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Optimal Risk Management Strategies in a General Perturbed Risk Process

Abouzar Bazyari

Department of Statistics, Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr, Iran.

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Abstract. This paper investigates the optimal risk management strategies in a general compound Poisson risk model consisting of safety loading of insurer and reinsurance to minimize the infinite-time ruin probability. Its price process is perturbed by a geometric Brownian motion with the drift and volatility of risky asset. In addition, we allow this company to buy proportional reinsurance to reduce the underlying risk and invest its surplus in a risky asset whose price is driven by correlated Brownian motions. We focus on the possibility of an insurance company utilizing optimal controls and study the optimization problem of minimizing the infinite-time ruin probability in a financial market. For the diffusion approximation of risk model, we obtain an analytic expression for the minimum nfinite-time ruin probability and the corresponding optimal controls by using the martingale approach. Since it is not easy to derive the explicit expression for the infinite-time ruin probability of perturbed risk model, we obtain the optimal strategies to maximize the Lundberg exponent when the claim amounts are identically distributed and have an exponentially decaying tail. Moreover, we study the effect of investment on the ruin probability in both perturbed risk models. Finally, some numerical examples are conducted to illustrate the effects of model parameters on the optimal risk management strategies and on the financial market.

Keywords. Diffusion Risk Process, Infinite-time Ruin Probability, Optimization Problem, Proportional Reinsurance, Risky Asset.

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Corresponding Author: Abouzar Bazyari (ab_bazyari@pgu.ac.ir)

1 Introduction

When the surplus process of an insurance company falls below zero, the company is said to have experienced ruin. Insurance companies customarily take precautions to avoid ruin. These precautions are referred to as control variables and include investments, capital injections or refinancing, portfolio selection, and reinsurance arrangements, to mention but a few. The ruin probability is an important risk measure and has been frequently considered in recent years and studying the ruin probabilities helps us to investigate behaviors of a company when it is in deficit.

The optimal portfolio selection problem is of practical importance in actuarial science and mathematical finance. In this paper, we will find the optimal investment and proportional reinsurance strategies to minimize the infinite-time ruin probability of an insurer who faces a claim process that follows Brownian motion with drift. Furthermore, we will try to investigate more directly the effect of investment on the infinite-time ruin probability in a perturbed compound risk model and its diffusion approximation.

Merton (1971) used the stochastic optimal control method in continuous finance to obtain a closed-form solution to the problem of optimal portfolio strategy under specific assumptions about asset returns and investor preferences. Browne (1995) minimized the infinite-time ruin probability by finding the optimal investment strategy in the financial market. Schmidli (2001) found the optimal reinsurance strategy for an insurer facing a claim process that follows Brownian motion with drift but, otherwise, assumed that the surplus was not invested in any financial instrument. Hipp and Plum (2000) and Chen et al. (2010) considered the optimal reinsurance-investment problems for insurers in the sense of minimizing the ruin probability. Zhang and Siu (2012), Yi et al. (2013) and Liang and Bayraktar (2014) focused on seeking the optimal reinsurance-investment strategies to maximize the expected utility from terminal wealth. Bai and Guo (2008) studied the optimal proportional reinsurance and investment for maximizing exponential utility and minimizing the ruin probability with no-shorting constraint. Liang and Long (2015), considered the problem of minimizing the absolute ruin probability in a diffusion approximation model of the insurance company's surplus where the insurance company can invest in financial market, and in which the risky asset follows Black-Scholes model.

Zeng et al. (2016) investigated the optimal reinsurance-investment strategies in different situations under the mean-variance criterion. Zhang et al. (2016) analyzed the optimal investment and reinsurance strategies for insurers with a generalized mean-variance premium principle in diffusion model. While from the reinsurers point of view, Wang and Peng (2017) considered the optimal reinsurance strategy that minimizes the reinsurer's risk measure of his total loss when the risk is measured by distortion risk measures and premiums are calculated under the distortion premium principle. The optimal investment and reinsurance strategies for an insurance company to minimize the ruin probability, for the risk process with Brownian motion, and correlated risk model with common Poisson shocks are given in Liang and Young (2018) and Xu et al. (2018), respectively. Zhang et al. (2018) investigated the optimal quota-share reinsurance strategies and showed that the reinsurer's safety loading plays a key role in determining the optimal retained proportion. Han et al. (2022) studied an optimal reinsurance-investment problem in a risk model with two dependent classes of insurance business and showed that the optimal reinsurance strategy is in the form of pure quota-share reinsurance under the variance premium principle. Bazyari (2025) investigated the optimization problem on the infinite-time ruin probability when the insurance company allows to buy quota-share reinsurance with a risky asset simultaneously.

Motivated by the above papers, we consider a general perturbed compound Poisson risk model involving two correlated Brownian motions and its diffusion approximation model using the aggregate claim process. It is supposed that the insurance company can control allocations of the surplus into a risky asset and reinsurance purchases. We aim to find the optimal investment and reinsurance strategies to minimize the infinite-time ruin probability in both risk models.

The main novelty of this paper is to consider the possibility for an insurance company to rely on optimal controls and study the optimization problem in the financial market in order to minimize the infinite-time ruin probability in two risk models, (i) a general perturbed compound Poisson risk model consisting of safety loading of insurer and reinsurance with the drift and volatility of risky asset, (ii) the diffusion approximation of risk model. We will show that investment with the diffusion approximation model is better than the perturbed compound Poisson risk model with risky asset and reinsurance, which leads to decreasing the ruin probability.

The rest of this paper is organized as follows. In Section 2, we give the perturbed compound risk model with risky asset and reinsurance, where the price process is perturbed by the expected instantaneous rate of risky asset and asset's volatility. Moreover, the diffusion approximation of the perturbed compound Poisson risk model is presented. In Section 3, the optimal strategies in both models to minimize the ruin probabilities are obtained. In Section 4, the effect of investment on the infinite-time ruin probabilities in both models are studied by mathematical approaches. Numerical examples to illustrate the effects of model parameters on the optimal risk management strategies are given in Section 5. Concluding remarks are presented in Section 6.

2 Risk Model and Financial Market

In this study, we give a general perturbed compound Poisson risk model and its diffusion approximation with reinsurance and investment strategies to minimize the infinite-time ruin probabilities. The company operates in a continuously evolving environment, where it receives premiums continuously and faces claims modeled by a compound Poisson process. To maximize its profitability, the insurance company dynamically adjusts its risk exposure through purchase of reinsurance, and invest its surplus in a risky asset. The optimization problem is to determine the optimal reinsurance and investment strategies that maximize the profitability of insurance company.

We formulate our risk model and stochastic optimal control problems. In the sequel, we will work on a probability space $(\Omega, \mathcal{F}, P)$, endowed with the information filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which carries all stochastic quantities to be defined in the following. Let us start with compound Poisson risk model (also known as Cramér-Lundberg model or classical risk model), which is an actuarial model used to analyse the risk of an insurance portfolio as

$$dR_t = cdt - dS_t, \qquad t \geq 0, \tag{2.1}$$

with an initial deterministic surplus $R_0 = u$, the surplus process increases linearly due to premiums that are collected continuously over time at a constant rate $c > 0$, and $S_t = \sum_{i=1}^{N(t)} X_i$ represents the aggregate claims up to time t, where we assume that it is a compound Poisson process, $\Big\{N(t), t \geq 0\Big\}$ is a renewal process denoting the number of claims up to time t, which is a Poisson counting process with intensity λ, claim amounts $\Big\{X_i, i \geq 1\Big\}$ are a sequence of positive independent and identically distributed (i.i.d) random variables with common distribution $g(x)$ with mean $\mu = E(X_i)$, $i \geq 1$, and common generation function $M_X(l) = E(e^{lX_1})$, for $l \in (-\infty, \infty)$. Moreover, we assume that the claim number process $\Big\{N(t), t \geq 0\Big\}$ is independent of the claim amounts $\Big\{X_i, i \geq 1\Big\}$.

Now, we approximate the compound Poisson risk process (2.1) in terms of reinsurance contract with the standard Brownian motions and constant reinsurance premiums. To spread risk in the portfolio, each insurer purchases proportional reinsurance. More precisely, we allow the insurance company to choose proportional reinsurance with level $q \in (0, 1)$. If we assume that θ_1 is the safety loading of the reinsurer, then $\frac{1+\theta_1}{\lambda\mu(1-q)}$ is the premium rate for the reinsurance and $\theta_2 = \frac{c}{\lambda\mu} - 1$ is the safety loading of the insurer. With these notations, the perturbed actuarial model to analyse the risk in terms of safety loading of reinsurance is given by

$$dU_t^q = \Big(q(1+\theta_1) - (\theta_1 - \theta_2)\Big)\lambda\mu dt + \sigma B_{0t} - qdS_t, \qquad t \geq 0, \tag{2.2}$$

with the initial surplus $U_0^q = u$, where $\sigma \geq 0$ is a constant, $\Big\{B_{0t}, t \geq 0\Big\}$ is a standard Brownian motion which is independent of the claim number process $\Big\{N(t), t \geq 0\Big\}$ and claim amounts $\Big\{X_i, i \geq 1\Big\}$. In equation (2.2), σB_{0t} represents the additional diffusion in the insurance market or fanancial environment.

We assume that the net profit condition holds, i.e.

$$\lambda\mu\Big[\Big(q(1+\theta_1) - (\theta_1 - \theta_2)\Big) - q\Big] > 0.$$

We need the inequality $q > 1 - \frac{\theta_2}{\theta_1}$, otherwise the ruin probability of insurance company will be one for any $u \geq 0$.

As similar to equation (1) in Bazyari (2025), we consider the insurance portfolio whose price process P is perturbed by the following geometric Brownian motion:

$$dP_t = \zeta dP_t + \nu P_t dB_{1t}, \qquad t \geq 0, \tag{2.3}$$

where ζ and ν are positive constants and represent the expected instantaneous rate of risky asset and asset's volatility or standard deviation of its returns, respectively. These constants are used to the model when the asset's price movements over time, often within stochastic differential equations. The process $\{B_{1t}, t \geq 0\}$ is another standard Brownian motion independent of the claim number process $\{N(t), t \geq 0\}$ and $\{X_i, i \geq 1\}$, and have the correlation coefficient ρ, $-1 < \rho < 1$, with the process $\{B_{0t}, t \geq 0\}$. Moreover, we assume that the insurance company invests the amount $\beta > 0$ in the risky asset over time.

Definition 2.1. The pair of strategy (β, q) is said to be admissible if

i) it is $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable, in which $\{\mathcal{F}_t\}_{t\geq 0}$ is the augmentation of $\sigma\Big(U_t^q : t \geq 0\Big)$

ii) the value of risky asset β be finite and the strategy q lies in the interval $[0, 1]$.

The conditions for the admissibility set (β, q) are consistent with the Definitions 2.1 and 2.2 in Bazyari (2025).

If the insurance company is available for a proportional loading of $\theta_1 > \theta_2$, then its portfolio subject to these two controls (risky asset and reinsurance) follows the dynamic setting

$$dU_t^{\beta,q} = \Big(q(1+\theta_1) - (\theta_1 - \theta_2)\Big)\lambda\mu dt + \sigma dB_{0t} - q dS_t + \beta(\zeta dt + \nu dB_{1t}), \qquad t \geq 0, \tag{2.4}$$

with the initial surplus $U_0^{\beta,q} = u$. If we denote the time of ruin by $\tau = \inf\Big\{t : U_t^{\beta,q} \leq 0\Big\}$, then the infinite-time ruin probability is given by

$$\psi(u) = P\Big(\tau < \infty \Big| U_0^{\beta,q} = u\Big).$$

Moreover, if $\mathcal{A} = (0, \infty) \times (1 - \frac{\theta_2}{\theta_1}, 1)$ denotes the set of all admissible strategies, then the minimum of infinite-time ruin probability is defined by

$$\psi^m(u) = \inf_{(\beta,q)\in\mathcal{A}} P\Big(\tau < \infty \Big| U_0^{\beta,q} = u\Big).$$

2.1 Diffusion Approximation of the Compound Poisson Risk Model

In this subsection, we consider the insurance company whose reserves dynamics follow a diffusion perturbed risk model. More precisely, we model the classical Cramér-Lundberg process by a new aggregate claim process which follows by a Brownian motion with drift as

$$\hat{S}_t = \lambda\mu t - \lambda\mu_2 B_{2t}, \tag{2.5}$$

where $\mu_2 = E(X_1^2)$ and $\left\{B_{2t}, t \geq 0\right\}$ is another standard Brownian motion which is independent of $\left\{B_{0t}, t \geq 0\right\}$ and $\left\{B_{1t}, t \geq 0\right\}$. By replacing S_t in equation (2.4) with $\hat{S}_t$ and simplifying the equation, the new dynamic setting of insurance company obeys the equation

$$dD_t^{\beta,q} = \left(q\theta_1 - \theta_1 + \theta_2\right)\lambda\mu dt + \sigma dB_{0t} + q\lambda\mu_2 dB_{2t} + \beta(\zeta dt + \nu dB_{1t}), \qquad t \geq 0, \tag{2.6}$$

with the initial surplus $D_0^{\beta,q} = u$, and the retention level $q \in \left(1 - \frac{\theta_2}{\theta_1}, 1\right)$. For this diffusion approximation risk model, let $\tau_D = \inf\left\{t : D_t^{\beta,q} \leq 0\right\}$ denotes the time of ruin, then the infinite-time ruin probability is given by

$$\psi_D(u) = P\left(\tau_D < \infty \middle| D_0^{\beta,q} = u\right), \tag{2.7}$$

and the minimum of infinite-time ruin probability is defined by

$$\psi_D^m(u) = \inf_{(\beta,q)\in\mathcal{A}} P\left(\tau_D < \infty \middle| D_0^{\beta,q} = u\right). \tag{2.8}$$

3 Optimal Strategies in the Risk Models

In this section, we obtain a closed-form formula for the minimal ruin probability of dynamic setting risk model (2.6) over the infinite-time horizon and identify the corresponding optimal control strategies using the martingale method. Furthermore, we will derive the optimal strategies to maximize the Lundberg exponent of dynamic setting risk model (2.4). These methods involve analyzing a risk process and its behavior over time, with the goal of minimizing the probability of an insurer facing financial ruin.

3.1 Optimal Strategies in Diffusion Approximation of Risk Model

Theorem 3.1. *Consider the perturbed diffusion approximation risk model $dD_t^{\beta,q}$ in (2.6) and define*

$$L_D(\beta, q) = \frac{2\left[\beta\zeta + \left(q\theta_1 - \theta_1 + \theta_2\right)\lambda\mu\right]}{\beta^2\nu^2 + \sigma^2 + q^2\lambda\mu_2 + 2\beta\nu\sigma\rho}. \tag{3.1}$$

Then the process $\left\{e^{-L_D(\beta,q)D_t^{\beta,q}}, t \geq 0\right\}$ is a martingale and the exponential estimate of infinite-time ruin probability is given by

$$\psi_D(u) = e^{-L_D(\beta,q)u}. \tag{3.2}$$

Proof. From the diffusion approximation risk model (2.6), we have

$$\begin{aligned} E\Big(e^{-l(D_t^{\beta,q}-u)}\Big) &= \exp\Big\{-l\Big[\beta\zeta+\big(q\theta_1-\theta_1+\theta_2\big)\lambda\mu\Big] \\ &\quad +\frac{1}{2}l^2\Big(\beta^2\nu^2+\sigma^2+q^2\lambda\mu_2+2\beta\nu\sigma\rho\Big)\Big\}. \end{aligned}$$

Let

$$k(l) = -l\Big[\beta\zeta+\big(q\theta_1-\theta_1+\theta_2\big)\lambda\mu\Big]+\frac{1}{2}l^2\Big(\beta^2\nu^2+\sigma^2+q^2\lambda\mu_2+2\beta\nu\sigma\rho\Big). \tag{3.3}$$

Then the equation $k(r)=0$ has the positive root

$$L_D(\beta,q)=\frac{2\Big[\beta\zeta+\big(q\theta_1-\theta_1+\theta_2\big)\lambda\mu\Big]}{\beta^2\nu^2+\sigma^2+q^2\lambda\mu_2+2\beta\nu\sigma\rho},$$

and we have $E\Big(e^{-L_D(\beta,q)(D_t^{\beta,q}-u)}\Big)=1$, therefore, the process $\Big\{e^{-L_D(\beta,q)D_t^{\beta,q}}, t\geq 0\Big\}$ is a martingale. The positive root $L_D(\beta,q)$ is so called the Lundberg exponent. Using the martingale approach given in Grandell (1991, pp. 10-12), for any $u\geq 0$, we can derive the exponential estimate of infinite-time ruin probability by

$$\psi_D(u)=e^{-L_D(\beta,q)u},$$

and this completes the proof. □

Theorem 3.2. *Let the pair* (β^*,q^*) *be the optimal strategy which minimizes the infinite-time ruin probability* $\psi_D(u)$*, i.e.*

$$\psi_D^m(u)=\inf_{(\beta,q)\in(0,\infty)\times(1-\frac{\theta_2}{\theta_1},1)}\psi_D(u).$$

The optimal strategies that minimize the infinite-time ruin probability are given by

$$\beta^*=\frac{\zeta}{l\nu^2}-\frac{\sigma\rho}{\nu}, \qquad q^*=\min\Big\{\frac{\mu\theta_1}{l\mu_2},1\Big\},$$

and the exponential estimate of infinite-time ruin probability is

$$\psi_D(u)=e^{-L_D u},$$

where L_D *is given by*

$$L_D=\begin{cases} l^+, & \text{if } q^*<1, \\ \\ l_1, & \text{if } q^*=1, \end{cases}$$

with

$$l^+=\frac{(\theta_2-\theta_1)\lambda\mu-\zeta\sigma\rho/\nu\pm\Big(\Big[(\theta_2-\theta_1)\lambda\mu-\zeta\sigma\rho/\nu\Big]^2+\sigma^2(1-\rho^2)\Big(\zeta/\nu^2+\lambda\mu^2\theta_1^2/\mu_2^2\Big)\Big)^{\frac{1}{2}}}{\sigma^2(1-\rho^2)},$$

and

$$l_1 = \frac{\theta_2\lambda\mu - \zeta\sigma\rho/\nu + \Big(\Big[\theta_2\lambda\mu - \zeta\sigma\rho/\nu\Big]^2 + \Big(\sigma^2(1-\rho^2) + \lambda\mu_2\Big)\frac{\zeta^2}{\nu^2}\Big)^{\frac{1}{2}}}{\sigma^2(1-\rho^2) + \lambda\mu_2}.$$

Proof. Finding the pair (β^*, q^*) is equivalent to finding the pair (β^*, q^*) which minimizes the Lundberg exponent. Therefore, we try to find L_D such that

$$L_D = \sup_{(\beta,q)\in(0,\infty)\times(1-\frac{\theta_2}{\theta_1},1)} L_D(\beta, q).$$

The function in (3.3) is non-negative at $l = l_D$, i.e., L_D is a solution to the equation

$$\sup_{(\beta,q)\in(0,\infty)\times(1-\frac{\theta_2}{\theta_1},1)} \Big\{ -l\Big[\beta\zeta + \Big(q\theta_1 - \theta_1 + \theta_2\Big)\lambda\mu\Big] + \frac{1}{2}l^2\Big(\beta^2\nu^2 + \sigma^2 + q^2\lambda\mu_2 + 2\beta\nu\sigma\rho\Big)\Big\}. \tag{3.4}$$

With differentiating of the function $k(l)$ with respect to β, we get

$$\beta^* = \frac{\zeta}{l\nu^2} - \frac{\sigma\rho}{\nu},$$

and the minimum of function $k(l)$ with respect to q is given by

$$q_{max} = \frac{\mu\theta_1}{l\mu_2}.$$

Therefore, the optimal strategy of reinsurance is given by $q^* = q_{max} \wedge 1 = \min\Big\{q_{max}, 1\Big\}$. Now, let $q^* = q_{max} < 1$, then putting β^* and q^* into the equation (3.4), we obtain the following two roots:

$$l = \pm\frac{(\theta_2 - \theta_1)\lambda\mu - \zeta\sigma\rho/\nu \pm \Big(\Big[(\theta_2 - \theta_1)\lambda\mu - \zeta\sigma\rho/\nu\Big]^2 + \sigma^2(1-\rho^2)\Big(\zeta/\nu^2 + \lambda\mu^2\theta_1^2/\mu_2^2\Big)\Big)^{\frac{1}{2}}}{\sigma^2(1-\rho^2)}.$$

On the other hand, since we want to minimize the infinite-time ruin probability, only the positive root of l is valid. Therefore, the minimum infinite-time ruin probability is given by

$$\psi_D^m(u) = e^{-l^+u},$$

where l^+ is the positive root of l. In this case, the optimal reinsurance strategy is $q_{max} = \frac{\mu\theta_1}{l^+\mu_2}$. Let $q^* = 1$, and put β^* and q^* into the equation (3.4), we obtain

$$l_1 = \frac{\theta_2\lambda\mu - \zeta\sigma\rho/\nu + \Big(\Big[\theta_2\lambda\mu - \zeta\sigma\rho/\nu\Big]^2 + \Big(\sigma^2(1-\rho^2) + \lambda\mu_2\Big)\frac{\zeta^2}{\nu^2}\Big)^{\frac{1}{2}}}{\sigma^2(1-\rho^2) + \lambda\mu_2},$$

and the minimum infinite-time ruin probability is

$$\psi_D^m(u) = e^{-l_1u}.$$

This completes the proof. $\square$

3.2 Optimal Strategies in Compound Poisson Risk Model with Risky Asset and Reinsurance

In this subsection, we derive the optimal strategies to maximize the Lundberg coefficient of dynamic setting risk model (2.4).

Let $L_o(\beta, q)$ be the Lundberg exponent in this risk model. Then $L_o(\beta, q)$ satisfies the following equation:

$$\begin{aligned}
&\Big(\beta\zeta + \lambda\mu(1+\theta_1)q - \lambda\mu(\theta_1-\theta_2)\Big)l - \frac{1}{2}\Big(\nu^2\beta^2 + \sigma^2 + 2\nu\beta\sigma\rho\Big)l^2 \\
&\quad -\lambda\Big(\int_0^\infty e^{qlx_1} g(x)dx - 1\Big) \\
&= \Big(\beta\zeta + \lambda\mu(1+\theta_1) - \lambda\mu(\theta_1-\theta_2)\Big)l - \frac{1}{2}\Big(\nu^2\beta^2 + \sigma^2 + 2\nu\beta\sigma\rho\Big)l^2 \\
&\quad -\lambda\Big(M_X(ql) - 1\Big) = 0, \qquad (3.5)
\end{aligned}$$

where $M_X(ql) = \int_0^\infty e^{qlx_1} g(x)dx$ is the moment generating function of random variable X.

We try to maximize the Lundberg exponent L_o, i.e. to obtain the value of L_o such that

$$L_o = \sup_{(\beta,q)\in(0,\infty)\times(1-\frac{\theta_2}{\theta_1},1)} L_o(\beta, q).$$

We solve this problem using the same approach as in Section 3.1. Suppose that the non-positive value $l = L_o$ is the solution to equation (3.5), then we have

$$\begin{aligned}
\sup_{(\beta,q)\in(0,\infty)\times(1-\frac{\theta_2}{\theta_1},1]} &\Big\{\Big(\beta\zeta + \lambda\mu(1+\theta_1) - \lambda\mu(\theta_1-\theta_2)\Big)l - \frac{1}{2}\Big(\nu^2\beta^2 + \sigma^2 + 2\nu\beta\sigma\rho\Big)l^2 \\
&-\lambda\Big(M_X(ql) - 1\Big)\Big\} = 0. \qquad (3.6)
\end{aligned}$$

In order to find the Lundberg exponent, we assume that the claim amounts are identically distributed having an exponentially decaying tail with $\bar{g}(x) = 1 - g(x)$. It mean that there exists $l_\infty > 0$ such that $\lim_{l\to l_\infty} M_X(l) = \infty$ (for more details see for instance Asmussen and Albrecher, 2010). Then the maximum of equation (3.5) over the strategy β is given by

$$\beta^* = \frac{\zeta}{l\nu^2} - \frac{\sigma\rho}{\nu}.$$

Putting β^* into the equation (3.6), we have

$$\begin{aligned}
\sup_{q\in(1-\frac{\theta_2}{\theta_1},1]} &\Big\{\Big(-\frac{\zeta\sigma\rho}{\nu} + \lambda\mu(1+\theta_1)q - \lambda\mu(\theta_1-\theta_2)\Big)l \\
&+\frac{\zeta}{2\nu^2} - \frac{1}{2}\sigma^2(1-\rho^2)l^2 - \lambda\Big(M_X(ql) - 1\Big)\Big\} = 0. \qquad (3.7)
\end{aligned}$$

Let

$$\begin{aligned} h(q) &= \Big(-\frac{\zeta\sigma\rho}{\nu} + \lambda\mu(1+\theta_1)q - \lambda\mu(\theta_1-\theta_2)\Big)l + \frac{\zeta}{2\nu^2} - \frac{1}{2}\sigma^2(1-\rho^2)l^2 \\ &\quad -\lambda\Big(M_X(ql)-1\Big), \end{aligned} \tag{3.8}$$

then by differentiating $h(q)$ with respect to q, we get the equality

$$(1+\theta_1)\mu = E\Big(Xe^{qlX}\Big) = \frac{\partial}{\partial qr}M_X(qr) = M'_X(qr). \tag{3.9}$$

If $s = qr$, then the equation (3.9) becomes

$$(1+\theta_1)\mu = M'_X(s), \tag{3.10}$$

which is the same as equation (5) in Hald and Schmidli (2004).
We note that, if the claim amounts $\Big\{X_i, i \geq 0\Big\}$ are exponential distributed, then the positive solution of (3.10) is given by

$$s = \frac{1}{\mu}\Big(1 - \sqrt{\frac{1}{1+\theta_1}}\Big). \tag{3.11}$$

Lemma 3.1. *Suppose that s be the unique positive solution to equation (3.10), then the inequality*

$$M_X(s) < 1 + (1+\theta_1)\mu s,$$

holds.

Proof. Define the function $z(l)$ as

$$z(l) = (1+\theta_1)\mu l - M_X(l) + 1,$$

then by differentiating $z(l)$ with respect to l, we have

$$z'(l) = (1+\theta_1)\mu - M'_X(l),$$

therefore, from (3.10), $z'(s) = 0$. On the other hand,

$$z''(l) = E\Big(X^2e^{lX}\Big) = M''_X(l) \leq 0,$$

which means that $z(l)$ is a concave function and its maximum is obtained at $l = s$. Therefore, $z(s) > 0$, i.e. $M_X(s) < 1 + (1+\theta_1)\mu s$, and this completes the proof. □

In the next Theorem, the optimal strategies and maximal value of Lundberg exponent for original compound risk model are obtained.

Theorem 3.3. *Consider the perturbed risk model* $dU_t^{\beta,q}$ *in (2.4) for identically distributed claim amounts* $\{X_i, i \geq 1\}$ *with exponentially decaying tail. Let* s *be the positive root of equation (3.9) and* q_0 *denotes the argument where* $h(q)$ *in (3.8) attains its maximum, then* $q_0(s)$ *is given by*

$$q_0(s) = \frac{\zeta\sigma\rho/\nu + \lambda\mu(\theta_1 - \theta_2)s + (K)^{\frac{1}{2}}}{2\left[\lambda\mu(1+\theta_1)s - \lambda\left(M_X(s) - 1\right) + \zeta^2/2\nu^2\right]}, \tag{3.12}$$

where

$$K = \left[\zeta\sigma\rho/\nu + \lambda\mu(\theta_1 - \theta_2)^2\right]s^2 + 2\sigma^2(1-\rho^2)s^2\left[\lambda\mu(1+\theta_1)s - \lambda\left(M_X(s)-1\right) + \zeta^2/2\nu^2\right]. \tag{3.13}$$

Moreover, the optimal strategies to maximize the Lundberg exponent are

$$\beta^* = \frac{\zeta}{L_o\nu^2} - \frac{\sigma\rho}{\nu}, \tag{3.14}$$

and

$$q^* = \min\left(q_0(s), 1\right), \tag{3.15}$$

where the maximal value of Lundberg exponent for original compound risk model with $q^* < 1$ *is given by the unique positive*

$$L_o = \frac{\lambda\mu(\theta_1 - \theta_2) - \zeta\sigma\rho/\nu + (B)^{\frac{1}{2}}}{\sigma^2(1-\rho^2)}, \tag{3.16}$$

with

$$B = \left[\lambda\mu(\theta_1 - \theta_2) - \zeta\sigma\rho/\nu\right]^2 - 2\sigma^2(1-\rho^2)\left[\lambda\left(M_X(s) - 1\right) - \lambda\mu(1+\theta_1)s - \frac{\zeta^2}{2\nu^2}.\right.$$

Furthermore, the maximal value of Lundberg exponent for original compound risk model with $q^* = 1$ *is given by the unique positive solution of equation*

$$\left((1+\theta_2)\lambda\mu - \frac{\zeta\sigma\rho}{\nu}\right)l + \frac{\zeta^2}{2\nu^2} - \frac{1}{2}\sigma^2(1-\rho^2)l^2 = \lambda\left(M_X(l) - 1\right). \tag{3.17}$$

Proof. Since q_0 denotes the argument function $h(q)$, where $h(q)$ in (3.8) attains its maximum value, then $s = q_0 l$. Putting $l = \frac{s}{q_0}$ into the equation (3.7), we get

$$\left(-\frac{\zeta\sigma\rho}{\nu} - \lambda\mu(\theta_1 - \theta_2)\right)sq_0 + \left(\lambda\mu(1+\theta_1)s - \lambda\left(M_X(s) - 1\right) + \frac{\zeta^2}{2\nu^2}\right)q_0^2$$
$$-\frac{1}{2}\sigma^2(1-\rho^2)s^2 = 0.$$

Therefore,

$$q_0(s) = \pm\frac{\zeta\sigma\rho/\nu + \lambda\mu(\theta_1 - \theta_2)s + (K)^{\frac{1}{2}}}{2\left[\lambda\mu(1+\theta_1)s - \lambda\left(M_X(s) - 1\right) + \zeta^2/2\nu^2\right]},$$

where K is given in (3.13). On the other hand, from Lemma 3.1, we know that

$$\lambda\Big(M_X(s)-1\Big) < \lambda\mu(1+\theta_1)s.$$

Thus by omitting the negative root, the equation (3.12) will be obtained. If we know the distribution of claim amount X, then its moment generating and the closed form expression of s will be obtained. Thus the optimal strategy q^* can be given by $q^* = \min\Big\{q_0(s), 1\Big\}$. Substituting $q^* < 1$ into the equation (3.7) and simplifying, we have

$$\begin{aligned}&\Big(-\frac{\zeta\sigma\rho}{\nu}+\lambda\mu(\theta_1-\theta_2)\Big)l+\lambda\mu(1+\theta_1)q+\frac{\zeta^2}{2\nu^2}-\frac{1}{2}\sigma^2(1-\rho^2)l^2\\&=\lambda\Big(M_X(ql)-1\Big).\end{aligned} \tag{3.18}$$

To obtain the unique positive solution of equation (3.18), we define the function

$$h_2(l)=\Big(-\frac{\zeta\sigma\rho}{\nu}+\lambda\mu(\theta_1-\theta_2)\Big)l+\frac{\zeta^2}{2\nu^2}-\frac{1}{2}\sigma^2(1-\rho^2)l^2.$$

Since $h_2(l)$ is a concave function, then the equation $h_2(l) = 0$ has two positive and negative roots. Thus by Lemma 3.1, the equation (3.18) has a unique positive solution L_o which is given in (3.16).

Furthermore, substituting $q^* = 1$ into the equation (3.7) and simplifying the equation (3.17) will be obtained. Define the functions

$$h_3(l)=\Big((1+\theta_2)\lambda\mu-\frac{\zeta\sigma\rho}{\nu}\Big)l+\frac{\zeta^2}{2\nu^2}-\frac{1}{2}\sigma^2(1-\rho^2)l^2,$$

and $h_4(l) = \lambda\Big(M_X(l)-1\Big)$. It is easy to check that $h_3(l)$ is a concave function, and the equation $h_3(l) = 0$ has two positive and negative roots. On the other hand, since $h_4(l)$ is an increasing convex function and $h_4(0) = 0$, then $h_3(l)$ and $h_4(l)$ have an unique point of intersection at some $l > 0$. Therefore, the equation (3.17) has an unique positive solution and this completes the proof. □

We note that, if the pair (β^*, q^*) is the optimal strategy that minimizes the infinite-time ruin probability of original compound risk model, then for any $u \geq 0$ the inequality

$$\psi^m(u) \leq e^{-L_o u} \leq e^{-L_o(\beta,q)u},$$

holds.

Theorem 3.4. *Assume that the claim amounts* $\Big\{X_i, i\geq 0\Big\}$ *are exponential distributed, then the optimal strategies to maximize the Lundberg exponent are*

$$\beta^* = \frac{\zeta}{L_o\nu^2}-\frac{\sigma\rho}{\nu},$$

and

$$q^* = \min\Big(\frac{1-\sqrt{\frac{1}{1+\theta_1}}}{\mu L_o}, 1\Big),$$

where the maximal value of Lundberg exponent with risky asset and investment for $q^* < 1$ *is given by the unique positive*

$$L_o = \frac{\lambda\mu(\theta_1-\theta_2) - \zeta\sigma\rho/\nu + \big(B\big)^{\frac{1}{2}}}{\sigma^2(1-\rho^2)},$$

with

$$B = \Big[\lambda\mu(\theta_2-\theta_1) - \zeta\sigma\rho/\nu\Big]^2 + 2\sigma^2(1-\rho^2)\Big[\lambda\Big(2+\theta_1-2\sqrt{1+\theta_1}\Big) + \frac{\zeta^2}{2\nu^2}.$$

Furthermore, the maximal value of Lundberg exponent for original compound risk model with $q^* = 1$ *is given by the unique positive solution of equation*

$$\Big((1+\theta_2)\lambda\mu - \frac{\zeta\sigma\rho}{\nu}\Big)l + \frac{\zeta^2}{2\nu^2} - \frac{1}{2}\sigma^2(1-\rho^2)l^2 = \frac{\lambda\mu l}{1-\mu l}.$$

Proof. Putting the equation (3.11) into the Theorem 3.3, the results will be obtained easily. □

3.3 Some Inequalities on the Infinite-time Ruin Probabilities

In this subsection, we give some inequalities on the infinite-time ruin probabilities in the original compound risk model and diffusion approximation risk model.

Theorem 3.5. *Let* L_o *and* L_D *be the Lundberg exponents in the original compound Poisson risk model and diffusion approximation of risk model, respectively. Then for any arbitrary strategy* (β, q)*, the inequality*

$$L_o(\beta,q) \le L_D(\beta,q), \tag{3.19}$$

holds.

Proof. Using Theorem 3.1 and equation (3.5), the Lundberg exponents $L_D(\beta,q)$ and $L_o(\beta,q)$ satisfy the following equations

$$\Big(\beta\zeta + (\theta_2-\theta_1)\lambda\mu + q\theta_1\lambda\mu\Big)l - \frac{1}{2}\Big(\beta^2\nu^2 + \sigma^2 + 2q\nu\sigma\rho\Big)l^2 = \frac{1}{2}q^2\lambda\mu_2 l^2, \tag{3.20}$$

and

$$\Big(\beta\zeta + (1+\theta_1)q\lambda\mu - (\theta_1-\theta_2)\lambda\mu\Big)l - \frac{1}{2}\Big(\beta^2\nu^2 + \sigma^2 + 2q\nu\sigma\rho\Big)l^2 = \lambda\Big(M_X(ql) - 1\Big), \tag{3.21}$$

respectively. The equation (3.21) can be written as

$$\Big(\beta\zeta + (\theta_2-\theta_1)\lambda\mu + q\theta_1\lambda\mu\Big)l - \frac{1}{2}\Big(\beta^2\nu^2 + \sigma^2 + 2q\nu\sigma\rho\Big)l^2 = \lambda\Big(M_X(ql) - 1\Big) - \lambda\mu q l, \tag{3.22}$$

which the left hand side of the equation (3.22) is the same as left hand side of the equation (3.20). Define the function

$$H_5(l) = \Big(\beta\zeta + (\theta_2 - \theta_1)\lambda\mu + q\theta_1\lambda\mu\Big)l - \frac{1}{2}\Big(\beta^2\nu^2 + \sigma^2 + 2q\nu\sigma\rho\Big)l^2,$$

then from equation $H_5(l) = 0$, we obtain the following two roots

$$r_1 = 0,$$

and

$$r_2 = \frac{2\Big(\beta\zeta + (\theta_2 - \theta_1)\lambda\mu + q\theta_1\lambda\mu\Big)}{\beta^2\nu^2 + \sigma^2 + 2q\nu\sigma\rho} > 0.$$

Now, we define two functions $g_1(l)$ and $g_2(l)$ as

$$g_1(l) = \frac{1}{2}q^2\lambda\mu_2 l^2, \qquad \textit{and} \qquad g_2(l) = \lambda\Big(M_X(ql) - 1\Big) - \lambda\mu q l,$$

respecively, then as shown in Figure 1, $g_1(l) \leq g_2(l)$ and $L_o(\beta, q) \leq L_D(\beta, q)$.

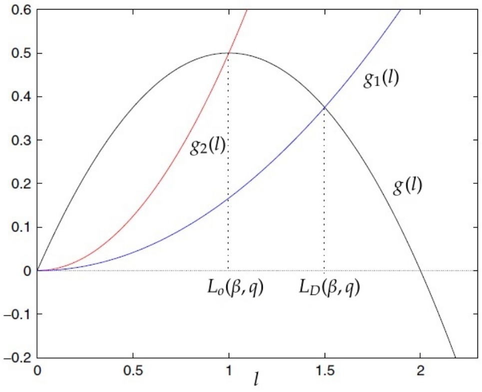

Figure 1: The functions $g_1(l)$ and $g_2(l)$ with Lundberg exponents

Furthermore, to show the inequality (3.19) by mathematical approach, we have the equalities

$$g_1(l) = \frac{1}{2}q^2\lambda\mu_2 l^2 = \lambda\int_0^\infty \frac{1}{2}l^2q^2z^2 g(z)dz, \tag{3.23}$$

and

$$\begin{aligned} g_2(l) &= \lambda\Big(M_X(ql) - 1\Big) - \lambda\mu q l \\ &= \lambda\int_0^\infty \Big(e^{qlz} - 1 - qlz\Big)g(z)dz. \end{aligned} \tag{3.24}$$

Therefore, from (3.23) and (3.24), we have

$$g_2(l) - g_1(l) = \lambda \int_0^\infty \Big(e^{qlz} - 1 - qlz - \frac{1}{2}l^2q^2z^2\Big)g(z)dz.$$

On the other hand, since $e^{qlz} - 1 - qlz - \frac{1}{2}l^2q^2z^2 \geq 0$, for all $l \geq 0$, then $g_1(l) \leq g_2(l)$ and $L_o(\beta, q) \leq L_D(\beta, q)$, and this completes the proof. □

Theorem 3.6. *Let L_o be the maximum Lundberg exponent of compound risk model (2.4) and $\psi_D(u)$ be the minimum finite time ruin probability of risk model (2.6), then the inequality*

$$\psi_D^m(u) \leq e^{-L_o u},$$

holds.

Proof. From Theorem (3.4) for any arbitrary strategy (β, q), the inequality (3.19) holds, therefore, we have

$$L_o \leq L_o(\beta^*, q^*) \leq L_D(\beta^*, q^*) \leq L_D,$$

thus

$$\psi_D^m(u) = e^{-L_D u} \leq e^{-L_o u},$$

and this completes the proof. □

4 The Effect of Investment on the Ruin Probability

In this section, we give a comparison for theoretical results with the investment a risky asset and the case without investment in the perturbed compound risk model and diffusion approximation risk model when $q^* < 1$ (although using the same method, we can derive the same results when $q^* = 1$). We will show that in both models with investment, the infinite-time ruin probability decreases.

Let L_w and L_{wo} be the maximum Lundberg exponents in both models in the case with investment and without investment, respectively. We will show that $L_w \geq L_{wo}$. First consider the diffusion approximation risk model (2.6), then from Section 3.1, we know that L_w and L_{wo} satisfy the following equations

$$\sigma^2(1 - \rho^2)l^2 + 2\Big(\frac{\zeta}{\nu}\sigma\rho - (\theta_2 - \theta_1)\lambda\mu\Big)l - \Big(\frac{\zeta^2}{\nu^2} + \frac{\lambda\mu^2\theta_1^2}{\mu_2}\Big) = 0, \tag{4.1}$$

and

$$\sigma^2 l^2 - 2(\theta_2 - \theta_1)\lambda\mu l - \frac{\lambda\mu^2\theta_1^2}{\mu_2} = 0, \tag{4.2}$$

respectively. We can see that the equation (4.2) is the special case of equation (4.1) with $\zeta = 0$ and $\rho = 0$.
To prove the inequality $L_w \geq L_{wo}$, define two functions $M_1(l)$ and $M_2(l)$ as

$$M_1(l) = \sigma^2(1-\rho^2)l^2 + 2\Big(\frac{\zeta}{\nu}\sigma\rho - (\theta_2-\theta_1)\lambda\mu\Big)l - \Big(\frac{\zeta^2}{\nu^2} + \frac{\lambda\mu^2\theta_1^2}{\mu_2}\Big),$$

and

$$M_2(l) = \sigma^2 l^2 - 2(\theta_2-\theta_1)\lambda\mu l - \frac{\lambda\mu^2\theta_1^2}{\mu_2},$$

respectively. Then as shown in Figure 2, for all $l \geq 0$, $M_1(l) \leq M_2(l)$ and $L_w \geq L_{wo}$.

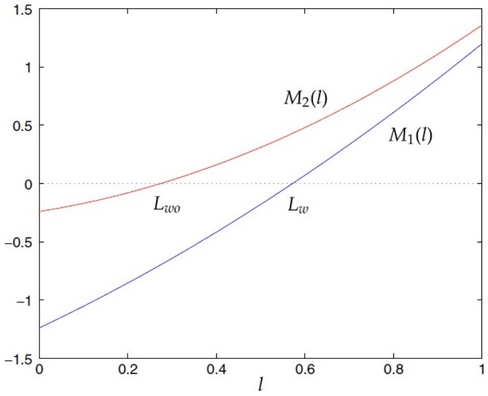

Figure 2: The functions $M_1(l)$ and $M_2(l)$ with maximum Lundberg exponents

Furthermore, to see that the inequality $M_1(l) \leq M_2(l)$ holds, by mathematical approach we get

$$M_1(l) - M_2(l) = -\sigma^2\rho^2 l^2 + 2\frac{\zeta}{\nu}\sigma\rho l - \frac{\zeta^2}{\nu^2} = -\Big(\sigma\rho l - \frac{\zeta}{\nu}\Big)^2 \leq 0,$$

and therefore, $L_w \geq L_{wo}$.

Now, we consider the compound risk model with risky asset and reinsurance. From Section 3.2, we know that L_w and L_{wo} satisfy the following equations

$$-\frac{1}{2}\sigma^2(1-\rho^2)l^2 - \Big(\frac{\zeta}{\nu}\sigma\rho - (\theta_2-\theta_1)\lambda\mu\Big)l + \frac{1}{2}\frac{\zeta^2}{\nu^2} + \lambda\mu(1+\theta_1)s - \lambda\Big(M_X(s)-1\Big) = 0, \quad (4.3)$$

and

$$-\frac{1}{2}\sigma^2 l^2 + (\theta_2-\theta_1)\lambda\mu l + \lambda\mu(1+\theta_1)s - \lambda\Big(M_X(s)-1\Big) = 0, \quad (4.4)$$

respectively. We can see that the equation (4.4) is the special case of equation (4.3) with $\zeta = 0$ and $\rho = 0$.
To prove the inequality $L_w \geq L_{wo}$, define two functions $N_1(l)$ and $N_2(l)$ as

$$N_1(l) = -\frac{1}{2}\sigma^2(1-\rho^2)l^2 - \Big(\frac{\zeta}{\nu}\sigma\rho - (\theta_2-\theta_1)\lambda\mu\Big)l + \frac{1}{2}\frac{\zeta^2}{\nu^2} + \lambda\mu(1+\theta_1)s - \lambda\Big(M_X(s)-1\Big),$$

and

$$N_2(l) = -\frac{1}{2}\sigma^2 l^2 + (\theta_2-\theta_1)\lambda\mu l + \lambda\mu(1+\theta_1)s - \lambda\Big(M_X(s)-1\Big),$$

respectively. Then as shown in Figure 3, for all $l \geq 0$, $N_1(l) \geq N_2(l)$ and $L_w \geq L_{wo}$.

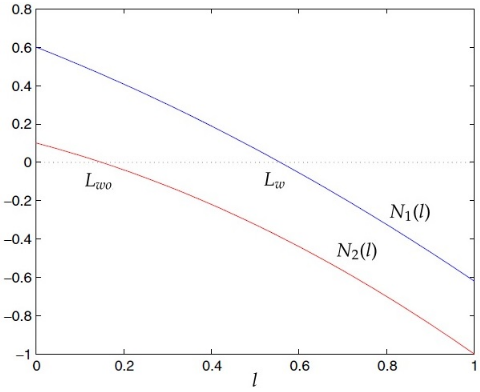

Figure 3: The functions $N_1(l)$ and $N_2(l)$ with maximum Lundberg exponents

Furthermore, to see that the inequality $N_1(l) \geq N_2(l)$ holds, by mathematical approach we get

$$N_1(l) - N_2(l) = \frac{1}{2}\sigma^2\rho^2 l^2 - 2\frac{\zeta}{\nu}\sigma\rho l + \frac{1}{2}\frac{\zeta^2}{\nu^2} = -\Big(\sigma\rho l - \frac{\zeta}{\nu}\Big)^2 \geq 0,$$

and therefore, $L_w \geq L_{wo}$.

5 Numerical Illustrations

In this section, we give some numerical examples to illustrate the effects of model parameters on the optimal risk management strategies and on the financial market.

Examples 5.1. Assume that the claim amounts $\Big\{X_i, i \geq 0\Big\}$ are exponential distributed with parameter $\frac{1}{2}$. Moreover, let $\lambda = 2$, $\theta_1 = 0.4$, $\theta_2 = 0.8$, $\mu_2 = 2$ and $\sigma = 0.5$. The results are shown in Figures 4-6. In these figures, the effects of model parameters on the optimal risk management strategies are investigated. In Figure 4, we assume that $\zeta = 0.5$ and $\nu = 0.6$, but in Figure 5, we change the volatility of risky asset to $\nu = 2$.

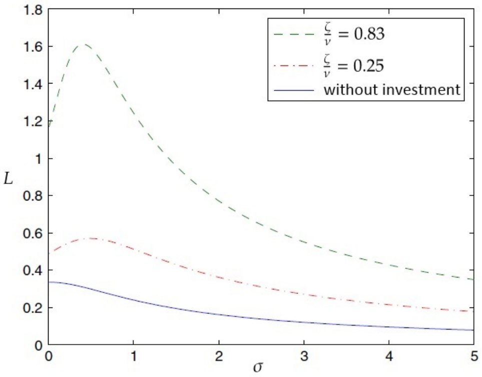

Figure 4: The effect of ν and L in the compound risk model with risky asset and investment

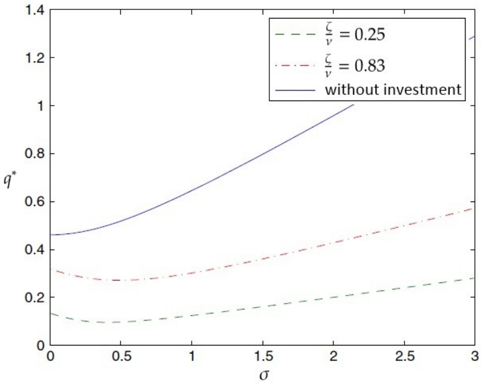

Figure 5: The effect of ν and q^* in the compound risk model with risky asset and investment

From Figure 4 with $\rho = -0.6$, we see that the Lundberg exponents with investment are larger than the case without investment, which is consistence with the result given in Section 4. Moreover, a larger $\frac{\zeta}{\nu}$ yields a larger Lundberg exponent. From Figure 5 with $\rho = -0.6$, we see that the larger $\frac{\zeta}{\nu}$ yields a lower retention level by insurance company. Moreover, the retention level with investment is less than the case without investment.

From Figure 6, we see that a larger asset's volatility ν yields less investment. Whereas, the larger expected instantaneous rate of return of the risky asset ζ will not necessarily yields more investment.

For further investigation, the effect of correlation coefficient on the optimal strategies β^* and q^* are studied and the results reported in Table 1.

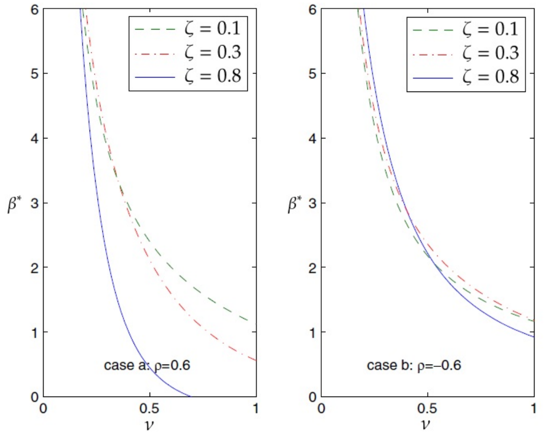

Figure 6: The effect of ν and β^* in the compound risk model with risky asset and investment

Table 1: Effect of correlation coefficient on the optimal strategies

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
β^*	8.92	7.25	6.82	5.45	4.70	3.92	3.28	2.04	1.46	0.56	0.12
q^*	0.07	0.09	0.15	0.19	0.29	0.38	0.47	0.58	0.70	0.83	0.91

From Table 1, we see that the larger correlation coefficient ρ yields less risky asset, whereas the insurance company has to choose larger proportional reinsurance.

Examples 5.2. Assume that the claim amounts $\{X_i, i \geq 0\}$ are exponential distributed with parameter $\frac{1}{2}$ and $\psi(u)$ be the ruin probability in the compound risk model with risky asset and investment. Moreover, let $\lambda = 2$, $\theta_1 = 0.4$, $\theta_2 = 0.8$, $\mu_2 = 2$, $q = 1$, $\beta = 1$, $\zeta = 0$ and $\rho = 0.7$. The results are shown in Figure 7. In this figure, we compare the infinite-time ruin probability $\psi(u)$ with dashed line, the upper bound $e^{-L_o u}$ with thin line, and the infinite-time ruin probability $\psi_D(u)$ with heavy line.

In Figure 7(a) with the volatility of risky asset $\nu = 5$, we observe that $\psi_D(u) < \psi(u) < e^{-L_o u}$ for any $u \geq 0$, which means than with the given diffusion approximation of risk model (2.6), the infinite-time ruin probability decreases. In Figure 7(b) with the volatility of risky asset $\nu = 0.15$, we observe that the results in three cases are almost the same and the graphs overlap. That is to say, when the volatility of risky asset is very small, then $\psi(u)$, $\psi_D(u)$ and $e^{-L_o u}$ are the same. In fact, in this case, $\psi_D(u)$ and $e^{-L_o u}$ are good estimations for the ruin probability $\psi(u)$.

6 Concluding Remarks

Computing the ruin probabilities as a variety of different risk model types are considered in many papers. This paper investigated optimal risk management strategies combining investment in a risky asset and purchasing proportional reinsurance to minimize the infinite-time ruin probability of an insurance company. The underlying risk model is a general perturbed compound Poisson process, where the price process is disturbed by a geometric Brownian motion with specific drift and volatility. A diffusion approximation of this model is also examined. In Theorem 3.1, we obtained the exponential estimate of infinite-time ruin probability for the perturbed risk model (2.6). Theorem 3.2 derived the optimal strategies in diffusion approximation of risk model. The optimal strategies in compound Poisson risk model with risky asset and reinsurance are given in Theorem 3.3. Theorem 3.4 obtained the optimal strategies when the claim amounts are exponential distributed. Moreover, we studied some inequalities on the infinite-time ruin probabilities and investigated the effect of investment on the ruin probability in perturbed compound risk model and diffusion approximation risk model. The theoretical results showed that in both models with investment, the infinite-time ruin probability decreases. Finally, the effects of model parameters on the optimal risk management strategies and on the financial market are studied with two numerical examples. These examples confirm that the obtained theoretical results for ruin probability are excellent and completely reliable.

For future research, the heavy-tailed distributions such as Log-normal distribution or Pareto distribution can be used. In this case, a comparison for theoretical results with the investment a risky asset and the case without investment in both risk models could be interesting.

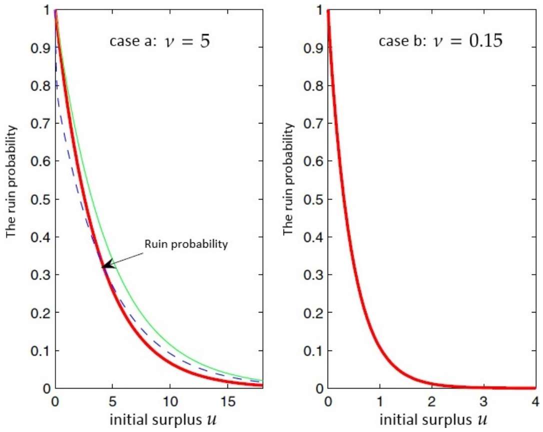

Figure 7: The effect of initial surplus on the ruin probabilities

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