pakes generalized Linnik distribution, characterized by its characteristic function (CHF):

$$\varphi(t) = \frac{1}{1+\nu\log(1+|t|^{\alpha})}, \quad 0 < \alpha \leq 2, \quad \nu > 0, \tag{1.5}$$

The distribution is denoted by $GPGL(\alpha, \nu)$. Notably:

- when $\nu = 1$, it simplifies to the geometric $\alpha-$Laplace distribution

  and

- Seetha Lekshmi and Jose (2006) developed autoregressive processes with *GPGL* marginal distributions

Recent extensions in this domain include Jayakumar and Kalyanaraman (2007) that introduced autoregressive processes with diverse marginal distributions and Torricelli et al. which investigated processes related to the tempered positive Linnik distribution, characterized by its Laplace transform as below:

$$L(s) = (\frac{1}{1+sgn(\gamma)\lambda((\theta+s)^{\gamma})-\theta\gamma})^{\delta}, Re(s) > 0,$$

with parameter space for $(\gamma, \lambda, \delta, \theta)$ given by

$$S = \{(-\infty, 1] \setminus \{0\} \times \mathbb{R}_{+} \times \mathbb{R}_{+} \times \mathbb{R}_{+}\} \cup \{(0, 1] \times \mathbb{R}_{+} \times \mathbb{R}_{+} \times \{0\}\}.$$

Recent developments in generalized Linnik distributions include:

- Korolev et al. (2020) presented novel mixture representations of the generalized Linnik distribution in terms of Normal, Laplace, exponential, and stable laws, establishing relationships between the mixing distributions.
- Stanislavsky and Werron (2021) investigated confined random motion governed by Laplace and Linnik statistics. While extensive research exists on heavy-tailed distributions, a significant limitation persists: most studies focus exclusively on symmetric cases ($\beta = 0$, where β represents the skewness parameter). This restriction is particularly problematic given that real-world signals typically exhibit skewness, as evidenced in financial time series, computer network traffic, queue service times, hydrological and meteorological data, geophysical signals, urban vehicle noise and telecommunications noise (Dance & Kuruoglu, 1999).

In this work, we make several key contributions to the theory of skewed heavy-tailed distributions:

(i) We propose an extension of the Linnik distribution incorporating a skewness parameter β (with $|\beta| \leq 1$).

(ii) Based on this extension, we develop a first-order autoregressive (AR(1)) model (Section 2).

(iii) We introduce a novel class of geometric skewed Linnik distributions and rigorously examine their properties (Section 3).

(iv) We generalize these results to construct kth-order autoregressive processes (Section 4).

(v) We validate our approach through comprehensive real-data analysis and estimation (Section 5).

2 A New Class of Generalized Skewed Linnik Distributions

In this section, the generalized skewed Linnik distribution is introduced and some of its properties are studied.

Definition 2.1. A random variable W on $R = (-\infty, +\infty)$ is said to follow the generalized skewed Linnik distribution and write $W \stackrel{d}{=} \mathrm{GSL}(\alpha, \beta, \nu)$ if it involves the CHF

$$\varphi(t) = \frac{1}{(1 + |t|^{\alpha}\, w(t, \alpha, \beta))^{\nu}}, \quad 0 < \alpha \leq 2, \quad -1 \leq \beta \leq 1, \quad \nu > 0, \tag{2.1}$$

where

$$w(t, \alpha, \beta) = \begin{cases} 1 - i\beta \tan \frac{\pi\alpha}{2}\, \mathrm{sign}(t), & \alpha \neq 1, \\ 1 + i\beta \ln |t|\, \mathrm{sign}(t), & \alpha = 1. \end{cases} \tag{2.2}$$

In fact $\exp(-|t|^{\alpha}\, w(t, \alpha, \beta))$ is the CHF of a stable distribution (see Samorodnitsky and Taqqu, 1994 page 5).

Remark 1. If $\beta = 0$, the generalized skewed Linnik distribution reduces to Pakes generalized Linnik distribution.

Remark 2. If $\nu = 1$, the generalized skewed Linnik distribution gives a geometric stable distribution or a skewed Linnik distribution with α and β parameters. In Theorem (2.1), we present stochastic representation for generalized skewed Linnik random variable.

Theorem 2.1. *If V and Y are independent random variables in such a way that V has the gamma distribution with the Laplace-Stieltjes transform $(1+t)^{-\nu}$ and Y has a stable distribution having the CHF $g(t) = \exp(-|t|^{\alpha} w(t, \alpha, \beta))$, then $V^{1/\alpha} Y \stackrel{d}{=} W(\alpha, \beta, \nu)$, where $W(\alpha, \beta, \nu)$ denotes a random variable having the CHF (2.1).*

Proof. We define W as $V^{1/\alpha} Y$. Then we have:

$$\begin{aligned}\varphi_W(t) &= E(\exp(itV^{1/\alpha}Y)) \\ &= E[E(\exp(itV^{1/\alpha}Y|V)] \\ &= E[\exp(-|t|^{\alpha} w(t,\alpha,\beta)V)] \\ &= \frac{1}{(1+|t|^{\alpha} w(t,\alpha,\beta))^{\nu}}.\end{aligned}$$

□

This theorem defines a family of generalized skewed Linnik law GSL(α, β, ν), $W \stackrel{d}{=}$ GSL$(\alpha, \beta, \upsilon)$. This representation can be used to simulate the new skewed Linnik law GSL(α, β, ν).

For $\beta = 0$, this mixture representation defines the generalized symmetric Linnik law which was given by Pakes (1998).

The above representation plays a fundamental role in deriving the following Theorem, which talks about its moments.

Theorem 2.2. *Let $W \stackrel{d}{=} GSL(\alpha, \beta, \upsilon)$ with $0 < \alpha < 2$. Then,*

$$E|W|^p < \infty \quad \text{for any} \quad 0 < p < \alpha, \tag{2.3}$$

$$E|W|^p = \infty \quad \text{for any} \quad p \geq \alpha. \tag{2.4}$$

We omit the proof of the Theorem. In fact it follows from the same moment property for stable distributions.

As an application, we consider the generalized skewed Linnik first-order autoregressive process (GSLAR(1)), constructed through $\{X_n; n \geq 1\}$ where X_n satisfies the equation

$$X_n = \phi X_{n-1} + \varepsilon_n, \qquad |\phi| < 1, \tag{2.5}$$

where ε_n is a sequence of independent and identically distributed random variables such that X_n is a stationary Markovian with the generalized skewed Linnik marginal distribution. In terms of the CHF, it can be framed as

$$\varphi_{X_n}(t) = \varphi_{\varepsilon_n}(t) \varphi_{X_{n-1}}(\phi t).$$

Assuming stationarity, we have for $\alpha \neq 1$,

$$\begin{aligned}\varphi_{\varepsilon_n}(t) &= \frac{\varphi_{X_n}(t)}{\varphi_{X_{n-1}}(\phi t)} = \frac{[1+\left|\phi t\right|^{\alpha} w(\phi t,\alpha,\beta)]^{\nu}}{[1+|t|^{\alpha} w(t,\alpha,\beta)]^{\nu}} = \frac{[1+\left|\phi\right|^{\alpha} |t|^{\alpha} w(t,\alpha,\beta)]^{\nu}}{[1+|t|^{\alpha} w(t,\alpha,\beta)]^{\nu}} \\ &= [\left|\phi\right|^{\alpha} + (1-\left|\phi\right|^{\alpha})\frac{1}{(1+|t|^{\alpha} w(t,\alpha,\beta))}]^{\nu}.\end{aligned}$$

Hence for $\left|\phi\right| < 1$, the innovation $\{\varepsilon_n\}$ can be regarded as the ν-convolution of random variables $U_n s$ yielding

$$U_n = \begin{cases} 0 & \text{with probability} \quad |\phi|^{\alpha}, \\ \text{GS} & \text{with probability} \quad 1-|\phi|^{\alpha}, \end{cases}$$

where GSs are identically and independently distributed geometric stable random variables.

The sample path of the generalized skewed Linnik first-order autoregressive process for various values of α, β, ϕ when $\nu = 2$, is illustrated in Fig. 1. For each AR(1) process the autocorrelation function is also given in Fig. 1. The first-order sample autocorrelation is approximately equal to ϕ.

We simulated the generalized skewed Linnik autoregressive process using the known coefficient. Then we used the simulated values and minimized the following function to estimate ϕ as the unknown coefficient

$$\sum_{n=1}^{k} \left|X_n - \phi X_{n-1}\right|^{p}, \tag{2.6}$$

where $0 < p < \alpha$. For $k = 100$ and

(i) $\alpha = 1.5, \beta = 0.5, \nu = 2, \phi = 0.4$, the estimated value of ϕ is 0.38 ($\hat{\phi} = 0.38$),

(ii) $\alpha = 1.75, \beta = -0.75, \nu = 2, \phi = 0.2$, the estimated value of ϕ is 0.27,

(iii) $\alpha = 0.5, \beta = 0.5, \nu = 2, \phi = 0.3$, the estimated value of ϕ is 0.29.

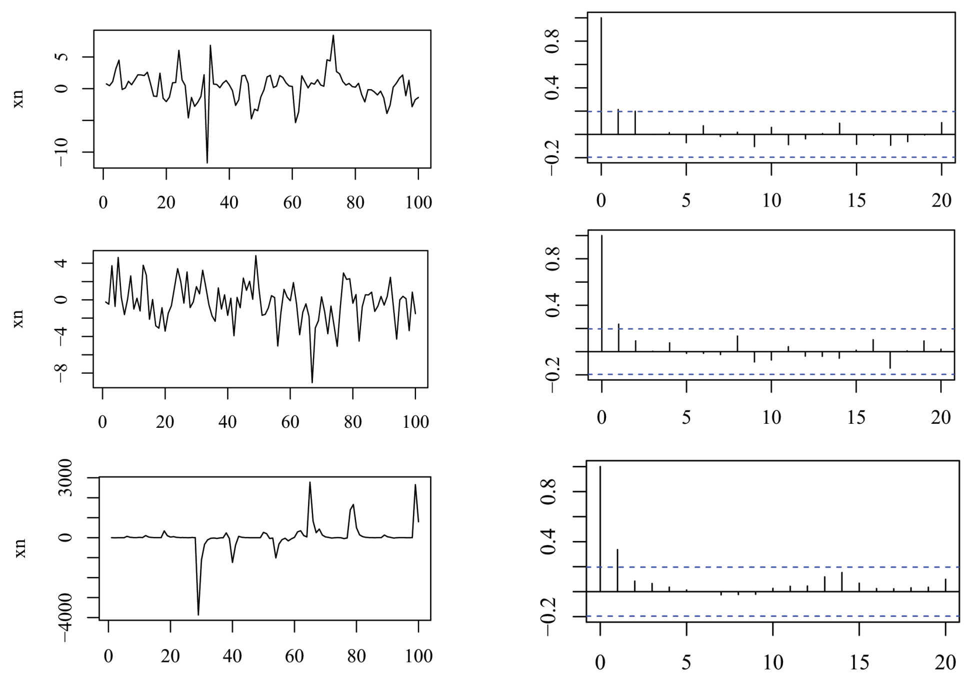

Figure 1: The sample paths of $AR(1)$ processes for $\nu = 2$ and for (a) $\alpha = 1.5$, $\beta = 0.5$ and $\phi = 0.4$, (b) $\alpha = 1.0$, $\beta = -0.75$ and $\phi = 0.2$ and (c) $\alpha = 0.5$, $\beta = 0.5$ and $\phi = 0.3$.

3 A New Class of Generalized Geometric Skewed Linnik Distributions

A random variable X on $R = (-\infty, +\infty)$ is said to follow the generalized geometric skewed Linnik distribution and write $X \stackrel{d}{=} \text{GGSL}(\alpha, \beta, \nu)$ if it involves the CHF

$$\varphi(t) = \frac{1}{1 + \nu \log(1 + |t|^{\alpha} w(t, \alpha, \beta))}, \quad 0 < \alpha \le 2, \quad -1 \le \beta \le 1, \quad \nu > 0, \tag{3.1}$$

where $w(t, \alpha, \beta)$ is as (2.2).

Remark 3. In this remark we consider some special cases of the (3.1):

i) If $\nu = 1$, and $\beta = 0$, (3.1) reduces to the geometric α- Laplace distribution.

ii) If $\beta = 0$, (3.1) reduces to the Geometric Pakes generalized Linnik distribution.

iii) The generalized geometric skewed Linnik distribution is geometrically and infinitely divisible.

The relation between the generalized skewed α-Laplace distribution and its random sum is discussed in the following theorem.

Theorem 3.1. *Suppose that $X_1, X_2, ...$ are iid generalized skewed α-Laplace random variables and let $N(p)$ be a geometric random variable with mean $1/p$. If $Y = X_1 + X_2 + ... + X_{N(p)}$, then $Y \stackrel{d}{=} GGSL(\alpha, \beta, 1/p)$.*

Proof. The CHF of Y is

$$\begin{aligned}\varphi_Y(t) &= \sum_{k=1}^{\infty} [\varphi_X(t)]^k p(1-p)^{k-1} \\ &= \frac{(p/(1+\log(1+|t|^\alpha w(t,\alpha,\beta)))}{1-((1-p)/(1+\log(1+|t|^\alpha w(t,\alpha,\beta)))} \\ &= \frac{1}{1+(1/p)\log(1+|t|^\alpha w(t,\alpha,\beta))}.\end{aligned}$$

Hence $Y \stackrel{d}{=}$ GGSL$(\alpha, \beta, 1/p)$. □

The limiting distribution of the generalized geometric sum of the generalized Pakes is presented in the next theorem.

Theorem 3.2. *The generalized geometric skewed Linnik distribution is the limiting distribution of the generalized geometric sum of the generalized Pakes.*

Proof. We have $[1 + |t|^\alpha w(t, \alpha, \beta)]^{-\nu/n} = \{1 + [1 + |t|^\alpha w(t, \alpha, \beta)]^{\nu/n} - 1\}^{-1}$ as the CHF of a probability distribution because the new generalization of Pakes generalized Linnik distribution is infinitely divisible. Hence in accordance with (Pillai (1990), Lemma 3.2)

$$\varphi_n(t) = \{1 + [1 + n\,|t|^\alpha w(t, \alpha, \beta)]^{\nu/n} - 1\}^{-1},$$

is the CHF of the geometric sum of the identically and independently distribution as the new generalization of Pakes generalized Linnik random variables. Limit is taken via $n \to \infty$

$$\begin{aligned}\varphi(t) &= \lim_{n\to\infty} \varphi_n(t) \\ &= \{1 + [1 + \lim_{n\to\infty} n\,|t|^\alpha w(t, \alpha, \beta)]^{\nu/n} - 1\}^{-1} \\ &= [1 + \nu log(1 + |t|^\alpha w(t, \alpha, \beta))]^{-1}.\end{aligned}$$

□

The following Theorem is useful for generating GGSL$(\alpha, \beta, \upsilon)$ random variable.

Theorem 3.3. *If X and Y are independent random variables in such a way that X has the geometric gamma distribution with the Laplace transform $1/(1 + \nu\log(1 + t))$ and Y has a stable distribution having the CHF $g(t) = \exp(-|t|^\alpha w(t, \alpha, \beta))$, then $X^{1/\alpha}Y \stackrel{d}{=} Z$ where $Z \stackrel{d}{=} GGSL(\alpha, \beta, \upsilon)$.*

Proof. Let $Z{=}X^{1/\alpha}Y$

$$\begin{aligned}\varphi_Z(t) &= E(\exp(itX^{1/\alpha}Y))\\ &= \int_0^{+\infty}\varphi_Y(tx^{1/\alpha})dF(x), \quad \text{where} \quad F(\cdot) \text{ is the distribution of } X\\ &= \int_0^{+\infty}\exp(-x\,|t|^\alpha\, w(tx,\alpha,\beta)dF(x)\\ &= \frac{1}{1+\upsilon\log(1+|t|^\alpha\, w(t,\alpha,\beta))}.\end{aligned}$$

□

The above representation plays a fundamental role in deriving the following theorem.

Theorem 3.4. *Let* $Z \stackrel{d}{=} X^{1/\alpha}Y$ *with* $0<\alpha<2$*. Then,*

$$E\,|Z|^p < \infty, \quad \textit{for any} \quad 0<p<\alpha, \tag{3.2}$$

$$E\,|Z|^p = \infty, \quad \textit{for any} \quad p\geq\alpha. \tag{3.3}$$

4 Generalized Geometric Skewed Linnik Processes

In this section we can develop a first-order new autoregressive process with a new family of geometric Pakes generalized Linnik marginal distributions. Let $\{X_n\}$ be an autoregressive process given by

$$X_n = \begin{cases}\varepsilon_n & \text{with probability} \quad p,\\ X_{n-1}+\varepsilon_n & \text{with probability} \quad 1-p,\end{cases} \tag{4.1}$$

where $0<p<1$ and $\{\epsilon_n\}$ represent a sequence of independent and identically distributed random variables. This tractable form of autoregressive model through exponential variables is discussed by Lawrance and Lewis (1981).

Now we construct an $AR(1)$ process with a stationary marginal distribution as the generalized geometric skewed Linnik distribution GGSL(α,β,ν).

Theorem 4.1. *Consider the stationary autoregressive process* $\{X_n\}$ *with a structure given by* (4.1)*. A necessary and sufficient condition for* $\{X_n\}$ *and* $\{\epsilon_n\}$ *to be identically distributed, except for a scale change, is that* $\{X_n\}$ *must be marginally distributed as a generalized geometric skewed Linnik distribution.*

Proof. Denoting the CHF of X_n through $\varphi_{X_n}(t)$ and that of ϵ_n through $\varphi_{\epsilon_n}(t)$, (4.1) in terms of the CHF is

$$\varphi_{X_n}(t) = p\varphi_{\epsilon_n}(t)+(1-p)\varphi_{X_{n-1}}(t)\varphi_{\epsilon_n}(t).$$

Under the assumption of stationarity, it reduces to the form

$$\varphi_X(t) = p\varphi_\epsilon(t) + (1-p)\varphi_X(t)\varphi_\epsilon(t).$$

Writing $\varphi_X(t) = \frac{1}{1+v\log(1+|t|^\alpha w(t,\alpha,\beta))}$, as a solution for $\varphi_\epsilon(t)$, we obtain $\varphi_\epsilon(t) = \frac{1}{1+pv\log(1+|t|^\alpha w(t,\alpha,\beta))}$. Hence it follows that $\epsilon_n \stackrel{d}{=} GGSL(\alpha,\beta,pv)$.

Conversely assume that X_n and ϵ_n are independent and identically distributed, except for a scale change. That is $\varphi_\epsilon(t) = \varphi_X(kt)$, where k is a constant. But $\varphi_\epsilon(t) = \frac{1}{1+v\log(1+|t|^\alpha w(t,\alpha,\beta))}$. Therefore we have

$$\begin{aligned}\varphi_X(t) &= \frac{p\varphi_\epsilon(t)}{1-(1-p)\varphi_\epsilon(t)} = \frac{p/[1+v\log(1+|t|^\alpha\, w(t,\alpha,\beta))]}{1-(1-p)/[1+v\log(1+|t|^\alpha\, w(t,\alpha,\beta))]}\\ &= \frac{1}{1+(v/p)\log(1+|t|^\alpha\, w(t,\alpha,\beta))}.\end{aligned}$$

Taking $k = 1/p$, it follows that $X \stackrel{d}{=} GSL(\alpha,\beta,\nu/p)$. □

The structure in (4.1) can be rewritten as $X_n = I_n X_{n-1} + \epsilon_n$, where $P[I_n = 0] = 1 - P[I_n = 1] = p$, $0 < p < 1$. Then the joint CHF of (X_n, X_{n-1}) of this structure is given by

$$\begin{aligned}\varphi_{X_{n-1},X_n}(s,t) &= E(e^{isX_{n-1}+itX_n})\\ &= E(e^{isX_{n-1}+it(I_nX_{n-1}+\epsilon_n)})\\ &= E(e^{i(s+tI_n)X_{n-1}})\varphi_{\epsilon_n}(t)\\ &= \frac{1}{1+vp\log(1+|t|^\alpha\, w(t,\alpha,\beta))}\Big[\frac{p}{1+v\log(1+|s|^\alpha\, w(s,\alpha,\beta))} + \\ &\quad \frac{1-p}{1+v\log(1+|s+t|^\alpha\, w(s+t,\alpha,\beta))}\Big].\end{aligned}$$

From the expression it is evident that $\varphi_{X_{n-1},X_n}(s,t) \neq \varphi_{X_{n-1},X_n}(t,s)$ and hence the process is not time reversible.

4.1 Generalization to the *kth* Order of the Geometric Skewed Linnik Autoregressive Process

Lawrance and Lewis (1982) constructed the higher-order analogs of the autoregressive equation (4.1) via the structure

$$X_n = \begin{cases}\epsilon_n, & w.p. \quad p,\\ X_{n-1}+\epsilon_n, & w.p. \quad p_1,\\ X_{n-2}+\epsilon_n, & w.p. \quad p_2,\\ \cdots & \\ X_{n-k}+\epsilon_n, & w.p. \quad p_k,\end{cases} \tag{4.2}$$

where $\sum_{i=1}^{k} p_i = 1 - p$, $0 \leq p_i, p \leq 1$, and ϵ_n is independent of $\{X_{n-1}, X_{n-2}, ...\}$.

In terms of the CHF can be computed as $\varphi_{X_n}(t) = p\varphi_{\epsilon_n}(t) + p_1\varphi_{X_{n-1}}(t)\varphi_{\epsilon_n}(t) + p_2\varphi_{X_{n-2}}(t)\varphi_{\epsilon_n}(t) + ... + p_k\varphi_{X_{n-k}}(t)\varphi_{\epsilon_n}(t)$.
Under the assumption of stationary, we obtain,

$$\varphi_{\epsilon}(t) = \frac{\varphi_X(t)}{1 - (1-p)\varphi_X(t)}.$$

This demonstrates that the results developed in the first part of Section 4 are valid in this case as well. This defines a stationary autoregressive process of order k through the geometric skewed Linnik marginal distribution.

5 A Real Data Analysis

The standard Box-Jenkins methodology for Gaussian time series analysis comprises three key stages: model identification, parameter estimation, and diagnostic checking. However, these stages are fundamentally interconnected, as diagnostic checks typically reduce to verifying whether the fitted residuals constitute a white noise process - effectively returning the analysis to the model identification phase. In this work, we focus primarily on ARMA model identification within the stable distribution framework, with secondary attention to parameter estimation techniques. Our contribution is threefold:

i) We synthesize previously scattered theoretical results into a unified framework.

ii) We evaluate their practical performance through extensive simulation studies.

iii) We identify several surprising and theoretically consequential empirical findings.

Our analysis reveals that while standard Gaussian Box-Jenkins techniques can be formally extended to the Linnik setting, their practical implementation requires significant modifications and careful interpretation. A crucial diagnostic for infinite variance processes involves monitoring the behavior of the sample variance: S_n^2 = variance of first n observations Key diagnostic patterns:

1. Finite variance case: S_n^2 converges to σ^2 as n tends to ∞,

2. infinite variance case: S_n^2 diverges as n n tends to ∞.

Notably, this test remains valid for both independent and correlated data structures.

Figure 2 shows this test for the global average Sea Surface Temperature (SST) anomaly data, multiplied by 100, for the month of November from 1980 to 2024. The data were downloaded from http://jisao.washington.edu. The divergence of S_n^2 as n increases, as well as the irregularity of the graph, are clearly visible.

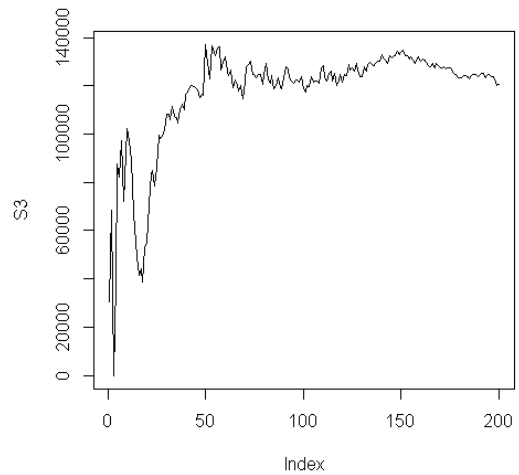

Figure 2: Figure of S_n^2 for the global average Sea Surface Temperature data

5.1 Model identification

The first step to fit data $\{X_1, X_2, \cdots, X_n\}$ into a ARMA(p, q) model is to determine lag parameters p and q.

In the finite variance case model, identification techniques are based on an analysis of the sample autocorrelation function $\hat{\rho}_k$ (ACF) and sample partial autocorrelation function $\hat{\phi}_{kk}$ (PACF). Although second moments are infinite in the generalized skewed Linnik case, the tools used are also samples of ACF and PACF. The graph of the data shows that the data do not seem to have emerged from stationary time series (Figure 3). One lag difference shows stationarity with some sharp spikes, which reconfirms that the data are heavy-tailed (Figure 4). By doing a search over all possible models, and choosing according to AIC, BIC and –logilk (function "auto.arima" in R library "forecast") One can observe that the data follow an ARIMA(2 ,1 ,0). Table 1 shows AIC, BIC and –logilk for different models.

Table 1: AIC, BIC and $-\log ilk$ for different models.

Model	AIC	BIC	$-\log ilk$
ARIMA(2, 1, 0)	22952.36	22900.17	11481.33
ARIMA(1, 1, 0)	23046.37	23015.18	11593.26
ARIMA(1, 1, 1)	23145.89	23117.14	11623.17
ARIMA(2, 2, 0)	23315.07	23119.53	11645.18
ARIMA(2, 2, 1)	23423.18	23254.15	11973.15
ARIMA(1, 2, 0)	23619.26	23353.21	12156.04

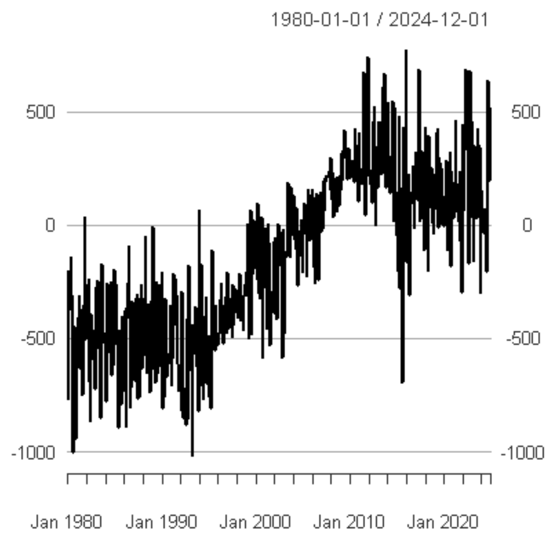

Figure 3: Time series plot of the global average Sea Surface Temperature data

5.1.1 Estimation of the Distribution Parameters

A popular technique for designing estimators is to find the formula for the moments of a distribution in terms of distribution parameters. This technique is not applicable to GSL(α, β, ν) distribution because for $\alpha < 2$ the distribution has infinite variance, and for $0 < \alpha \leq 1$ it has infinite mean. Halvarsson (2013) showed that an alternative is to consider fractional moments. The unsigned and signed $q-$th fractional moments are defined through

$$\mu_{|W|^q} = E[|W|^q] = \int_{-\infty}^{\infty} |w|^q f(w) dw,$$

$$\mu_{W^{<q>}} = E[W^{<q>}] = \int_{-\infty}^{\infty} sign(w)|w|^q f(w) dw,$$

where $f(w)$ is the density function of the random variable W and $sign(w)$ is the signum function. The empirical counterparts of $\hat{\mu}_{|W|^q}$ and $\hat{\mu}_{W^{<q>}}$ are given by

$$\hat{\mu}_{|W|^q} = \frac{1}{n} \sum_{i=1}^{n} |W_i|^q,$$

$$\hat{\mu}_{|W|^{<q>}} = \frac{1}{n} \sum_{i=1}^{n} sign(W_i)|W_i|^q.$$

To find $\hat{\mu}_{|W|^q}$ and $\hat{\mu}_{W^{<q>}}$ when $f(w)$ is distributed after GSL(α, β, ν), one can follow Theorem 5.1.

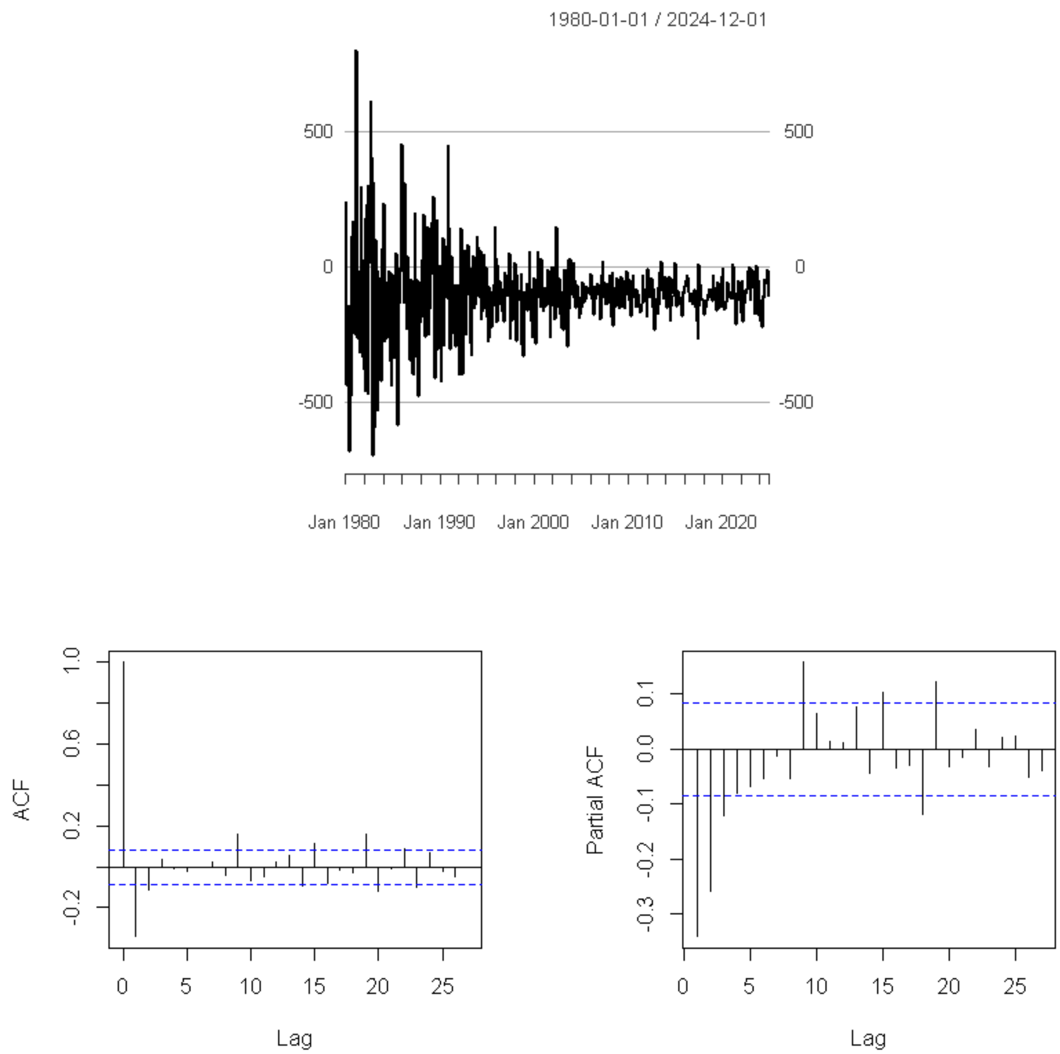

Figure 4: Time series plot with one difference for the global average Sea Surface Temperature data, with ACF and PACF.

Theorem 5.1. *Let $q' = min(1, \alpha)$, then for $q \in (-q', \alpha)$ and $\alpha \neq 1$ the qth unsigned and signed fractional moments of $W \stackrel{d}{=} GSL(\alpha, \beta, \nu)$ can be formulated as*

$$\mu_{|W|^q} = E[|W|^q] = \frac{\Gamma(\nu + \frac{q}{\alpha})\Gamma(1 - \frac{q}{\alpha})}{\Gamma(\nu)\Gamma(1-q)} \frac{cos(\frac{q\theta}{\alpha})}{|cos\theta|^{q/\alpha} cos(\frac{q\pi}{2})}, \qquad q \in (-1, \alpha),$$

$$\mu_{W^{<q>}} = E[W^{<q>}] = \frac{\Gamma(\nu + \frac{q}{\alpha})\Gamma(1 - \frac{q}{\alpha})}{\Gamma(\nu)\Gamma(1-q)} \frac{sin(\frac{q\theta}{\alpha})}{|cos\theta|^{q/\alpha} sin(\frac{q\pi}{2})}, \qquad q \in (-2, -1) \cup (-1, \alpha),$$

where $\Gamma(\cdot)$ is the gamma function and $\theta = arctan(\beta tan(\frac{\alpha\pi}{2}))$.

Proof. Dance and Kuruoglu (1999) obtained the fractional moments for skewed α-stable random variables.

According to Theorem 2.1, $W \overset{d}{=} V^{1/\alpha}Y$ with V and Y is given in the Theorem. Therefore we have:

$$\begin{aligned} E[|W|^q] &= E[V^{q/\alpha}]E[|Y|^q] \\ &= \frac{\Gamma(\nu+\frac{q}{\alpha})}{\Gamma(\nu)}\frac{\Gamma(1-\frac{q}{\alpha})}{\Gamma(1-q)}\frac{cos(\frac{q\theta}{\alpha})}{|cos\theta|^{q/\alpha}cos(\frac{q\pi}{2})}, \quad q \in (-1,\alpha), \end{aligned}$$

$$\begin{aligned} E[W^{<q>}] &= E[V^{q/\alpha}]E[[Y^{<q>}] \\ &= \frac{\Gamma(\nu+\frac{q}{\alpha})}{\Gamma(\nu)}\frac{\Gamma(1-\frac{q}{\alpha})}{\Gamma(1-q)}\frac{sin(\frac{q\theta}{\alpha})}{|cos\theta|^{q/\alpha}sin(\frac{q\pi}{2})}, \quad q \in (-2,-1)\cup(-1,\alpha). \quad \square \end{aligned}$$

While fractional moments estimators are typically designed for independent data, our analysis deals with correlated AR(2) processes. To address this limitation, we employ systematic sampling (1 of 4) to generate uncorrelated subsequences, as confirmed by generalized runs tests. This transformation allows valid application of fractional moments estimation. Using $q = 0.5$, we obtain the parameter estimates $\alpha = 1.3517$, $\beta = -0.0841$ and $\nu = 168.3519$. It should be noted that the estimation involves four potential estimators (from signed and unsigned fractional moments at $q = 0.5$) for three parameters. We resolve this overdetermination by excluding the signed fractional moment estimator for $q = -0.5$, maintaining parameter identifiability while preserving estimation robustness.

5.1.2 Prameter Estimation for ARMA Models (Whittle Estimator)

To estimate the parameters of an ARMA model, Whittle suggested a procedure that is based on a periodogram (see Adler and et al 1998 and Mikosch and et al 1995). In this set up X_t is a causal and invertible ARMA(p, q)-process, i.e.

$$\Phi(B)X_t = \Theta(B)\varepsilon_t.$$

Becuase we assume causality and invertibility, the complex-valued polynomial $\Phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ and $\Theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$ have no common zeros and no zero on or inside the unit disk. The space Θ is defined for all $\theta \in R^{p+q}$ with the two aforementioned properties. For $\theta \in \Theta$ the power transfer function is defined:

$$g(\lambda;\theta) = \frac{|\Theta(e^{-i\lambda})|^2}{|\Phi(e^{-i\lambda})|^2}.$$

The self-normalized periodogram is denoted via

$$I_{n,X}(\lambda) = \frac{|\sum_{t=1}^{n} X_t e^{-i\lambda}|^2}{\sum_{t=1}^{n} X_t^2}, \qquad \lambda \in (-\pi,\pi]$$

$\overline{\sigma}^2_{n,X} = \frac{1}{n}\sum_j \frac{I_{n,X}(\lambda_j)}{g(\lambda_j,\theta)}$, is defined where summation is taken over $\lambda_j = \frac{2\pi j}{n} \in (-\pi, \pi]$. Minimizing $\overline{\sigma}^2_{n,X}$ with respect to $\theta \in \Theta$ leads to the Whittle estimators given by

$$\hat{\theta}_n = argmin_{\theta\in\Theta}\overline{\sigma}^2_{n,X}(\theta).$$

According to this procedure, for the AR(2) model ϕ_1 and ϕ_2 are estimated by -0.517 and -0.302 respectively for GSL(1.3517, −0.0841, 160.3519).

Future Work:

The extension of the method to ARMA processes would be interested as a future work.

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