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# Shrinkage Estimation for Spherically Symmetric Distributions under Modified Balanced Loss Functions

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**Abstract.** We focus on the problem of estimating the average vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)$ of a random vector $\mathbf{X} \in \mathbb{R}^d$ which follows a spherically symmetric distribution. We consider modified balanced loss functions of the form:
**(i)** $L_{\omega,\boldsymbol{\delta}_0,\rho}(\boldsymbol{\delta},\boldsymbol{\theta}) = \omega\rho(\|\boldsymbol{\delta}-\boldsymbol{\delta}_0\|^2)+(1-\omega)\rho(\|\boldsymbol{\delta}-\boldsymbol{\theta}\|^2)$ and **(ii)** $\ell(\omega\,\|\boldsymbol{\delta}-\boldsymbol{\delta}_0\|^2+(1-\omega)\|\boldsymbol{\delta}-\boldsymbol{\theta}\|^2)$. Here, $\boldsymbol{\delta}_0$ represents a target estimator of $\boldsymbol{\theta}$, $\omega \in [0,1]$ and $\rho$ and $\ell$ are increasing and concave functions. If $d \geq 4$ and the target estimator is $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$, we provide conditions on the parameter $a$ for Baranchik type estimators $\boldsymbol{\delta}_{a,S}(\mathbf{X}) = \left(1 - a(1-\omega)S(\|\mathbf{X}\|^2)/\|\mathbf{X}\|^2\right)\mathbf{X}$ and reach the minimaxity. These conditions are derived using the radial properties of spherically symmetric distributions, which do not require the existence of a probability density for the random vector $\mathbf{X}$. Furthermore, we extend the obtained results to the case of robust shrinkage estimators of the form $\boldsymbol{\delta}_{\omega,g}(\mathbf{X}) = \mathbf{X} + a(1-\omega)g(\mathbf{X})$, where $g(\cdot)$ is a weakly differentiable and satisfying some conditions. Additionally, we conduct a simulation study in order to show the effectiveness and the usefulness of the obtained results.

**Keywords.** Modified Balanced Loss Function, Baranchik Type Estimators, Shrinkage Estimation, Dominance, Spherically Symmetric Distribution, Robust Schrinkage Estimators.

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## 1 Introduction

Stein(1956) was the pioneer in demonstrating the inadmissibility of the best invariant estimator $\mathbf{X} = (X_1, \ldots, X_d)$ for a normal mean $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)$ under the quadratic loss function. He established that estimators of the form:

$$\boldsymbol{\delta}_{a,b}(\mathbf{X}) = \left(1 - a/(b + \|\mathbf{X}\|^2)\right)\mathbf{X},$$

dominate $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$ when $d \geq 3$, given that $a$ is sufficiently small and $b$ is sufficiently large. Subsequently, extensive research has been devoted to improving the best invariant estimator for a location vector $\boldsymbol{\theta}$ by relaxing the normality assumption, exploring alternatives to the loss function, or considering more versatile estimators. James and Stein (1961), for his part, he introduced a specific class of estimators:

$$\boldsymbol{\delta}_a(\mathbf{X}) = \left(1 - a/\|\mathbf{X}\|^2\right)\mathbf{X},$$

dominate $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$ under the quadratic loss function when $0 < a < 2(d-2)$, but assuming that $\mathbf{X}$ follows a normal distribution with identical covariance matrix $\boldsymbol{I}_d$. This result has been validated and extended by subsequent research, in particular the work of Cellier et al. (1989), Cellier and Fourdrinier (1995), Fourdrinier and Strawderman (1996), Fourdrinier and Ouassou (2000), Fourdrinier et al. (2003), Jozani et al. (2006), Fourdrinier et al. (2008).

Brandwein and Strawderman (1979), Brandwein and Strawderman (1978, 1980) and Brandwein et al. (1993) demonstrated that the dominance result holds even when the distribution of $\mathbf{X}$ is spherically symmetric ($SS_d(\boldsymbol{\theta})$) and $d \geq 4$. On the other hand Baranchik(1970) introduced a class of estimators to estimate the mean of a multivariate normal distribution. Indeed, it is the class of estimators, denoted $\boldsymbol{\delta}_{a,b,S}(\mathbf{X})$, which are in the form:

$$\boldsymbol{\delta}_{a,b,S}(\mathbf{X}) = \left(1 - aS(\|\mathbf{X}\|^2)/(b + \|\mathbf{X}\|^2)\right)\mathbf{X}.$$

This class includes various choices for the function $S(t)$, such as the simple form $S(t) = t/(t+b)$ with $b \geq 0$. Then, Brandwein et al. (1991) and Strawderman proposed a more generalized form of shrinkage estimators, which are expressed as:

$$\boldsymbol{\delta}_g(\mathbf{X}) = \mathbf{X} + a\ g(\mathbf{X}).$$

They used the divergence theorem to show that these estimators dominate $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$ , especially under the quadratic loss function and concave quadratic loss functions.

Then, Brandwein et al. (1991) established certain conditions on $\boldsymbol{\delta}_g(\mathbf{X})$ to dominate $\mathbf{X}$, which are dominated by $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$ and are minimax where we assume that there exists a non-positive function $h(.)$ such that $h(\mathbf{X})$ is subharmonic and $E_{R,\boldsymbol{\theta}}\left[R^2 h(\mathbf{U})\right]$ does not increase with $\mathbf{U} \sim U_{R,\boldsymbol{\theta}}$, $E_{\boldsymbol{\theta}}\,[\|h(\mathbf{X})\|] < \infty$ and such that $g(\mathbf{X})$ is weakly differentiable and satisfying:

$$\textbf{(i)}\ div(g) \le h,$$

$$\textbf{(ii)}\ \|g\|^2 + 2h \le 0 \ \text{ and } 0 < a < 1/[dE_0(\|\mathbf{X}\|^2)].$$

Notice that the James-Stein estimator:

$$\boldsymbol{\delta}_a(\mathbf{X}) = (1 - a/\|\mathbf{X}\|^2)\mathbf{X},$$

with specific choices of $g(\mathbf{X}) = -\mathbf{X}/\|\mathbf{X}\|^2$ and $h(\mathbf{X}) = -b/\|\mathbf{X}\|^2$ where $0 \le b \le d-2$, as well as the Baranchik estimator:

$$\boldsymbol{\delta}_{a,S}(\mathbf{X}) = \left(1 - aS(\|\mathbf{X}\|^2)/\|\mathbf{X}\|^2\right)\mathbf{X},$$

for which $g(\mathbf{X}) = -\left(S(\|\mathbf{X}\|^2)/\|\mathbf{X}\|^2\right)\mathbf{X}$ and $h(\mathbf{X}) = -bS(\|\mathbf{X}\|^2)/\|\mathbf{X}\|^2$ where $\frac{1}{2} \le b \le d-2$, can be considered as special cases of robust shrinkage estimators.

The Balanced Loss Functions (BLFs), introducted by Zellner (1994), is formulated to reflect two criteria, namely goodness of fit and the estimation precision. A goodness of fit criterion such as the sum of squared residuals in regression problems leads to an estimator which gives a good fitted and unbiased estimator; however, it may not be as accurate as a biased estimator. It is therefore necessary to provide a framework that meets this need. The BLFs are appealing as they combine proximity of a given estimator $\boldsymbol{\delta}$ to both a target estimator $\boldsymbol{\delta}_0$ and the unknown parameter $\boldsymbol{\theta}$:

$$L_{\omega,\boldsymbol{\delta}_0}(\boldsymbol{\delta},\boldsymbol{\theta}) = \omega\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\|^2 + (1-\omega)\|\boldsymbol{\delta} - \boldsymbol{\theta}\|^2,$$

where $\boldsymbol{\theta} \in \mathbb{R}^d$ and $\omega \in [0,1]$ is the weight given to the proximity of $\boldsymbol{\delta}$ to $\boldsymbol{\delta}_0$ and $\boldsymbol{\delta}(\mathbf{X})$ is a given estimator of $\boldsymbol{\theta}$. Marchand and Strawderman (2020) introduced a novel class of loss functions called Modified Balanced Loss Functions (MBLFs), denoted as in **(i)** and **(ii)**, where $\rho(\cdot) \ge 0$ and $l(\cdot) \ge 0$ are increasing and concave functions and satisfy a completely monotone property. Notice that the researchers focused their investigation on the estimation of the vector mean of a mixture of scales of the normal distribution. Notably, these MBLFs can be considered as a natural extension of Zellner's balanced loss function, where the functions $\rho(t)$ and $\ell(t)$ take the simple form of $t$ for all non-negative values of $t$ in the real numbers $\mathbb{R}_+$. Hobbad et al. (2021) generalized this result and showed that the baranchik type estimator $\boldsymbol{\delta}_{a,S}(\mathbf{X}) = (1 - a(1-\omega)S(\|\mathbf{X}\|^2)/|\mathbf{X}\|^2)\mathbf{X}$ to dominate the estimator $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$ for the class of spherically symmetric densities under MBLFs **(i)** and **(ii)** where $\rho$ and $\ell$ are increasing and concave functions only.

The rest of this paper is organized as follows. In section 2, we give some conditions for the Baranchik type estimator (2.1) dominates $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$ under MBLFs **(i)** and **(ii)**, when the random vector $\mathbf{X}$ with spherically symmetric distribution without necessarily needing a probability density function. For this same model, we generalize, in section 3, the results of section 2 and those of Hobbad et al. (2021) to robust shrinking estimators for spherically symmetric distributions with or without probability density fnction. A practical study on simulated data is carried out in section 4. Finally, the proofs of all the lemmas and established theorems are presented in section 5. These proofs are based on the general properties of the spherically symmetric distributions, such as the law of the radius, the uniform distribution on a sphere and the concepts of superharmony and concavity. On the other hand, the proofs provided by Hobbad et al. (2021) are limited to spherically symmetric laws with a probability density and which are specifically built on the use of the latter.

## 2 Baranchik-type Estimators for Location Parameters under a Modified Loss Function

In this section, we aim to investigate the estimation problem of a $d$-dimensional parameter $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)$ when the observed vector $\mathbf{X} = (X_1, X_2, \ldots, X_d)$ has a spherically symmetric distribution without necessarily needing a density around $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_d)$, considering two specific types of MBLFs **(i)** and **(ii)**. Our main objective is to establish a sufficient condition for the parameter $a$ such that the Baranchik-type estimator, defined as:

$$\boldsymbol{\delta}_{a,S}(\mathbf{X}) = \left(1 - a(1-\omega)S(\|\mathbf{X}\|^2)/\|\mathbf{X}\|^2\right)\mathbf{X}, \tag{2.1}$$

dominates the estimator $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$. Here, the function $S$ is assumed to be twice differentiable almost everywhere, satisfying the conditions:

$$0 \leq S(.) \leq 1\,,\ S(.) \neq 0\,,\ S'(.) \geq 0 \text{ and } S''(.) \leq 0. \tag{2.2}$$

### 2.1 With Respect to the Loss Function $\omega\rho(\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\|^2) + (1-\omega)\rho(\|\boldsymbol{\delta} - \boldsymbol{\theta}\|^2)$

In the subsequent analysis, we will examine the minimax properties of Baranchik-type estimators for a location vector $\boldsymbol{\theta}$ when the observed vector $\mathbf{X}$ follows a spherically symmetric distribution without necessarily needing a density, considering the MBLFs **(i)**. We make the assumption that the function $\rho$ satisfies the following conditions:

$$\mathbf{H1}: \ \rho(0) = 0,\ 0 < \rho'(0) < +\infty, \text{ and } \rho \text{ is concave on } [0, +\infty].$$

In the following, we give examples of functions $\rho$ satisfying the assumption **H1**: $\rho(t) = t$, $\rho(t) = 1 - \exp(-t/\alpha)$ with $\alpha > 0$, $\rho(t) = \log(1+t)$, $\rho(t) = (1 + t/\gamma)^\beta$ with $\gamma > 0,\ \beta \in (0,1)$, $\rho(t) = r^2 t/(rt+1)$ with $r > 0$, $\rho(t) = \arctan(t)$, and $\rho(t) = \tanh(t)$. However, it is important to note that the last two examples, ($v$) and ($vi$), do not exhibit complete monotonicity. In other words, $(-1)^k \rho^k(t) \geq 0$ for $t > 0$ and $k \in \mathbb{N}$ is not satisfied for these two cases.
In order to show the domination and minimaxity results, we introduce the following lemma as a means of proof.

**Lemma 1.** *Let* $\mathbf{X}$ *be a random vector with a spherically symmetric distribution around* $\boldsymbol{\theta}$*, and let* $g(.)$ *be a weakly differentiable function such that* $E_{\boldsymbol{\theta}}\left[|(\mathbf{X}-\boldsymbol{\theta})^\top g(\mathbf{X})|\right] < \infty$*. Then*

$$E_{\boldsymbol{\theta}}\left[\rho'(\|\mathbf{X}-\boldsymbol{\theta}\|^2)(\mathbf{X}-\boldsymbol{\theta})^\top g(\mathbf{X})\right] = \frac{1}{d}E\left[r^2\rho'(r^2)\int_{B_{r,\boldsymbol{\theta}}} \nabla g(\mathbf{X})\, dV_{r,\boldsymbol{\theta}}(\mathbf{X})\right],$$

*where $E_{\boldsymbol{\theta}}$ denotes the expectation with respect to the radius distribution and where $V_{r,\boldsymbol{\theta}}(.)$ is the uniform distribution on $B_{r,\boldsymbol{\theta}}$ (the ball of radius $r$ centered at $\boldsymbol{\theta}$).*

The relevance of the lemma in the current study lies in the fact that if $\mathbf{X}$ is distributed according to a spherically symmetric distribution with parameter $\boldsymbol{\theta}$, then the conditional distribution of $\mathbf{X}$ given $R$ follows a uniform distribution on the sphere $\|\mathbf{X}-\boldsymbol{\theta}\|^2 = R^2$.

**Lemma 2.** *Let $\mathbf{X} \sim SS_d(\boldsymbol{\theta})$. For $d \geq 4$ and $g(\mathbf{x}) = -a\left(S(\|\mathbf{x}\|^2)/\|\mathbf{x}\|^2\right)\mathbf{x}$, we have*

$$\begin{aligned} &E_{\boldsymbol{\theta}}\left[(1-\omega)\rho'(\|\mathbf{X}-\boldsymbol{\theta}\|^2)[2(1-\omega)g(\mathbf{X}).(\mathbf{X}-\boldsymbol{\theta}) + (1-\omega)^2\|g(\mathbf{X})\|^2]\right] \\ &\leq \; E\left[(1-\omega)^2 a\rho'(R^2)\left(a(1-\omega)/R^2 - \frac{2(d-2)}{d}\right)E_{R,\boldsymbol{\theta}}\left[R^2\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}\right]/\|\mathbf{X}-\boldsymbol{\theta}\| = R]\right], \end{aligned}$$

*where $SS_d(\boldsymbol{\theta})$ denotes the spherically symmetric distribution arround $\boldsymbol{\theta}$ in $\mathbb{R}^d$.*

We give the main dominance theorem of this section.

**Theorem 2.1.** *Assume that $d \geq 4$, $\mathbf{X}$ has a spherically symmetric distribution arround $\boldsymbol{\theta}$ ($\mathbf{X} \sim SS_d(\boldsymbol{\theta})$) and that the function $\rho$ satisfies* **H1**. *For estimating $\boldsymbol{\theta}$ under loss **(i)** with $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$, the estimator $\boldsymbol{\delta}_{a,S}(\mathbf{X})$ in (2.1) where the function $S$ satisfies (2.2), dominates $\boldsymbol{\delta}_0(\mathbf{X})$ provided*

$$0 < a < \frac{d-2}{d} \times \frac{E\left[\rho'(R^2)R^2\right]}{\omega\rho'(0)E\left[R^{-2}\right] + (1-\omega)E\left[\rho'(R^2)R^{-2}\right]}, \tag{2.3}$$

*where $E$ denotes the expectation with respect to the distribution of the radius $R$.*

**Example 1 (Student distribution).** *Suppose that $\mathbf{X} \in \mathbb{R}^d$ is distributed according to a Multivariate Student distribution with unknown location parameter $\boldsymbol{\theta}$, scale $\mathbf{I}_d$ and $\nu > 0$ degrees of freedom, denoted by $t_d(\boldsymbol{\theta}, \mathbf{I}_d, \nu)$, with the following pdf $f(x) = \dfrac{\Gamma(\frac{d+\nu}{2})}{(\pi\nu)^{\frac{d}{2}}\Gamma(\frac{\nu}{2})}\left(1 + \|\mathbf{x}\|^2/\nu\right)^{-\frac{d+\nu}{2}}$.*
*The distribution is the mixture of multivariate normal distributions with the inverse gamma distribution as the weight function given by $G(t) = \dfrac{\nu^{\frac{\nu}{2}} \times t^{\frac{\nu}{2}-1} \times exp(-\frac{\nu t}{2})}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}$.*
*For $\rho(t) = 1 - exp(-t/\alpha)$, we have $\rho'(t) = \alpha^{-1}exp(-t/\alpha)$, then from (2.1), we obtain :*

$$a_0 = \frac{d-2}{d} \times \frac{\nu(\nu-1)}{r^2\left[\omega\,(1/\alpha + 1/r)^{\nu+1} + (1-\omega)\,(1/\alpha + 1/r)^2\right]}.$$

**Example 2 (Symmetric Kotz type distribution).** *Let $\mathbf{X}$ be the random vector having a symmetric Kotz type distribution with unknown location parameter $\boldsymbol{\theta}$, scale $I_d$, with the following pdf*

$$f(x) = C_d(\|\mathbf{x}\|^2)^{N-1}exp(-r(\|\mathbf{x}\|^2)^s) \;\; \textit{where } r, s > 0 \; 2N + d > 2, \tag{2.4}$$

*and $C_d$ is a normalizing constant. For $s = 1$ and $\nu = N + d/2 - 1 > 0$, the radial density of $\|\mathbf{X} - \boldsymbol{\theta}\|^2 = R^2$ can be expressed as $h(u) = \dfrac{r^\nu}{\Gamma(\nu)} u^{\nu-1} e^{-ru}$ , $u > 0$ which is a gamma distribution Gamma$(\nu, r)$. From (2.1) we obtain*

$$a_0 = \frac{d-2}{d} \times \frac{\nu(\nu-1)}{r^2 \left[\omega\,(1/\alpha + 1/r)^{\nu+1} + (1-\omega)\,(1/\alpha + 1/r)^2\right]}.$$

**Example 3 (Uniform distribution on a sphere).** *For the uniform distribution on the sphere $S_{m,\boldsymbol{\theta}}$ ($\mathbf{X} \sim U(S_{m,\boldsymbol{\theta}})$) and $\rho(t) = \log(1+t)$, we have $\rho'(t) = \dfrac{1}{1+t}$ and $R^2 = \|\mathbf{X} - \boldsymbol{\theta}\|^2 = m^2$.*

**Example 4 (Uniform distribution on a ball).** *Let $\mathbf{X}$ be the random vector having the uniform distribution on the ball $B_{m,\boldsymbol{\theta}}$ ( $\mathbf{X} \sim U(B_{m,\boldsymbol{\theta}})$), with $d \geq 4$ and $R^2 = \|\mathbf{X} - \boldsymbol{\theta}\|^2$ having the density $h(t) = \dfrac{d}{2m^d} t^{\frac{d}{2}-1} \mathbb{I}_{(0,m^2)}(t)$. According to the choice $\rho(t) = \log(1+t)$, $d = 4$ and from (2.1), we obtain*

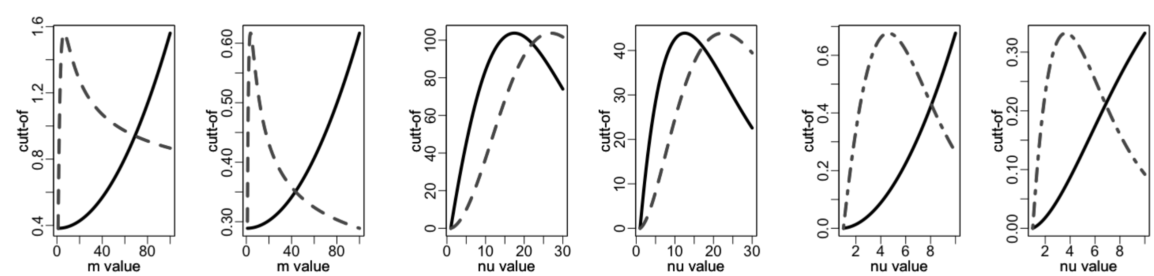
$$a_0(m) = \frac{1}{2} \times \frac{m^4/2 - m^2 + \log(1+m^2)}{m^2\omega + (1-\omega)\log(1+m^2)}.$$

**Remark 2.2.** *Suppose that $\mathbf{X}$ is spherically symmetric distributed arround $\boldsymbol{\theta}$ with density $f(\|\mathbf{x} - \boldsymbol{\theta}\|^2)$. The condition of domination founded by Hobbad et al.(2021) is:*

$$0 < a < \frac{2\,K^2\,(d-2)/d}{\{\,\omega\rho'(0) + K(1-\omega)\}\; E\,(\rho'(R^2)R^{-2}\,)}, \text{ where } K = E(\rho'(R^2)). \tag{2.5}$$

*If $\rho(t) = t$, the domination condition in 2.1 becomes $0 < a < (d-2)/d \times E\left[R^2\right]/E\left[R^{-2}\right]$ and the condition found in Hobbad et al. (2021) is $0 < a < (d-2)/d \times 2/E\left[R^{-2}\right]$. We deduce that if $E\left[R^2\right] < 2$ then the bound of the risk without density is lower than that with density if not we obtain the opposite.*

In the following figure 1, we conduct a comparison between the bound found in this paper from (2.1) for spherically symmetric distributions (without density) and the bound found in Hobbad et al. (2021) (with density) under the MBLFs (i).




(a) (b) (c)

Figure 1: (a) Uniform in the ball cases (b) Student distribution cases (c) Kotz distribution cases : without density (–), with density (- -) for $\omega = 1/4$ and $\omega = 3/4$.

In an alternative manner, these figures provide clear evidence that the bound established in Theorem (2.1) and the one derived in Hobbad et al. (2021) cannot be directly compared when a probability density is present. The superiority of one bound over the other is not consistently favored, it varies depending on the situation. However, the contribution of our work lies in the fact that we are able to obtain this bound even in scenarios where a probability density is not available.

## 2.2 With Respect to the Loss Function $\ell(\omega\|\boldsymbol{\delta}-\boldsymbol{\delta}_0\|^2+(1-\omega)\|\boldsymbol{\delta}-\boldsymbol{\theta}\|^2)$

In the following paragraph, our focus is on analyzing the minimax properties of Baranchik-type estimators for the location vector $\boldsymbol{\theta}$ of a random vector $\mathbf{X}$. The distribution of $\mathbf{X}$ is assumed to be spherically symmetric without necessarily needing a density. We specifically examine the estimators under MBLFs **(ii)**. We assume that the function $\ell$ satisfies

$$\mathbf{H2}: \ \ell(\cdot)\geq 0, \ \ell'(\cdot)>0 \text{ and } \ \ell \ \text{ is concave.}$$

In contrast to the previous section, the requirement for complete monotonicity on $\ell'$ in Marchand and Straw- derman (2020) is relaxed in this context when choosing $\rho$. Examples of loss functions $\ell$ that satisfy condition **H2** include the instances of functions $\rho$ that satisfy condition **H1** from the previous section. However, in this case, there is no requirement for finiteness of $\ell'(0)$, allowing for the inclusion of other losses such as $L^{\frac{q}{2}}$ losses with $\ell(t)=t^q, 0<q<1$ that also satisfy **H2**. Another interesting option is provided by Kubokawa et al.(2017), where $\ell(t)=2Q(\sqrt{t})-1$ with $Q$ representing a cumulative distribution function (c.d.f.) on $\mathbb{R}$ with an even and unimodal density $Q'$. We give a preparatory lemma which establishes a relation between the differences of loss by exploiting the concavity of $\ell$.

**Lemma 3.** *Suppose that $\mathbf{X}$ has a spherically symmetric distribution. For the problem of estimating $\boldsymbol{\theta}$ under the MBLFs **(ii)**, we have*

$$\boldsymbol{\delta}L_{\omega,\boldsymbol{\delta}_0,\ell}(\boldsymbol{\theta},\boldsymbol{\delta})\leq(1-\omega)^2\ell'((1-\omega)\|\boldsymbol{\delta}_0-\boldsymbol{\theta}\|)\boldsymbol{\delta}L_{0,\boldsymbol{\delta}_0,\ell}(\boldsymbol{\theta},\boldsymbol{\delta}).$$

The next theorem gives conditions for minimaxity of estimators $\boldsymbol{\delta}_{a,S}(\mathbf{X})$ given above by (2.1) for general spherically symmetric distributions.

**Theorem 2.3.** *Suppose that $\mathbf{X}$ has a spherically symmetric distribution ($\mathbf{X}\sim SS_d(\boldsymbol{\theta})$) and that the function $\ell$ satisfies **H2**. For $d\geq 4$, the estimator $\boldsymbol{\delta}_{a,S}(\mathbf{X})$ given above by (2.1) dominates $\boldsymbol{\delta}_0(\mathbf{X})=\mathbf{X}$ under MBLFs **(ii)** for all $\boldsymbol{\theta}$ with strict inequatity for some $\boldsymbol{\theta}$ provided*

$$0<a<2\frac{d-2}{d}\cdot\frac{E\left[R^2\ell'((1-\omega)R^2)\right]}{(1-\omega)E\left[\ell'((1-\omega)R^2)R^{-2}\right]\times E\left[R^2\right]}.$$

**Example 5 ( Uniform distribution on a sphere).** *For the uniform distribution on the sphere $S_{m,\boldsymbol{\theta}}(\mathbf{X}\sim U(S_{m,\boldsymbol{\theta}}))$ and $\ell(t)=t^q$ with $0<q<1$, we have $\ell'(t)=qt^{q-1}$ and $R^2=\|\mathbf{X}-\boldsymbol{\theta}\|^2=m^2$. Hence, the cut-off point from (2.3) is*

$$a_0(m)=\frac{d-2}{d}\times\frac{2m^2}{1-\omega}.$$

**Example 6** (**Uniform distribution on a ball).** *Let* $\mathbf{X}$ *be the random vector having the uniform distribution on the ball* $B_{m,\boldsymbol{\theta}}$ *(* $\mathbf{X} \sim U(B_{m,\boldsymbol{\theta}})$*), with* $d \geq 4$ *and* $R^2 = \|\mathbf{X} - \boldsymbol{\theta}\|^2$ *having the density* $h(t) = \dfrac{d}{2m^d} t^{\frac{d}{2}-1} \mathbb{I}_{(0,m^2)}(t)$*. From the choice of* $\ell(t) = t^q$ *with* $0 < q < 1$ *and from (2.3), we obtain the cut-off point* $a_0$ *is*

$$a_0(m) = \frac{2(d-2)}{d} \times \frac{d(2q+d-4)m^2}{(1-\omega)(d+2)(2q+d)}.$$

**Example 7** (**Symmetric Kotz type distribution).** *Suppose that* $\mathbf{X}$ *possesses a symmetric Kotz type distribution with unknown location parameter* $\boldsymbol{\theta}$*, scale* $\mathbf{I}_d$*. For* $s = 1$*, the density of the radius* $R^2$ *is* $h(u) = \dfrac{r^\nu}{\Gamma(\nu)} u^{\nu-1} e^{-ru}$ *,* $u > 0$*. When* $\ell(t) = t^q$ *with* $0 < q < 1$ *and from (2.3), we obtain the cut-off point* $a_0$ *as follows*

$$a_0 = \frac{2(d-2)}{d} \times \frac{(\nu+q-1)(\nu+q-2)}{(1-\omega)(\nu+1)}.$$

**Example 8** (**Student distribution).** *Suppose that* $\mathbf{X} \in \mathbb{R}^d$ *is distributed according to a multivariate student distribution with unknown location parameter* $\boldsymbol{\theta}$*, scale* $\mathbf{I}_d$ *and* $\nu > 0$ *degrees of freedom. From* $\ell(t) = t^q$ *with* $0 < q < 1$ *and (2.3), we get the cut-off point* $a_0$ *as follows*

$$a_0 = \frac{2(d-2)}{d} \times \frac{4(\nu/2+q-1)(\nu/2+q-2)}{\nu^2(1-\omega)}.$$

**Remark 2.4.** *Suppose that* $\mathbf{X}$ *is spherically symmetric distributed arround* $\boldsymbol{\theta}$ *with density* $f(\|\mathbf{x} - \boldsymbol{\theta}\|^2)$*. The condition of domination found by Hobbad et al.(2021) is :*

$$0 < a < 2\frac{d-2}{d} \cdot \frac{E\left[\ell'((1-\omega)R^2)\right]}{(1-\omega)E\left[\ell'((1-\omega)R^2)R^{-2}\right]}. \tag{2.6}$$

*In case* $\ell(t) = t$*, the domination condition in the theorem (2.3) and in Hobbad et al.(2021) becomes* $0 < a < (d-2)/d \times 2/((1-\omega)E\left[R^{-2}\right])$ *and for* $\omega = 0$*, it becomes* $0 < a < (d-2)/d \times 2/E\left[R^{-2}\right]$*. As* $\left\{\ell'((1-\omega)R^2)\right\}$ *is decreasing in* $R$ *according to* ***H2*** *and by using the covariance inequality, we obtain* $E\left[R^2\ell'((1-\omega)R^2)\right] \leq E\left[R^2\right] \times E\left[\ell'((1-\omega)R^2)\right]$*. So,*

$$2\frac{d-2}{d} \cdot \frac{E\left[R^2\ell'((1-\omega)R^2)\right]}{(1-\omega)E\left[\ell'((1-\omega)R^2)R^{-2}\right] \times E\left[R^2\right]} \leq 2\frac{d-2}{d} \cdot \frac{E\left[\ell'((1-\omega)R^2)\right]}{(1-\omega)E\left[\ell'((1-\omega)R^2)R^{-2}\right]}.$$

*This illustrates that the bound derived in Hobbad et al.(2021) exceeds the bound obtained from Theorem 2.3, primarily due to the inclusion of a density function, which provides additional information.*

The following figure 2 illustrate a comparison between the bound found in this paper from (2.3) for spherically symmetric random vectors (without density) and the bound found in Hobbad et al. (2021)(with density) under MBLFs (ii).

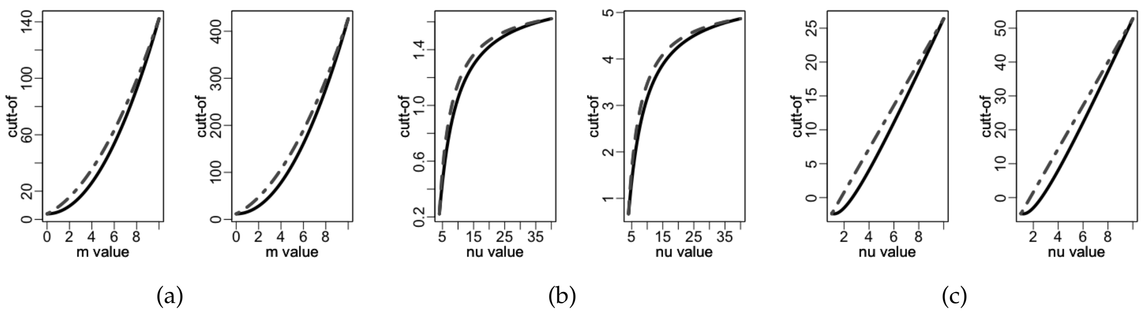


Figure 2: (a) Uniform in the ball cases (b) Student distribution cases (c) Kotz distribution cases for $\omega = 1/4$ and $\omega = 3/4$. (–) without density, (- -) with density.

# 3 Robust Shrinkage Estimators for Location Parameters under a Modified Balanced Loss Functions

Within this section, our objective is to investigate the estimation problem for the mean vector $\boldsymbol{\theta}$ of the random vector $\mathbf{X} \in \mathbb{R}^d$ having a spherically symmetric distribution, and suppose that $E_0\left[\|\mathbf{X}\|^2\right] < \infty$ and $E_0\left[1/\|\mathbf{X}\|^2\right] < \infty$. To accomplish this, we will introduce a broader class of estimators, of which the Baranchik class of estimators represents a specific subset. The proposed estimator is of the following form:

$$\boldsymbol{\delta}_{\omega,g}(\mathbf{X}) = \mathbf{X} + a(1-\omega)g(\mathbf{X}). \tag{3.1}$$

Let us consider the existence of a nonpositive function $h(\cdot)$ with the following properties: $h(\mathbf{X})$ is subharmonic, the expectation $E_{R,\boldsymbol{\theta}}\left[R^2 h(\mathbf{U})\right]$ is decreasing, where $\mathbf{U} \sim U_{R,\boldsymbol{\theta}}$, and $E_{\boldsymbol{\theta}}\left[|h(\mathbf{X})|\right]$ is finite. Moreover, we assume that $g$ possesses weak differentiability and satisfies the subsequent condition: $div(g) \leq h$ and $\|g\|^2 + 2h \leq 0$.

Subsequently, we aim to determine a specific condition on the parameter $a$ that guarantees the minimax property of our estimator and its dominance over the usual estimator $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$ under MBLFs **(i)** and **(ii)** where $\rho$ and $\ell$ are increasing and concave functions only.

## 3.1 With Respect to the Loss Function $\omega\rho(\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\|^2) + (1-\omega)\rho(\|\boldsymbol{\delta} - \boldsymbol{\theta}\|^2)$

Let us consider a spherically symmetrically distributed random vector $\mathbf{X}$ with a location parameter $\boldsymbol{\theta}$, which may or may not have a density ($\mathbf{X} \sim SS_d(\boldsymbol{\theta})$). Our goal is to determine the values of $a$ for which the estimator $\boldsymbol{\delta}_{\omega,g}(\mathbf{X}) = \mathbf{X} + a(1-\omega)g(\mathbf{X})$ dominate $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$ under MBLFs **(i)**. This MBLFs incorporates the target estimator $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$. We make the assumption that the $\rho$ function employed in MBLFs **(i)** fulfills the following conditions:

$$\mathbf{H1}: \ \rho(0) = 0, \ 0 < \rho'(0) < +\infty, \text{ and } \rho \text{ is concave.}$$

**Lemma 4.** *Consider a random vector* $\mathbf{X}$ *that obeys a spherically symmetric distribution with a density of the form* $f(\|\mathbf{x}-\boldsymbol{\theta}\|^2)$, *we can deduce the following outcome:*

$$E_{\boldsymbol{\theta}}\left[\rho\left(-h(\mathbf{X})\right)\right] \leq \rho'(0)E_{\boldsymbol{\theta}}\left[-h(\mathbf{X})\right] \leq \rho'(0)E^{*}_{\boldsymbol{\theta}}\left[-h(\mathbf{Y})\right]. \tag{3.2}$$

*Where* $\mathbf{Y} \sim f^{*}(\|\mathbf{y}-\boldsymbol{\theta}\|^2)$, $K = \int \rho'(r^2)f(r^2)dr$ *and* $f^{\star}(r^2) = \rho'(r^2)f(r^2)/K$.

**Theorem 3.1.** *Let* $\mathbf{X}$ *be a random vector that follows a spherically symmetric distribution with a density of the form* $\mathbf{X} \sim f(\|\mathbf{x}-\boldsymbol{\theta}\|^2)$. *Assuming that the function* $\rho$ *satisfies condition* ***H1*** *and* $d \geq 4$. *The estimator* $\boldsymbol{\delta}_{\omega,g}(\mathbf{X})$ *dominates* $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$ *under MBLFs* ***(i)*** *for all values of* $\boldsymbol{\theta}$, *and there exists at least one value of* $\boldsymbol{\theta}$ *where the domination is strict if the parameter a satisfies the inequality:*

$$0 < a < \frac{K/d}{\left[\omega\rho'(0) + K(1-\omega)\right]E^{\star}_{0}\left(\|\mathbf{Y}\|^{-2}\right)},$$

*where* $E^{\star}_0$ *denotes the expectation value when* $\boldsymbol{\theta} = 0$, *with respect to the density of Lemma 4.*

Subsequently, we will investigate the conditions for dominance in distributions that may not possess a density function. In such cases, we will leverage the inherent properties of spherically symmetric distributions.

The following results focus on the computation of conditional expectations based on the radius of a spherically symmetric distribution in $\mathbb{R}^d$, centered at $\boldsymbol{\theta} \in \mathbb{R}^d$. These expectations can be expressed as integrals involving the uniform distribution $U_{R,\boldsymbol{\theta}}$ defined on the sphere $S_{R,\boldsymbol{\theta}} = \{\mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{x}-\boldsymbol{\theta}\| = R\}$.

Let $E_{R,\boldsymbol{\theta}}[\varphi]$ denote the expectation of a function $\varphi$ with respect to $U_{R,\boldsymbol{\theta}}$. In order to obtain the expectation with respect to the entire distribution, denoted as $E_{\boldsymbol{\theta}}[\varphi]$, we apply the following procedure: first, we compute the expectation $E$ with respect to the distribution of the radius, and then we take the expectation $E[E_{R,\boldsymbol{\theta}}[\varphi]]$.

The following theorem provides conditions for the minimaxity of estimators of the form $\boldsymbol{\delta}_{\omega,g}(\mathbf{X}) = \mathbf{X}+a(1-\omega)g(\mathbf{X})$ in the case of general spherically symmetric distributions. It is important to note that this result does not require the existence of a density function and instead relies on the properties of the radial distribution.

**Theorem 3.2.** *Consider a random vector* $\mathbf{X}$ *with a spherically symmetric distribution centered at* $\boldsymbol{\theta}$. *Assume that* $E_0[\|\mathbf{X}\|^2] < \infty$ *and* $E_0[1/\|\mathbf{X}\|^2] < \infty$. *Let* $\rho$ *be a function that satisfies condition* ***H1***. *If* $d \geq 4$, *the estimator* $\boldsymbol{\delta}_{\omega,g}(\mathbf{X}) = \mathbf{X} + a(1-\omega)g(\mathbf{X})$ *dominates* $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$ *under MBLFs* ***(i)*** *for all* $\boldsymbol{\theta}$. *Moreover, there exists at least one value of* $\boldsymbol{\theta}$ *where the dominance is strictly established, meaning the inequality is strict. This dominance is guaranteed if the following condition is satisfied:*

$$0 < a < \frac{E\left[\rho'(R^2)\right]/d}{\omega\rho'(0)E\left[1/R^2\right] + (1-\omega)E\left[\rho'(R^2)/R^2\right]}.$$

### 3.2 With Respect to the Loss Function $\ell(\omega\|\boldsymbol{\delta}-\boldsymbol{\delta}_0\|^2)+(1-\omega)\|\boldsymbol{\delta}-\boldsymbol{\theta}\|^2)$

In this paragraph, our focus is on examining a random vector $\mathbf{X}\in\mathbb{R}^d$ having a spherically symmetric distribution arround $\boldsymbol{\theta}$. Our objective is to establish the conditions on $a$ that lead to the dominance of estimator $\boldsymbol{\delta}_{\omega,g}(\mathbf{X})=\mathbf{X}+a(1-\omega)g(\mathbf{X})$ over estimator $\boldsymbol{\delta}_0(\mathbf{X})=\mathbf{X}$ under the MBLF **(ii)**, where the function $\ell$ satisfies

$$\mathbf{H2}:\ \ell(\cdot)\geq 0,\ \ell'(\cdot)>0 \text{ and } \ell \text{ is concave,}$$

which incorporates the target estimator $\boldsymbol{\delta}_0(\mathbf{X})=\mathbf{X}$.
We will first present the theorem that gives the domination conditions for spherically symmetric distributions with a density of the form $f(\|\mathbf{x}-\boldsymbol{\theta}\|^2)$.

**Theorem 3.3.** *Assume that $\mathbf{X}$ follows a spherically symmetric distribution centered at $\boldsymbol{\theta}$, with a density function $f(\|\mathbf{x}-\boldsymbol{\theta}\|^2)$. We also assume that $E_0[\|\mathbf{X}\|^2]<\infty$ and $E_0[1/\|\mathbf{X}\|^2]<\infty$. Additionally, suppose that the function $\ell$ satisfies condition **H2**. For $d\geq 4$, the estimator $\boldsymbol{\delta}_{\omega,g}(\mathbf{X})=\mathbf{X}+a(1-\omega)g(\mathbf{X})$, dominates $\boldsymbol{\delta}_0(\mathbf{X})=\mathbf{X}$ under MBLFs **(ii)** for all $\boldsymbol{\theta}$, with strict inequatity for some $\boldsymbol{\theta}$ provided*

$$0<a<1/d(1-\omega)E^{\star}\left[\|\mathbf{Z}\|^{-2}\right],$$

*where* $Z\sim f^{*}(\|\mathbf{x}-\boldsymbol{\theta}\|^2)=\dfrac{\ell'((1-\omega)\|\mathbf{x}-\boldsymbol{\theta}\|^2)f(\|\mathbf{x}-\boldsymbol{\theta}\|^2)}{\displaystyle\int_{\mathbb{R}^d}\ell'((1-\omega)\|\mathbf{x}-\boldsymbol{\theta}\|^2)f(\|\mathbf{x}-\boldsymbol{\theta}\|^2)dx}.$

Next, we will establish the conditions for dominance in the case of spherically symmetric distributions without necessarily needing a density.

**Lemma 5.** *Suppose $\mathbf{X}$ follows a spherically symmetric distribution arroud $\boldsymbol{\theta}$. Consider the case where $d\geq 4$, and let $g(\mathbf{X})$ be a function that maps $\mathbb{R}^d$ to $\mathbb{R}^d$ and is weakly differentiable. Additionally, assume that:*
- $\|g(\mathbf{X})\|^2/2\leq -h(\mathbf{X})\leq -\nabla^{\top}g(\mathbf{X})$.
- $-h(\mathbf{X})$ *is superharmonic and* $E_{\boldsymbol{\theta}}\left[R^2h(\mathbf{U})\right]$ *is increasing function of $R$, where $\mathbf{U}$ has a uniform distribution on the sphere of radius $R$ centered at $\boldsymbol{\theta}$. Then*

$$E_{\boldsymbol{\theta}}\left[\|\mathbf{X}+ag(\mathbf{X})-\boldsymbol{\theta}\|^2-\|\mathbf{X}-\boldsymbol{\theta}\|^2\right]\leq E_{\boldsymbol{\theta}}\left[\left(-2a^2/r^2+2a\frac{1}{d}\right)E_{r,\boldsymbol{\theta}}[r^2h(\mathbf{X})/\|\mathbf{X}-\boldsymbol{\theta}\|^2=r^2\right].$$

**Theorem 3.4.** *Suppose that $\mathbf{X}$ have a spherically symmetric distribution ($\mathbf{X}\sim SS_d(\boldsymbol{\theta})$) and that the function $\ell$ satisfy **H2**. For $d\geq 4$, the estimator $\boldsymbol{\delta}_{\omega,g}(\mathbf{X})$ dominates delta$_0(\mathbf{X})=\mathbf{X}$ under MBLFs **(ii)** for all $\boldsymbol{\theta}$ with strict inequatity for some $\boldsymbol{\theta}$ provided*

$$0<a<\frac{1}{d}\cdot\frac{E\left[R^2\ell'((1-\omega)R^2)\right]}{(1-\omega)E\left[\ell'((1-\omega)R^2)R^{-2}\right]\times E_{\boldsymbol{\theta}}\left[R^2\right]}.$$

**Remark 3.5.**

- *The four theorems in this section yield parameter bounds for the parameter a that are smaller than those obtained for the Baranchik estimators, both in Marchand and Strawderman(2020) and the previous section of this article. Specifically, the bounds for the estimator* $\boldsymbol{\delta}_{\omega,g}(\mathbf{X})$ *are equal to the product of the bounds for the Baranchik estimators* $\boldsymbol{\delta}_{a,S}(\mathbf{X})$ *multiplied by* $1/(2(d-2))$. *This can be attributed to the estimator* $\boldsymbol{\delta}_{\omega,g}(\mathbf{X})$, *which serves as a generalization of the Baranchik estimator and justifies the observed relationship between the bounds.*

- *The Baranchik estimator can be viewed as a specific instance of the estimator* $\boldsymbol{\delta}_{\omega,g}(\mathbf{X})$, *when we set the functions* $h(\mathbf{X}) = -bS(\|\mathbf{X}\|^2)/\|\mathbf{X}\|^2$ *where* $\frac{1}{2} \leq b \leq d-2$. *In the case where* $b = 2(d-2)$, *the bounds obtained for the Baranchik estimator coincide with those obtained using the estimator* $\boldsymbol{\delta}_{\omega,g}(\mathbf{X})$.

# 4 A Simulation Study

In this section, we consider a random vector $\mathbf{X}$ having a spherically symmetric distribution without having a density. We know that we can write $\mathbf{X} - \boldsymbol{\theta} = R\mathbf{U}$ with $R$ is a positive real random variable (the distribution of radius), $\mathbf{U}$ has a uniform distribution on the unit sphere and $R$ and $\mathbf{U}$ are independent. In order to simulate $\mathbf{X}$, we assume that $R$ is a discrete variable with a Poisson distribution, so it's not continuous with respect to the Lebesgue measure which indicates that $\mathbf{X}$ does not have a density. The following lemma allows us to simulate $\mathbf{U}$. Finally, we obtain a simulation of $\mathbf{X} = \boldsymbol{\theta} + R\mathbf{U}$.

**Lemma 6.** *Let* $N_1, \ldots, N_d$ *be i.i.d. random variable* $N(0,1)$. *we consider:* $T = \left(N_1^2 + \ldots + N_d^2\right)^{1/2}$. *The random vector* $\left(\frac{N_1}{T}, ..., \frac{N_d}{T}\right)$ *have the uniform distribution on a sphere* $S(0,1)$.

A Table 1 gives the normalized risk of $\boldsymbol{\delta}_{a,S}(\mathbf{X})$ $(\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_{a,S}(\mathbf{X}))/\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_0(\mathbf{X})))$ for various values of the dimension $d$ and of the parameter $\boldsymbol{\theta}$ using MBLFs (ii). We obtain these results by considering (i) $\omega = 1/2$, (ii) $\ell(t) = t^q, q = 1/2$, (iii) $R$ has the Poisson distribution of parameter $\lambda = 8$, (iv) $S(t) = 1, a = 1$ and (v) $\boldsymbol{\theta} = \|\boldsymbol{\theta}\|(1, 0, \ldots, 0)$.

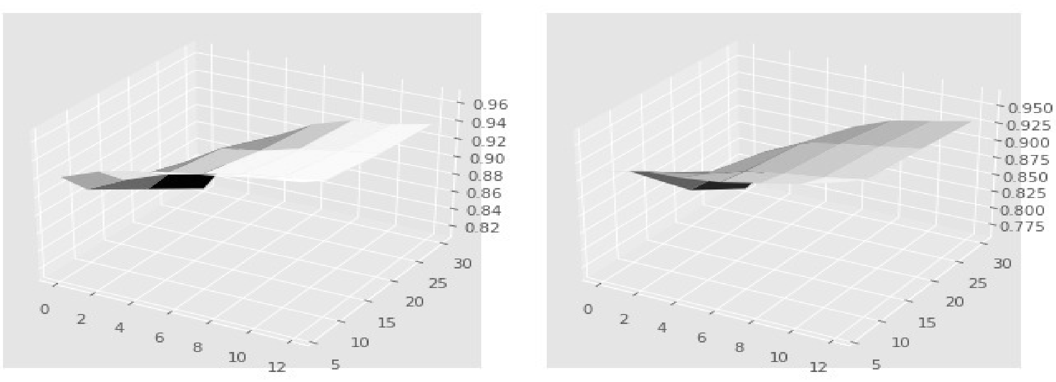
The following Table 2 gives the values of the normalized risk $\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_{a,S}(\mathbf{X}))/\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_0(\mathbf{X}))$ for various values of the dimension $d$ and of the parameter $\boldsymbol{\theta}$. The experimental values found under the following assumptions (i) $\boldsymbol{\theta} = \|\boldsymbol{\theta}\|(1, 0, \ldots, 0)$, (ii) $\rho(t) = \ln(1+t)$, (iii) $\omega = 1/3$, (iv) $s(t) = t/(1+t)$, (v) $a = 5$ and (vi), $R$ has the Poisson distribution of parameter $\lambda = 8$.

Table 1: the Risk ratios $(\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_{a,S}(\mathbf{X}))/\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_0(\mathbf{X})))$ for various values of $d$ and $\|\boldsymbol{\theta}\|$.

| $\|\boldsymbol{\theta}\|$ \ $d$ | d=6 | d=10 | d=15 | d= 20 | d=30 |
|---|---|---|---|---|---|
| 0 | 0.910 | 0.890 | 0.861 | 0.810 | 0.760 |
| 2 | 0.923 | 0.900 | 0.883 | 0.850 | 0.820 |
| 4 | 0.925 | 0.910 | 0.905 | 0.890 | 0.860 |
| 6 | 0.931 | 0.925 | 0.918 | 0.915 | 0.890 |
| 8 | 0.942 | 0.934 | 0.930 | 0.925 | 0.910 |
| 10 | 0.953 | 0.941 | 0.938 | 0.933 | 0.920 |
| 12 | 0.967 | 0.950 | 0.947 | 0.941 | 0.931 |

Table 2: The Risk ratios $\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_{a,S}(\mathbf{X}))/\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_0(\mathbf{X}))$ for various values of $d$ and of the parameter $\boldsymbol{\theta}$.

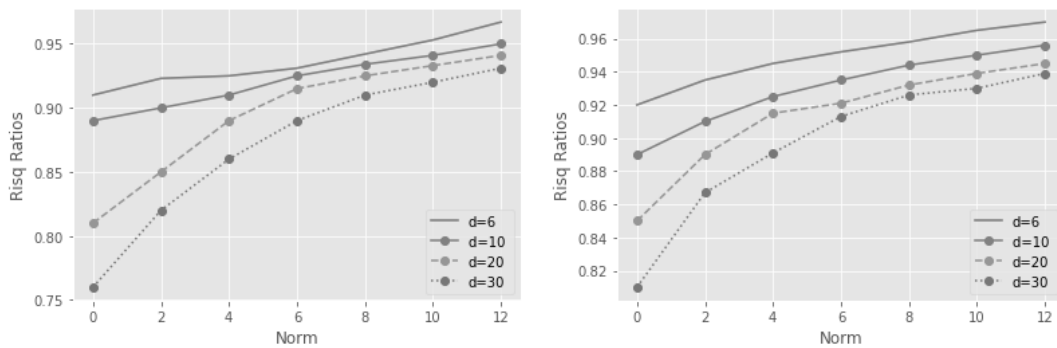
| $\|\boldsymbol{\theta}\|$ \ $d$ | d=6 | d=10 | d=15 | d= 20 | d=30 |
|---|---|---|---|---|---|
| 0 | 0.920 | 0.890 | 0.870 | 0.850 | 0.810 |
| 2 | 0.935 | 0.910 | 0.903 | 0.890 | 0.867 |
| 4 | 0.945 | 0.925 | 0.921 | 0.915 | 0.891 |
| 6 | 0.952 | 0.935 | 0.927 | 0.921 | 0.913 |
| 8 | 0.958 | 0.944 | 0.9390 | 0.932 | 0.926 |
| 10 | 0.965 | 0.950 | 0.943 | 0.939 | 0.930 |
| 12 | 0.970 | 0.957 | 0.951 | 0.945 | 0.939 |




(a) (b)

Figure 3: (a) Graph of risk ratios $\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_{a,S}(\mathbf{X}))/\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_0(\mathbf{X}))$ as function of the dimension $d$ and $\|\boldsymbol{\theta}\|$ under the loss (i). (b) Graph of risk ratios $\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_{a,S}(\mathbf{X}))/\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_0(\mathbf{X}))$ as function of the dimension $d$ and $\|\boldsymbol{\theta}\|$ under MBLFs loss (ii).

By using the normalized risk $\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_{a,S}(\mathbf{X}))/\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_0(\mathbf{X}))$ and $\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_{a,S}(\mathbf{X}))/\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_0(\mathbf{X}))$, the improvement of $\boldsymbol{\delta}_{a,S}$ on $\boldsymbol{\delta}_0$ are all the more significant (respectively, not significant) as this quantity is close to 0 (respectively, almost 1).




(a) (b)

Figure 4: (a) The graphs of Risk Ratios $\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_{a,S}(\mathbf{X}))/\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_0(\mathbf{X}))$ as a function of $\|\boldsymbol{\theta}\|$ for fixed values of $d$ under MBLFs (ii). (b) The graphs of Risk Ratios $\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_{a,S})/\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_0)$ as a function of $\|\boldsymbol{\theta}\|$ for fixed values of $d$ under MBLFs (i).

According to the Tables 1, 2 and Figures 3, 4, the normalized risk $\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_{a,S})/\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_0)$ and $\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_{a,S}(\mathbf{X}))/\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_0(\mathbf{X}))$ are a decreasing function of $d$ for fixed $\|\boldsymbol{\theta}\|$, and an increasing function of $\|\boldsymbol{\theta}\|$ for $d$ fixed. Thus, a significant improvement of the estimator $\boldsymbol{\delta}_{a,S}$ on $\boldsymbol{\delta}_0$ is obtained when $d$ is large and $\|\boldsymbol{\theta}\|$ is little.

## 4.1 Influence of the Poisson Parameter on the Risk Values

This table, contains the values of the risk $\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_{a,S}(\mathbf{X})$ under MBLFs (ii) by varying the values of lambda (the parameter of the radial Poisson distribution) using the following conditions $\ell(t) = t^q, q = 1/2, d = 10$ and $\boldsymbol{\theta} = (1, 0, \ldots, 0)$.

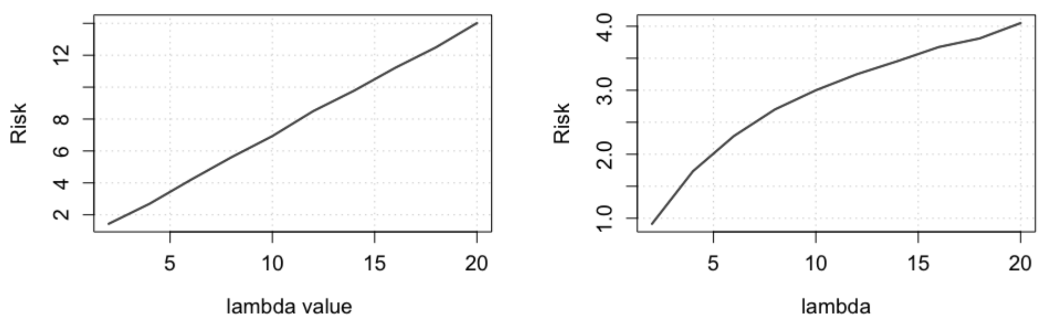
Table 3: The risk values $\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_{a,S}(\mathbf{X})$ as a function of the parameters $\lambda$ of the Poisson distribution under MBLFs (ii).

| $\lambda$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| Risk | 0.913 | 1.735 | 2.285 | 2.700 | 3.000 | 3.250 | 3.456 | 3.675 | 3.810 | 3.950 |

The following table gives the values of risk $\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_{a,S}(\mathbf{X})$ in terms of the lambda parameter of the Poisson distribution of the variable $R = \|\mathbf{X} - \boldsymbol{\theta}\|$, we have conduct the simulation in R language with the following conditions $d = 10$, $S(t) = t/(1 + t)$, $a = 1$, $\omega = 1/3$, $\boldsymbol{\theta} = (1, 0, \ldots, 0)$ and $\rho(t) = \ln(1 + t)$.

Table 4: The risk values $\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_{a,S}(\mathbf{X})$ as a function of the parameters of the Poisson distribution under MBLFs (i).

| $\lambda$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| Risk | 1.433 | 2.692 | 4.180 | 5.607 | 6.933 | 8.500 | 9.784 | 11.210 | 12.500 | 14.020 |




(a) (b)

Figure 5: (a) The risk values $\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_{a,S}(\mathbf{X})$ as a function of lambda. (b) The risk values $\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_{a,S}(\mathbf{X})$ as a function of lambda.

The Tables 3, 4 and graphs 5 showed that the risk increase when lambda increases, the growth is more important for MBLFs (ii).

## 4.2 Effect of Omega on the Value of Risk

We will give the risk values $\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_{a,S}(\mathbf{X})$ as a function of the parameter $\omega$ of MBLFs (ii) by implementing the simulations with the parameters $d = 10,\ \lambda = 10,\ \ell(t) = t^q,\ q = 1/2,\ S(t) = t/(1+t),\ a = 1$ and $\boldsymbol{\theta} = (1, 0 \ldots, 0)$.

Table 5: $\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_{a,S}(\mathbf{X})$ as a function of omega.

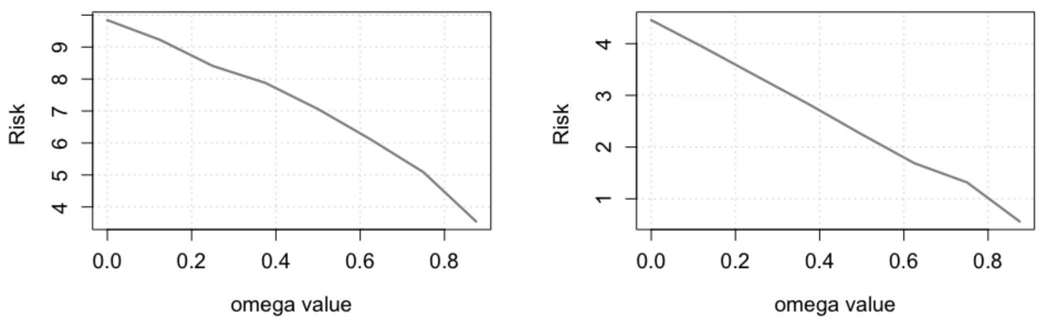
| $\omega$ | 0 | 1/8 | 2/8 | 3/8 | 4/8 | 5/8 | 6/8 | 7/8 |
|---|---|---|---|---|---|---|---|---|
| Risk | 9.842 | 9.225 | 8.410 | 7.879 | 7.060 | 6.110 | 5.087 | 3.545 |

the following table gives the values of the risk $\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_{a,S}(\mathbf{X})$ as a function of the parameter $\omega$ of the MBLF (i) under the following conditions $d = 10,\ \lambda = 10,\ a = 1/2,\ S(t) = t/(1+t),\ \boldsymbol{\theta} = (1, 0, ...., 0)$ and $\rho(t) = \ln(1+t)$.

Table 6: $\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_{a,S}(X)$ as a function of omega.

| $\omega$ | 0 | 1/8 | 2/8 | 3/8 | 4/8 | 5/8 | 6/8 | 7/8 |
|---|---|---|---|---|---|---|---|---|
| Risk | 4.455 | 3.925 | 3.374 | 2.825 | 2.246 | 1.688 | 1.317 | 0.561 |

In both cases, we notice that the risk decreases as a function of omega, which is illustrated in the following two graphs.




(a) (b)

Figure 6: (a) The Risk values $\mathcal{R}(\boldsymbol{\theta}, \ell, \boldsymbol{\delta}_{a,S}(\mathbf{X})$ as a function of omega. (b) The risk values $\mathcal{R}(\boldsymbol{\theta}, \rho, \boldsymbol{\delta}_{a,S}(\mathbf{X})$ as a function of omega.

## 5 Conclusion

This paper focuses on the estimation of the d-dimensional location vector for d-variate spherically symmetric distributions, even when these distributions do not necessarily have a density function. We consider MBLFs **(i)** and **(ii)** in this context. Previous work by Hobbad et al.(2021) introduced Baranchik-type estimators that dominate the usual estimator $\boldsymbol{\delta}_0(\mathbf{X}) = \mathbf{X}$ and achieve minimax performance based on the properties of density for spherically symmetric distributions with Lebesgue density $f(\|\mathbf{x} - \boldsymbol{\theta}\|^2)$.
In our study, we extend and build upon the findings of Hobbad et al. (2021), using the properties of spherically symmetric distributions in conjunction with the uniform distribution on a sphere, superharmonicity, and concavity. The cut-off points derived in (3.1) are smaller than those obtained by Hobbad et al. (2021). However, our results are applicable to a more general class of spherically symmetric distributions. It is worth noting that when $\omega = 0$, our cut-off point in (3.1) coincides with the one obtained by Brandwein and Strawderman(1980) in Theorem 2.1, which is $(d-2)/d \times 2/E\left[R^{-2}\right]$.
We proceeded to extend our findings to encompass robust shrinkage estimators. The bounds obtained for the estimator $\boldsymbol{\delta}_{\omega,g}(\mathbf{X})$ are equal to the product of the bounds of the Baranchik estimators $\boldsymbol{\delta}_{a,S}(\mathbf{X})$, multiplied by a factor of $1/(2(d-2))$.
Our next objective is to conduct a similar investigation for Bayesian estimators using MBLFs loss functions. By leveraging our existing results, we aim to enhance the performance of machine learning algorithms.

## 6 Proofs

**Proof of Lemma 1.** Let $h$ the radial distribution, we have:

$$\begin{aligned} E_{\boldsymbol{\theta}}\left[\rho'(\|\mathbf{X}-\boldsymbol{\theta}\|^2)(\mathbf{X}-\boldsymbol{\theta})^\top g(\mathbf{X})\right] &= \int_{\mathbb{R}^+}\int_{S_{r,\boldsymbol{\theta}}} \rho'(\|\mathbf{x}-\boldsymbol{\theta}\|^2)(\mathbf{x}-\boldsymbol{\theta})^\top g(\mathbf{X}) dU_{r,\boldsymbol{\theta}}(\mathbf{x})dh(r) \\ &= \int_{\mathbb{R}^+} \frac{r\rho'(r^2)}{\sigma_{r,\boldsymbol{\theta}}(S_{r,\boldsymbol{\theta}})} \int_{S_{r,\boldsymbol{\theta}}} \frac{(\mathbf{x}-\boldsymbol{\theta})^\top}{\|\mathbf{x}-\boldsymbol{\theta}\|} g(\mathbf{x}) d\sigma_{r,\boldsymbol{\theta}}(\mathbf{x})dh(r) \\ &= \int_{\mathbb{R}^+} \frac{r\rho'(r^2)}{\sigma_{r,\boldsymbol{\theta}}(S_{r,\boldsymbol{\theta}})} \int_{B_{r,\boldsymbol{\theta}}} \nabla g(\mathbf{x}) d\mathbf{x}\, dh(r), \quad \textit{(by Stokes' theorem)} \\ &= \frac{1}{d}\int_{\mathbb{R}^+} \rho'(r^2)r^2 \int_{B_{r,\boldsymbol{\theta}}} \nabla g(\mathbf{x}) dV_{r,\boldsymbol{\theta}}(\mathbf{x})dh(r), \end{aligned}$$

since the volume of $B_{r,\boldsymbol{\theta}}$ is $\lambda(B_{r,\boldsymbol{\theta}}) = r\sigma_{r,\boldsymbol{\theta}}(S_{r,\boldsymbol{\theta}})/d$.

**Proof of Lemma 2.** Due to radius conditioning, we get

$$\begin{aligned} & E_{\boldsymbol{\theta}}\left[(1-\omega)\rho'(\|\mathbf{X}-\boldsymbol{\theta}\|^2)[2(1-\omega)g(\mathbf{X}).(\mathbf{X}-\boldsymbol{\theta}) + (1-\omega)^2\|g(\mathbf{X})\|^2]\right] \\ = \quad & E\left[2(1-\omega)^2\rho'(R^2)E_{R,\boldsymbol{\theta}}\left[g(\mathbf{X}).(\mathbf{X}-\boldsymbol{\theta})/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right]\right] \\ + \quad & E\left[(1-\omega)^3\rho'(R^2)E_{r,\boldsymbol{\theta}}\left[\|g(\mathbf{X})\|^2/\|\mathbf{X}-\boldsymbol{\theta} = R\right]\right] \\ \le \quad & E\left[(1-\omega)^3\rho'(R^2)E_{R,\boldsymbol{\theta}}\left[\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right]\right] \\ - \quad & 2a(1-\omega)^2\frac{d-2}{d}\int_{\mathbb{R}^+} r^2\rho'(r^2)\int_{B_{r,\boldsymbol{\theta}}} \frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2} dV_{r,\boldsymbol{\theta}}(\mathbf{x})d\rho(r). \end{aligned}$$

Using subharmonicity of $h : \mathbf{x} \longrightarrow \dfrac{S(\|\mathbf{x}\|^2)}{\|\mathbf{x}\|^2}$, we get $\displaystyle\int_{B_{r,\boldsymbol{\theta}}} h(\mathbf{x})dV_{r,\boldsymbol{\theta}}(\mathbf{x})d\rho(r) \le \int_{S_{r,\boldsymbol{\theta}}} h(\mathbf{x})d\mathbf{U}_{r,\boldsymbol{\theta}}(\mathbf{x})d\rho(r)$. Hence,

$$\begin{aligned} & E_{\boldsymbol{\theta}}\left[(1-\omega)\rho'(\|\mathbf{X}-\boldsymbol{\theta}\|^2)[2(1-\omega)g(\mathbf{X}).(\mathbf{X}-\boldsymbol{\theta}) + (1-\omega)^2\|g(\mathbf{X})\|^2]\right] \\ \le \quad & E\left[(1-\omega)^2 a\rho'(R^2)\left(a(1-\omega)/R^2 - \frac{2(d-2)}{d}\right)E_{R,\boldsymbol{\theta}}[R^2\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\| = R]\right]. \end{aligned}$$

**Proof of Lemma 3.** Since $\ell(\cdot)$ is a concave function, for any points $\alpha$ and $\beta$, $\ell(\alpha+\beta)-\ell(a) \le \beta\ell'(\alpha)$. Thus, for $\alpha = L_{\omega,\boldsymbol{\delta}_0}(\boldsymbol{\theta},\boldsymbol{\delta}_0)$ and $\alpha+\beta = L_{\omega,\boldsymbol{\delta}_0}(\boldsymbol{\theta},\boldsymbol{\delta})$, we get

$$\boldsymbol{\delta} L_{\omega,\boldsymbol{\delta}_0,\ell}(\boldsymbol{\theta},\boldsymbol{\delta}) = L_{\omega,\ell,\boldsymbol{\delta}_0}(\boldsymbol{\theta},\boldsymbol{\delta}) - L_{\omega,\ell,\boldsymbol{\delta}_0}(\boldsymbol{\theta},\boldsymbol{\delta}_0) \le (1-\omega)^2\ell'((1-\omega)\|\boldsymbol{\delta}_0-\boldsymbol{\theta}\|^2)\boldsymbol{\delta} L_{0,\boldsymbol{\delta}_0}(\boldsymbol{\theta},\boldsymbol{\delta}).$$

**Proof of Theorem 2.1.** the difference in risk is :

$$
\begin{aligned}
\delta\mathcal{R}(\boldsymbol{\theta}) &= \mathcal{R}(\boldsymbol{\theta},\rho,\boldsymbol{\delta}) - \mathcal{R}(\boldsymbol{\theta},\rho,\boldsymbol{\delta}_0)\\
&\leq E_{\boldsymbol{\theta}}\left[\omega\rho'(0)(1-\omega)^2\|g(\mathbf{X})\|^2\right] + E_{\boldsymbol{\theta}}\Big[(1-\omega)\rho'(\|\mathbf{X}-\boldsymbol{\theta}\|^2)[2(1-\omega)g(\mathbf{X}).(\mathbf{X}-\boldsymbol{\theta})\\
&\quad +(1-\omega)^2\|g(\mathbf{X})\|^2]\Big]\\
&\leq E\left\{\frac{a^2\omega\rho'(0)(1-\omega)^2}{R^2}E_{R,\boldsymbol{\theta}}\left[R^2\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right]\right\}\\
&\quad +E\left\{(1-\omega)^2a\rho'(R^2)\left(a(1-\omega)/R^2 - \frac{2(d-2)}{d}\right)E_{R,\boldsymbol{\theta}}\left[R^2\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right]\right\}\\
&= a(1-\omega)^2E\left\{a\left(\omega\rho'(0)+(1-\omega)\rho'(R^2)\right)\frac{1}{R^2}E_{R,\boldsymbol{\theta}}\left[R^2\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\| = R^2\right]\right\}\\
&\quad +a(1-\omega)^2E\left\{-\rho'(R^2)R^2\frac{2(d-2)}{d}E_{R,\boldsymbol{\theta}}\left[\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right]\right\}. \qquad (6.1)
\end{aligned}
$$

By noticing that the expectation $E_{R,\boldsymbol{\theta}}\left[R^2S(\|\mathbf{X}\|^2)/\|\mathbf{X}\|^2 \mid \|\mathbf{X}-\boldsymbol{\theta}\| = R\right]$ is nondecreasing in $R$ and $\rho'(R^2)/R^2$ is nonincreasing in $R$. Since $\rho$ is concave then $\rho'$ is nonincreasing. By using the covariance inequality, we obtain an upper bound of the first term of the equatility 6.1

$$
E\left[a\left(\omega\rho'(0)+(1-\omega)\rho'(R^2)\right)\frac{1}{R^2}\right]\times E\left\{E_{R,\boldsymbol{\theta}}\left[R^2\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right]\right\}.
$$

For $d \geq 4$, the function $E_{R,\boldsymbol{\theta}}\left[\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right]$ is nonincreasing in $R$ and $R^2$ is increasing in $R$. So, by the covariance inequality we have

$$
\begin{aligned}
E\left\{E_{R,\boldsymbol{\theta}}\left[R^2\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right]\right\} &= E\left[R^2E_{R,\boldsymbol{\theta}}\left[\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right]\right]\\
&\leq E\left[R^2\right]\times E_{\boldsymbol{\theta}}\left[\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}\right].
\end{aligned}
$$

Similarly, employing the covariance inequality as previously mentioned, we derive an upper bound for the second term in the equation 6.1 $E\left[-\rho'(R^2)R^2\frac{2(d-2)}{d}\right]\times E_{\boldsymbol{\theta}}\left[\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}\right]$. Hence, an upper bound can be established for the difference in risk

$$
a(1-\omega)^2\left\{a\left(\omega\rho'(0)E\left[\frac{1}{R^2}\right]+(1-\omega)E\left[\frac{\rho'(R^2)}{R^2}\right]\right)-\frac{2(d-2)}{d}E\left[\rho'(R^2)R^2\right]\right\}\times E_{\boldsymbol{\theta}}\left[\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}\right].
$$

From the above, it follows immediately that 2.3 is a sufficient condition for the difference in risk to be negative for all $\boldsymbol{\theta}$.

**Proof of Theorem 2.3.** Since $\ell(\cdot)$ is a nondecreassing concave function, then by using lemma 7, we obtain an upper bound for the risk difference

$$\begin{aligned}
\delta\mathcal{R}(\boldsymbol{\theta}) &= E_{\boldsymbol{\theta}}\left[\boldsymbol{\delta}_{\omega,\ell,\boldsymbol{\delta}_0}(\boldsymbol{\theta},\boldsymbol{\delta}_{a,S})\right] \\
&\leq E_{\boldsymbol{\theta}}\left[(1-\omega)^2\ell'((1-\omega)\|\boldsymbol{\delta}_0-\boldsymbol{\theta}\|^2)\boldsymbol{\delta}_0(\boldsymbol{\theta},\boldsymbol{\delta})\right] \\
&= (1-\omega)^2\left\{E_{\boldsymbol{\theta}}\left[\ell'((1-\omega)\|\mathbf{X}-\boldsymbol{\theta}\|^2)\|g(\mathbf{X})\|^2\right]+E_{\boldsymbol{\theta}}\left[2\ell'((1-\omega)\|\mathbf{X}-\boldsymbol{\theta}\|^2)g(\mathbf{X})'(\mathbf{X}-\boldsymbol{\theta})\right]\right\} \\
&\leq a(1-\omega)^3E\left\{\ell'((1-\omega)R^2)\left(a(1-\omega)-\frac{2(d-2)}{d}R^2\right)E_{R,\boldsymbol{\theta}}\left[\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\|=R\right]\right\} \\
&= a(1-\omega)^3\left\{E\left[E_{R,\boldsymbol{\theta}}\left(R^2\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\|=R\right)\ell'((1-\omega)R^2)a(1-\omega)R^{-2}\right]+\right. \\
&\qquad \left.E\left[E_{R,\boldsymbol{\theta}}\left(\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\|=R\right)\times\frac{-2(d-2)R^2}{d}\ell'((1-\omega)R^2)\right]\right\},
\end{aligned} \tag{6.2}$$

where the last inequality follows from Lemma 7 and the second follows from (i). In the same way as the proof of the last theorem we use the covariance inequality. Given that $E_{R,\boldsymbol{\theta}}\left[R^2\left\{S(\|\mathbf{X}\|^2)/\|\mathbf{X}\|^2\right\}/\|\mathbf{X}-\boldsymbol{\theta}\|=R\right]$ is increasing in $R$, $a(1-\omega)R^{-2}\ell'\left((1-\omega)R^2\right)$ is nonincreasing in $R$, from the first term in equation (6.2) and covariance inequality, we derive that

$$\begin{aligned}
&E\left[E_{R,\boldsymbol{\theta}}\left(R^2\frac{S(\|\mathbf{X}\|^2)}{\|X\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\|=R\right)a(1-\omega)R^{-2}\ell'((1-\omega)R^2)\right] \\
&\quad\leq E\left[R^2E_{R,\boldsymbol{\theta}}\left(\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\|=R\right)\right]E\left[a(1-\omega)R^{-2}\ell'((1-\omega)R^2)\right].
\end{aligned} \tag{6.3}$$

According to $E_{R,\boldsymbol{\theta}}\left[\left\{S(\|\mathbf{X}\|^2)/\|\mathbf{X}\|^2\right\}/\|\mathbf{X}-\boldsymbol{\theta}\|=R\right]$ is nonincreasing in $R$ and $R^2$ is increasing in $R$ and by using the covariance inequality, we obtain an upper bound of the first expectation in the last inequality

$$E_{\boldsymbol{\theta}}\left[R^2E_{R,\boldsymbol{\theta}}\left(\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}/\|\mathbf{X}-\boldsymbol{\theta}\|=R\right)\right]\leq E_{\boldsymbol{\theta}}\left[\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}\right]\times E\left[R^2\right]. \tag{6.4}$$

Since $\left\{-2((d-2)/d)R^{-2}\ell'((1-\omega)R^2)\right\}$ is increasing in $R$ according to **H2** and by using the covariance inequality, we get an upper bound of the second expectation in (6.2)

$$E_{\boldsymbol{\theta}}\left[\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}\right]\times E\left[\frac{-2(d-2)R^2}{d}\ell'((1-\omega)R^2)\right]. \tag{6.5}$$

Combining (6.3), (6.4) and (6.5) with (6.2), we obtain

$$\begin{aligned}
\delta\mathcal{R}(\boldsymbol{\theta}) &\leq a(1-\omega)^3\left(a(1-\omega)E\left[\ell'((1-\omega)R^2)R^{-2}\right]\times E_{\boldsymbol{\theta}}\left[R^2\right]-2\frac{d-2}{d}E\left[R^2\ell'((1-\omega)R^2)\right]\right) \\
&\qquad\times E_{\boldsymbol{\theta}}\left[\frac{S(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2}\right].
\end{aligned}$$

This completes the proof of the theorem.

**Proof of Lemma 4.** By considering the expectation, the first inequality in (3.2) can be obtained from the concave function inequality $\rho(t) \leq \rho(0) + \rho'(0)t = \rho'(0)t$. This inequality holds since $\rho$ is a concave function with $\rho(0) = 0$, and we substitute $t$ with $-h(t)$. Let $h$ represent the density function of the radius $R = \|\mathbf{X} - \boldsymbol{\theta}\|$. In such a case, we can express it as follows:

$$\begin{aligned}
E_{\boldsymbol{\theta}}\left[-h(\mathbf{X})\right] &= E\left[E_{R,\boldsymbol{\theta}}\left(-h(\mathbf{X}) \;/\; \|\mathbf{X}-\boldsymbol{\theta}\| = R\right)\right] \\
&= \int_{\mathbb{R}_+} E_{r,\boldsymbol{\theta}}\left(-h(\mathbf{X}) \;/\; \|\mathbf{X}-\boldsymbol{\theta}\| = r\right) g(r)dr \\
&= \int_{\mathbb{R}_+} E_{r,\boldsymbol{\theta}}\left(-h(\mathbf{X}) \;/\; \|\mathbf{X}-\boldsymbol{\theta}\| = r\right) \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} r^{d-1} f(r^2)dr \\
&= \int_{\mathbb{R}_+} E_{r,\boldsymbol{\theta}}\left(-h(\mathbf{X}) \;/\; \|\mathbf{X}-\boldsymbol{\theta}\| = r\right) \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} r^{d-1} \frac{\rho'(r^2)f(r^2)}{K}\frac{K}{\rho'(r^2)}dr \\
&= \int_{\mathbb{R}_+} E_{r,\boldsymbol{\theta}}\left(-h(\mathbf{Y}) \;/\; \|\mathbf{Y}-\boldsymbol{\theta}\| = r\right) \frac{K}{\rho'(r^2)} g^*(r^2)dr \\
&= E^\star\left[E_{R,\boldsymbol{\theta}}\left(-h(\mathbf{Y}) \;/\; \|\mathbf{Y}-\boldsymbol{\theta}\| = R\right)\frac{K}{\rho'(R^2)}\right],
\end{aligned}$$

$g^*(r) = \left(2\pi^{d/2}/\Gamma\left(d/2\right)\right) r^{d-1} f^\star(r^2)$ is a density function of $\|\mathbf{Y} - \boldsymbol{\theta}\|$. According to lemma 9, for $d \geq 4$, the function $t \longmapsto -h(t)$ is super-harmonique then $E_{R,\boldsymbol{\theta}}\left(-h(\mathbf{Y}) \mid \|\mathbf{Y} - \boldsymbol{\theta}\| = R\right)$ is nonincreasing in $R$. Since $\rho$ is concave then $\rho'$ is non-increasing and that $K/\rho'(R^2)$ is increasing in $R$, hence with the covariance inequality we obtain

$$\begin{aligned}
& E^\star\left[E_{R,\boldsymbol{\theta}}\left(-h(\mathbf{Y}) \;/\; \|\mathbf{Y}-\boldsymbol{\theta}\| = R\right)\frac{K}{\rho'(R^2)}\right] \\
\leq\;& E_R^\star\left[\frac{K}{\rho'(R^2)}\right] E^\star\left[E_{R,\boldsymbol{\theta}}\left(-h(\mathbf{Y}) \;/\; \|\mathbf{Y}-\boldsymbol{\theta}\| = R\right)\right] = E_{\boldsymbol{\theta}}^\star\left[-h(\mathbf{Y})\right].
\end{aligned}$$

due to the fact that $E_R^\star\left[K/\rho'(R^2)\right] = 1$.

**Proof of Theorem 3.1.** We have:

$$\begin{aligned}
\delta\mathcal{R}_{\omega,\boldsymbol{\delta}_0,\rho}(\boldsymbol{\theta}) &= \mathcal{R}_{\omega,\boldsymbol{\delta}_0,\rho}(\boldsymbol{\delta}_{\omega,g},\boldsymbol{\theta}) - \mathcal{R}_{\omega,\boldsymbol{\delta}_0,\rho}(\boldsymbol{\delta}_0,\boldsymbol{\theta}) \\
&= E_{\boldsymbol{\theta}}\Big[\omega\rho(\|\mathbf{X} + a(1-\omega)g(\mathbf{X}) - \mathbf{X}\|^2) + (1-\omega)\rho(\|\mathbf{X} + a(1-\omega)g(\mathbf{X}) - \boldsymbol{\theta}\|^2) \\
&\quad -(1-\omega)\rho(\|\mathbf{X}-\boldsymbol{\theta}\|^2)\Big] \\
&= E_{\boldsymbol{\theta}}\Big[\omega\rho(\|a(1-\omega)g(\mathbf{X})\|^2) + (1-\omega)\rho(\|X - \boldsymbol{\theta} + a(1-\omega)g(\mathbf{X})\|^2) \\
&\quad -(1-\omega)\rho(\|\mathbf{X}-\boldsymbol{\theta}\|^2)\Big]
\end{aligned}$$

$$\begin{aligned}
&\leq E_{\boldsymbol{\theta}}\left\{\omega\rho'(0)\|a(1-\omega)g(\mathbf{X})\|^2 + (1-\omega)\left[\rho(\|\mathbf{X}-\boldsymbol{\theta}\|^2 + 2a(1-\omega)g(\mathbf{X}).(X-\boldsymbol{\theta})\right.\right.\\
&\qquad \left.\left. + a^2(1-\omega)^2\|g(\mathbf{X})\|^2) - \rho(\|\mathbf{X}-\boldsymbol{\theta}\|^2)\right]\right\}\\
&\leq E_{\boldsymbol{\theta}}\left\{a^2\omega\rho'(0)(1-\omega)^2\|g(\mathbf{X})\|^2 + (1-\omega)\rho'(\|\mathbf{X}-\boldsymbol{\theta}\|^2)\left[2a(1-\omega)\right.\right.\\
&\qquad \left.\left. g(\mathbf{X}).(\mathbf{X}-\boldsymbol{\theta}) + a^2(1-\omega)^2\|g(\mathbf{X})\|^2\right]\right\}\\
&= E_{\boldsymbol{\theta}}\left\{a^2\omega\rho'(0)(1-\omega)^2\|g(\mathbf{X})\|^2\right.\\
&\quad \left. + (1-\omega)K\times\frac{\rho'(\|\mathbf{X}-\boldsymbol{\theta}\|^2)}{K}\left[2a(1-\omega)g(\mathbf{X}).(\mathbf{X}-\boldsymbol{\theta}) + a^2(1-\omega)^2\|g(\mathbf{X})\|^2\right]\right\}\\
&= a(1-\omega)^2\left\{E_{\boldsymbol{\theta}}\left[a\omega\rho'(0)\|g(\mathbf{X})\|^2\right]\right.\\
&\quad \left. + KE^{\star}_{\boldsymbol{\theta}}\left[2.(\mathbf{Y}-\boldsymbol{\theta})g(\mathbf{Y}) + a(1-\omega)\|g(\mathbf{Y})\|^2\right]\right\},
\end{aligned} \tag{6.6}$$

the first inequality follows from $\rho(t) \leq \rho(0)+\rho'(0)t = \rho'(0)t$, $\rho(0) = 0$ and that the second follows from the concave function inequality: $\rho(t_1) - \rho(t_2) \leq \rho'(t_1)(t_1 - t_2)$, $\forall t_1, t_2 \geq 0$. According to the condition (i) and lemma 4 respectively together with (6.6), we have

$$\begin{aligned}
\frac{\delta\mathcal{R}_{\omega,\delta_0,\rho}(\boldsymbol{\theta})}{(1-\omega)^2} &\leq aE_{\boldsymbol{\theta}}\left[-2a\omega\rho'(0)h(\mathbf{X})\right] + KE^{\star}_{\boldsymbol{\theta}}\left[2g(\mathbf{Y}).(\mathbf{Y}-\boldsymbol{\theta})\right] + KE^{\star}_{\boldsymbol{\theta}}\left[-2a(1-\omega)h(\mathbf{Y})\right]\\
&\leq aE^{\star}_{\boldsymbol{\theta}}\left[-2a\omega\rho'(0)h(\mathbf{Y})\right] + KE^{\star}_{\boldsymbol{\theta}}\left[-2a(1-\omega)h(\mathbf{Y})\right]\\
&\quad + \frac{2K}{d}\int_{\mathbb{R}_+} r^2 \int_{B_{r,\boldsymbol{\theta}}} h(\mathbf{x})dV_{r,\boldsymbol{\theta}}(\mathbf{x})d\rho(r).
\end{aligned}$$

As $\mathbf{x} \longrightarrow -h(\mathbf{x})$ is superharmonic function then according to lemma 10 the last equation is bounded by

$$\begin{aligned}
&E^{\star}_{\boldsymbol{\theta}}\left[-2a\omega\rho'(0)h(\mathbf{Y})\right] + KE^{\star}_{\boldsymbol{\theta}}\left[-2a(1-\omega)h(\mathbf{Y})\right] + \frac{2K}{d}\int_{\mathbb{R}_+} r^2 \int_{S_{r,\boldsymbol{\theta}}} h(\mathbf{x})d\mathbf{U}_{r,\boldsymbol{\theta}}(\mathbf{x})d\rho(r)\\
&= E^{\star}\left\{2E^{\star}_{R,\boldsymbol{\theta}}\left[-h(\mathbf{Y})/\|\mathbf{Y}-\boldsymbol{\theta}\| = R\right]\left(a\omega\rho'(0) - \frac{K}{d}R^2 + K(1-\omega)a\right)\right\}\\
&= E^{\star}\left\{2E^{\star}_{R,\boldsymbol{\theta}}\left[-R^2h(\mathbf{Y})/\|\mathbf{Y}-\boldsymbol{\theta}\| = R\right]\left(a\frac{\omega\rho'(0) + K(1-\omega)}{R^2} - \frac{K}{d}\right)\right\}.
\end{aligned}$$

$E_{R,\boldsymbol{\theta}}\left[-R^2h(\mathbf{Y})/\|\mathbf{Y}-\boldsymbol{\theta}\| = R\right]$ is creasing in $R$ and $a\left[\omega\rho'(0) + K(1-\omega)\right]/R^2 - 2K(d-2)/d$ is non-increasing in $R$, an application of the covariance inequality leads to

$$\begin{aligned}
&\leq E^{\star}\left\{2E^{\star}_{R,\boldsymbol{\theta}}\left[-R^2h(\mathbf{Y})/\|\mathbf{Y}-\boldsymbol{\theta}\| = R\right]\right\}\times E^{\star}\left[\left(a\frac{\omega\rho'(0) + K(1-\omega)}{R^2} - \frac{K}{d}\right)\right]\\
&= E^{\star}\left\{2E^{\star}_{R,\boldsymbol{\theta}}\left[-R^2h(\mathbf{Y})/\|\mathbf{Y}-\boldsymbol{\theta}\| = R\right]\right\}\times\left(a(\omega\rho'(0) + K(1-\omega))E^{\star}\left[\frac{1}{R^2}\right] - \frac{K}{d}\right),
\end{aligned}$$

with $E^{\star}\left[\frac{1}{R^2}\right] = E^{\star}_{\boldsymbol{\theta}}\left[\frac{1}{\|\mathbf{Y}-\boldsymbol{\theta}\|^2}\right] = E^{\star}_0\left[\frac{1}{\|\mathbf{Y}\|^2}\right]$. This completes the proof of the theorem.

**Proof of Theorem 3.2.** We have:

$$\begin{aligned}
\delta\mathcal{R}(\boldsymbol{\theta}) &= \mathcal{R}(\boldsymbol{\theta},\rho,\boldsymbol{\delta}_{\omega,g}(\mathbf{X}))-\mathcal{R}(\boldsymbol{\theta},\rho,\boldsymbol{\delta}_0)\\
&\leq E_{\boldsymbol{\theta}}\left[a^2\omega\rho'(0)(1-\omega)^2\|g(\mathbf{X})\|^2\right]+E_{\boldsymbol{\theta}}\left[(1-\omega)\rho'(\|\mathbf{X}-\boldsymbol{\theta}\|^2)[2a(1-\omega)g(\mathbf{X}).(\mathbf{X}-\boldsymbol{\theta})\right.\\
&\quad\left.+a^2(1-\omega)^2\|g(\mathbf{X})\|^2]\right]\\
&\leq E_{\boldsymbol{\theta}}\left[a^2(1-\omega)^2\left(\omega\rho'(0)+(1-\omega)\rho'(\|\mathbf{X}-\boldsymbol{\theta}\|^2)\right)\|g(\mathbf{X})\|^2\right]\\
&\quad+E_{\boldsymbol{\theta}}\left[2a(1-\omega)^2\rho'(\|\mathbf{X}-\boldsymbol{\theta}\|^2)g(\mathbf{X}).(\mathbf{X}-\boldsymbol{\theta})\right]\\
&\leq 2a(1-\omega)^2E_{\boldsymbol{\theta}}\left[-a\left(\omega\rho'(0)+(1-\omega)\rho'(\|\mathbf{X}-\boldsymbol{\theta}\|^2)\right)h(\mathbf{X})\right]\\
&\quad+\frac{1}{d}E\left[R^2\rho'(R^2)\int_{B_{R,\boldsymbol{\theta}}}\nabla g(\mathbf{X})dV_{R,\boldsymbol{\theta}}(\mathbf{X})\right]\\
&\leq 2a(1-\omega)^2E\left[\left(-a\left(\omega\rho'(0)/R^2+(1-\omega)\frac{\rho'(R^2)}{R^2}\right)+\frac{1}{d}\rho'(R^2)\right)\right.\\
&\quad\left.\times E_{R,\boldsymbol{\theta}}\left[R^2h(\mathbf{X})/\|\mathbf{X}-\boldsymbol{\theta}\|=R\right]\right]\\
&\leq 2a(1-\omega)^2\left(a\left(\omega\rho'(0)E\left[1/R^2\right]+(1-\omega)E\left[\frac{\rho'(R^2)}{R^2}\right]\right)-\frac{1}{d}E\left[\rho'(R^2)\right]\right)\\
&\quad E\left[E_{R,\boldsymbol{\theta}}\left[-R^2h(\mathbf{X})/\|\mathbf{X}-\boldsymbol{\theta}\|=R\right]\right].
\end{aligned}$$

**Proof of Theorem 3.3** Let $L=\int_{\mathbb{R}^d}\ell'((1-\omega)\|x-\boldsymbol{\theta}\|^2)f(\|\mathbf{x}-\boldsymbol{\theta}\|^2)d\mathbf{x}$, so

$$\begin{aligned}
\delta\mathcal{R}(\boldsymbol{\theta}) &= E_{\boldsymbol{\theta}}\left[\boldsymbol{\delta}_{\boldsymbol{\theta},\boldsymbol{\delta}_0,\ell}(\boldsymbol{\theta},\boldsymbol{\delta}_{\omega,g})\right]\\
&\leq E_{\boldsymbol{\theta}}\left[(1-\omega)^2\ell'((1-\omega)\|\boldsymbol{\delta}_0-\boldsymbol{\theta}\|^2)\boldsymbol{\delta}_0(\boldsymbol{\theta},\boldsymbol{\delta}_{\omega,g})\right]\\
&= (1-\omega)^2E_{\boldsymbol{\theta}}\left[\ell'((1-\omega)\|\mathbf{X}-\boldsymbol{\theta}\|^2)(\|\boldsymbol{\delta}_{\omega,g}(\mathbf{X})-\boldsymbol{\theta}\|^2-\|\mathbf{X}-\boldsymbol{\theta}\|^2)\right]\\
&= (1-\omega)^2E_{\boldsymbol{\theta}}\left[\ell'((1-\omega)\|\mathbf{X}-\boldsymbol{\theta}\|^2)\left(\|\mathbf{X}+a(1-\omega)g(\mathbf{X})-\boldsymbol{\theta}\|^2-\|\mathbf{X}-\boldsymbol{\theta}\|^2\right)\right]\\
&= (1-\omega)^2\times LE^{\star}_{\boldsymbol{\theta}}\left[a^2(1-\omega)^2\|g(\mathbf{Z})\|^2+2a(1-\omega)g(\mathbf{Z}).(\mathbf{Z}-\boldsymbol{\theta})\right]\\
&= (1-\omega)^3\times L\times a\left\{E^{\star}_{\boldsymbol{\theta}}\left[a(1-\omega)\|g(\mathbf{Z})\|^2\right]+2E^{\star}_{\boldsymbol{\theta}}\left[g(\mathbf{Z}).(\mathbf{Z}-\boldsymbol{\theta})\right]\right\}\\
&= (1-\omega)^3\times L\times a\left\{E^{\star}_{\boldsymbol{\theta}}\left[-2a(1-\omega)h(\mathbf{Z})\right]+a\frac{1}{d}\int_{\mathbb{R}^+}r^2\int_{B_{r,\boldsymbol{\theta}}}div(g(\mathbf{z}))dV_{r,\boldsymbol{\theta}}(\mathbf{z})d\rho(r)\right\}\\
&\leq aL(1-\omega)^3\left\{E^{\star}_{\boldsymbol{\theta}}\left[-2a(1-\omega)h(\mathbf{Z})\right]+\frac{2}{d}\int_{\mathbb{R}^+}r^2\int_{S_{r,\boldsymbol{\theta}}}h(\mathbf{z})d\mathbf{U}_{r,\boldsymbol{\theta}}(\mathbf{z})d\rho(r)\right\}\\
&= aL(1-\omega)^3E^{\star}_{R}\left[E_{R,\boldsymbol{\theta}}\left[\left(-2a(1-\omega)+\frac{2}{d}R^2\right)h(\mathbf{Z})/\|\mathbf{Z}-\boldsymbol{\theta}\|=R\right]\right]\\
&= 2aL(1-\omega)^3E^{\star}\left[\left(a(1-\omega)\frac{1}{R^2}-\frac{1}{d}\right)E_{R,\boldsymbol{\theta}}\left[-R^2h(\mathbf{Z})/\|\mathbf{Z}-\boldsymbol{\theta}\|=R\right]\right], \qquad (6.7)
\end{aligned}$$

where the first inequality follows from lemma 3, the second follows frome (i) and the third follows from lemma 10. By notting that $E_{R,\boldsymbol{\theta}}\left[-R^2h(\mathbf{Z})/\|\mathbf{Z}-\boldsymbol{\theta}\|=R\right]$ is increasing in $R > 0$, $\left\{(1-\omega)\frac{1}{R^2} + \frac{1}{d}\right\}$ is non-increasing in $R$, we infer from (6.7) and the covariance inequality that

$$\delta\mathcal{R}(\boldsymbol{\theta}) \quad \leq \quad aL(1-\omega)^3\left(a(1-\omega)E^\star\left[\frac{1}{\|\mathbf{Z}\|^2}\right] - \frac{1}{d}\right)E^\star\left\{-R^2E_{R,\boldsymbol{\theta}}\left[h(\mathbf{Z})/\|\mathbf{Z}-\boldsymbol{\theta}\| = R\right]\right\},$$

this completes the proof of theorem.

**Proof of Lemma 5.**

$$\begin{aligned} E_{\boldsymbol{\theta}}\left[\|\mathbf{X} + ag(\mathbf{X}) - \boldsymbol{\theta}\|^2 - \|\mathbf{X}-\boldsymbol{\theta}\|^2\right] &= E_{\boldsymbol{\theta}}\left[-2a^2\|g(\mathbf{X})\|^2/2 + 2a(\mathbf{X}-\boldsymbol{\theta})^\top g(\mathbf{X})\right] \\ &\leq E_{\boldsymbol{\theta}}\left[-2a^2h(\mathbf{x})\right] \\ &\quad +2a\left[\frac{1}{d}\int_{\mathbb{R}^+} r^2 \int_{B_{r,\boldsymbol{\theta}}} \nabla g(\mathbf{X})dV_{r,\boldsymbol{\theta}}(\mathbf{x})d\rho(r)\right] \\ &\leq E_{\boldsymbol{\theta}}\left[-2a^2h(\mathbf{x})\right] \\ &\quad +2a\left[\frac{1}{d}\int_{\mathbb{R}^+} r^2 \int_{B_{r,\boldsymbol{\theta}}} h(\mathbf{x})dV_{r,\boldsymbol{\theta}}(\mathbf{x})d\rho(r)\right]. \end{aligned}$$

By subharmonicity of $h$, $\int_{B_{r,\boldsymbol{\theta}}} h(\mathbf{x})dV_{r,\boldsymbol{\theta}}(\mathbf{x})d\rho(r) \leq \int_{S_{r,\boldsymbol{\theta}}} h(\mathbf{x})d\mathbf{U}_{r,\boldsymbol{\theta}}(\mathbf{z})d\rho(r)$. Hence,

$$E_{\boldsymbol{\theta}}\left[\|\mathbf{X} + ag(\mathbf{X}) - \boldsymbol{\theta}\|^2 - \|\mathbf{X}-\boldsymbol{\theta}\|^2\right] \quad \leq \quad E_{\boldsymbol{\theta}}\left[\left(-2a^2/r^2 + 2a\frac{1}{d}\right)E_{r,\boldsymbol{\theta}}\left[r^2h(\mathbf{X})/\|\mathbf{X}-\boldsymbol{\theta}\|^2 = r^2\right]\right].$$

**Proof of Theorem 3.4.** Since $\ell(\cdot)$ is a increassing concave function, then by using lemma 3, we obtain an upper bound for the risk difference

$$\begin{aligned} &\delta\mathcal{R}(\boldsymbol{\theta}) = E_{\boldsymbol{\theta}}\left[\boldsymbol{\delta}_{\omega,\ell,\boldsymbol{\delta}_0}(\boldsymbol{\theta},\boldsymbol{\delta}_{\omega,g})\right] \\ \leq\ & E_{\boldsymbol{\theta}}\left[(1-\omega)^2\ell'((1-\omega)\|\boldsymbol{\delta}_0-\boldsymbol{\theta}\|^2)\delta_0(\boldsymbol{\theta},\boldsymbol{\delta})\right] \\ =\ & (1-\omega)^2\left\{E_{\boldsymbol{\theta}}\left[\ell'((1-\omega)\|\mathbf{X}-\boldsymbol{\theta}\|^2)\|g(\mathbf{X})\|^2\right] + E_{\boldsymbol{\theta}}\left[2\ell'((1-\omega)\|\mathbf{X}-\boldsymbol{\theta}\|^2)g(\mathbf{X})'(\mathbf{X}-\boldsymbol{\theta})\right]\right\} \\ \leq\ & 2a(1-\omega)^3E\left\{\ell'((1-\omega)R^2)\left(a(1-\omega) - \frac{1}{d}R^2\right)E_{R,\boldsymbol{\theta}}\left[-h(\mathbf{x})/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right]\right\} \\ =\ & 2a(1-\omega)^3\left\{E\left[E_{R,\boldsymbol{\theta}}\left(-R^2h(\mathbf{x})/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right)\ell'((1-\omega)R^2)a(1-\omega)R^{-2}\right]\right. \\ & \left.\quad +E\left[E_{R,\boldsymbol{\theta}}\left(-h(\mathbf{X})/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right)\times\frac{-R^2}{d}\ell'((1-\omega)R^2)\right]\right\}, \end{aligned} \tag{6.8}$$

where the last inequality follows from Lemma 3 and the second follows frome (i). In the same way as the proof of the last theorem we use the covariance inequality. By noting that $E_{R,\boldsymbol{\theta}}\left[-R^2h(\mathbf{x})/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right]$ is increasing in $R$ according to (iii) and $a(1-\omega)R^{-2}\ell'\left((1-\omega)R^2\right)$ is non-increasing in $R$ according to **H2**, we infer from the first therm in Equation (6.8) and covariance inequality that

$$\begin{aligned} & E\left[E_{R,\boldsymbol{\theta}}\left(-R^2h(\mathbf{X})/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right)a(1-\omega)R^{-2}\ell'((1-\omega)R^2)\right] \\ & \quad\leq\ E\left[R^2E_{R,\boldsymbol{\theta}}\left(-h(\mathbf{X})/\|\mathbf{X}-\boldsymbol{\theta}\| = R\right)\right]E\left[a(1-\omega)R^{-2}\ell'((1-\omega)R^2)\right]. \end{aligned} \tag{6.9}$$

Since $E_{R,\boldsymbol{\theta}}\left[-h(\mathbf{X})/\|\mathbf{X}-\boldsymbol{\theta}\|=R\right]$ is non-increasing in $R$ and $R^2$ is increasing in $R$ then, using the covariance inequality we get an upper bound of the first expectation in the last inequality

$$E_{\boldsymbol{\theta}}\left[R^2 E_{R,\boldsymbol{\theta}}\left(-h(\mathbf{x})/\|\mathbf{X}-\boldsymbol{\theta}\|=R\right)\right] \leq E_{\boldsymbol{\theta}}\left[-h(\mathbf{X})\right] \times E\left[R^2\right]. \tag{6.10}$$

As $\left\{-1/d)R^{-2}\ell'((1-\omega)R^2)\right\}$ is increasing in $R$ according to **H2**, then by using the covariance inequality we get an upper bound of the second expectation in (6.8)

$$E_{\boldsymbol{\theta}}\left[-h(\mathbf{x})\right] \times E\left[\frac{-R^2}{d}\ell'((1-\omega)R^2)\right]. \tag{6.11}$$

By combining (6.9), (6.10) and (6.11) with (6.8) we get

$$\begin{aligned} \boldsymbol{\delta}\mathcal{R}(\boldsymbol{\theta}) \quad \leq \quad & 2a(1-\omega)^3\left(a(1-\omega)E\left[\ell'((1-\omega)R^2)R^{-2}\right] \times E_{\boldsymbol{\theta}}\left[R^2\right] - \frac{1}{d}E\left[R^2\ell'((1-\omega)R^2)\right]\right) \\ & \times E_{\boldsymbol{\theta}}\left[-h(\mathbf{x})\right]. \end{aligned}$$

This completes the proof of the theorem.

## Appendix: Some Technical Lemmas

This Appendix gives some technical lemmas used in our paper.

**Lemma 7.** *If $S$ is a non-negative, differentiable and concave real valued function, then $S$ is increasing on $\mathbb{R}_+$ and the function $t \rightarrow S(t)/t$ is non-increasing on $\mathbb{R}_+^*$. Furthermore, if in addition $S$ is twice differentiable, then the function $\mathbf{x} \rightarrow S(\|\mathbf{x}\|^2)/\|\mathbf{x}\|^2$ is super-harmonic for $d \geq 4$.*

**Proof of Lemma 7.** $S$ is differentiable, its concavity is equivalent to $S(t) - S(t_0) \leq S'(t_0)(t-t_0)$ for any $t_0$ and any $t$, the first part of the lemma follows. Hence, if $S'(t_0) < 0$ for some $t_0 \geq 0$, then $\lim_{t\to\infty} S(t) = -\infty$ which contradicts the assumption that $S(t) \geq 0$ for any $t$. Therefore $S'(t) \geq 0$ for all $t \geq 0$. Similarly, for any $t > 0$, we have $S(0) - S(t) \leq -tS'(t)$. Hence $t^2\left(S(t)/t\right)' = tS'(t) - S(t) \leq -S(0) \leq 0$ so that $S(t)/t$ is non-increasing.
Finally by straightforward calculation the Laplacian of $S(\|\mathbf{x}\|^2)/\|\mathbf{x}\|^2$ is given by

$$\boldsymbol{\delta}\left[S(\|\mathbf{x}\|^2)/\|\mathbf{x}\|^2\right] \quad = \quad \frac{2}{\|\mathbf{x}\|^4}\left[2\|\mathbf{x}\|^4 S''(\|\mathbf{x}\|^2) + (p-4)\left(\|\mathbf{x}\|^2 S'(\|\mathbf{x}\|^2) - S(\|\mathbf{x}\|)^2\right)\right] \leq 0$$

since $S''(.) \leq 0$, $tS'(t) - S(t) \leq 0$ and $d \geq 4$, which is the desired result.

**Lemma 8.** *If $g$ is a super-harmonic function on $\mathbb{R}^p$ and $\mathbf{Z}$ is a random variable with a uniform distribution on the sphere centered at the origin and of radius $\tau$, then, for any $\boldsymbol{\theta} \in \mathbb{R}^p$, the expectation $E_{\boldsymbol{\theta}}\left[g(\boldsymbol{\theta}+\mathbf{Z})\right]$ is a non-increasing function of $\tau$.*

**Proof of Lemma 8.** See Theorem 2.30 pp. 54 in Du Plessis (1970) .

**Lemma 9.** *Let* $\mathbf{Y}$ *be a random variable,* $g(\mathbf{y})$ *and* $h(\mathbf{y})$ *any functions for which* $E[g(\mathbf{Y})]$, $E[h(\mathbf{Y})]$, *and* $E[g(\mathbf{Y})h(\mathbf{Y})]$ *exist. Then*
$(a)$ *If one of the functions* $g(.)$ *and* $h(.)$ *is non-increasing and the other is increasing, then*

$$E[g(\mathbf{Y})h(\mathbf{Y})] \leq E[g(\mathbf{Y})]E[h(\mathbf{Y})].$$

$(b)$ *If both functions are either increasing or non-increasing, then*

$$E[g(\mathbf{Y})h(\mathbf{Y})] \geq E[g(\mathbf{Y})]E[h(\mathbf{Y})].$$

**Proof of Lemma 9.** See Lehmann and Casella (1998).

**Lemma 10.** *If* $g(\mathbf{X})$ *is a superharmonic function* ($\nabla^2 g(\mathbf{X}) = \sum(\partial/\partial X_i^2)g(\mathbf{X}) < 0$) *and* $\mathbf{X}$ *has a uniform distribution on the sphere centered at* $\boldsymbol{\theta}$ *with radius* $R$*, then*

$$E_{\boldsymbol{\theta}}\left[g(\mathbf{X})\right] < E_{\boldsymbol{\theta}}\left[g(\mathbf{Z})\right],$$

*where* $Z \sim U(\|Z - \boldsymbol{\theta}\|^2 \leq R^2)$*. So*

$$\frac{1}{A(S)}\int_S g(\mathbf{X})dA(\mathbf{X}) \leq \frac{1}{M(B)}\int_B g(Z)dM(Y),$$

*where* $A(S)$ *and* $M(B)$ *represent the areas of the sphere* $S$ *and ball* $B$ *respectively.*

**Proof of Lemma 10.** Du Plessis (1970), page 54.

**Lemma 11.** *Giving* $\mathbf{X}$ *with a spherically symmetric distribution around* $\boldsymbol{\theta}$ *and* $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ *a increasing function. Then,* $E\left[R^2\frac{\beta(\|\mathbf{X}\|^2)}{\|\mathbf{X}\|^2} \mid \|\mathbf{X} - \boldsymbol{\theta}\| = R\right]$ *is a increasing function of* $R > 0$*.*

**Proof of Lemma 11.** See Brandwein and Strawderman (1980), pages 394-395 within the proof of their Theorem 3.3.1.

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## References

Baranchik, A. J. (1970). A Family of Minimax Estimators of the Mean of a Multivariate Normal Distribution. *Annals of Mathematical Statistics*, **41**(2), 642-645.

Brandwein, A. and Strawderman, W. (1978). Minimax Estimation of Location Parameters for Spherically Symmetric Unimodal Distributions Under Quadratic Loss. *The Annals of Statistics*, **6**(2), 377-416.

Brandwein, A. and Strawderman, W. (1979). Minimax estimation of mean of spherically symmetric distributions under general quadratic loss.*The Annals of Statistics*, **9**(2), 579-588.

Brandwein, A. and Strawderman, W. (1980). Minimax estimation of location parameters for spherically symmetric distributions with concave loss. *Annals of Statistics*, **8**(2), 279-284.

Brandwein and Ralescu and Strawderman (1993). Shrinkage estimators of the location parameter for certain spherically symmetric distributions. *Annals of the Institute of Statistical Mathematics*, **45**(3), 551-565.

Brandwein and Ralescu and Strawderman (1991). Generalizations of James-Stein Estimators Under Spherical Symmetry. *Annals of Statistics*, **19**(3), 639.

Cellier, D., and Fourdrinier, D. (1995). Shrinkage estimators under spherical symmetry for the general linear model. *Journal of Multivariate Analysis*, **52**(2), 338-351.

Cellier, D. and Fourdrinier, D., and Robert, C. (1989). Robust shrinkage estimators of the location parameter for elliptically symmetric distributions. *Journal of Multivariate Analysis*, **138**(12), 197-206.

Du Plessis, N. (1970). *An introduction to potential theory*, Wiley-Interscience, New York.

Fourdrinier, D. and Strawderman, W. E. (1996). A paradox concerning shrinkage estimators : should a known scale parameter be replaced by an estimated value in the shrinkage factor. *Journal of Multivariate Analysis*, **59**(2), 109-140.

Fourdrinier, D. and Ouassou, I. (2000). Estimation of the mean of a spherically symmetric distribution with constraints on the norm. *The Canadian Journal of Statistics*, **28**(2), 399-415.

Fourdrinier, D. and Ouassou, I. and Strawderman,W. E. (2003). Estimation of a parameter vector when some components are restricted. *Journal of Multivariate Analysis*, **86,**(1), 14-27.

Fourdrinier, D. and Ouassou, I. and Strawderman, W. E. (2008). Estimation of a mean vector under quartic Loss, *Journal of Statistical Planing and Inference*, **12**, 3841-3841.

Fourdrinier, D. and Strawderman, W. E. and Wells, M. T. (2018), *Shrinkage estimation*, JSpringer series in statistics.

Hobbad, L and Marchand, É. and Ouassou, I. (2021). On shrinkage estimation of a spherically symmetric distribution for balanced loss functions. *Journal of Multivariate Analysis*, **186**.

Jafari Jozani and M., Marchand, É. and Parsian, A. (2006). On estimation with weighted balanced-type loss function. *Statistics & Probability Letters*, **76**(8), 773-780.

James, W. and Stein, C. (1961). Estimation with quadratic loss. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, **1**, 361-379.

Kubokawa and Marchand and Strawderman (2017). A On predictive density estimation for location families under integrated absolute error loss. *Bernoulli*, **23**(3), 3197-3212.

Lehmann and Casella (1998), *Theory of point estimation*, second. ed., Springer, New York.

Marchand and Strawderman (2020). On shrinkage estimation for balanced loss functions. *Journal of Multivariate Analysis*, **175**.

Marchand and Strawderman (2020). On shrinkage estimation for balanced loss functions. *Journal of Multivariate Analysis*, **175**.

Stein, C. (1956). Inadmissibility of the usual estimator for the mean vector of a multivariate normal distribution. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, **1**, 197-206.

Zellner, A. (1994). Bayesian and non Bayesian estimation using balanced loss functions,*Bernoulli*, **23**(3), 371-390.