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A Note on a New Bivariate Copula Defined with a Piecewise Generator Function

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Abstract. Copulas are essential probability tools for characterizing the joint distribution of random variables. In this article, we contribute to the topic by studying special bivariate copulas. They have the property of being defined with piecewise components and are designed for the analysis of data that have dependence structures with distinct substructures in square zones. The theoretical properties of the copulas are studied, with emphasis on their mathematical validity and some dependence measures. In particular, it is shown that the Kendall tau coefficient has a simple expression that is governed by several parameters, demonstrating the flexibility of the approach. In addition, a real data example is provided to demonstrate the applicability of the copulas. Fair comparisons with other standard copulas motivate their use in other practical scenarios.

Keywords. Copulas, Piecewise Dependence Models, Dependence Measures, Data Analysis.

MSC: 62G07, 62H12.

1 Introduction

Copulas can be defined as multivariate functions that describe the dependence structure between random variables independently of their marginal distributions.They provide a powerful tool for modeling complex types of dependence in multivariate (probability) distributions. In a sense, they quantify how the joint probability of two or more

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variables is distributed relative to their marginal probabilities. To clarify the ideas, let us formalize the situation using mathematics in the bivariate (or two-dimensional) case. Consider two continuous univariate random variables, X and Y, with the corresponding cumulative distribution functions $F(x) = P(X \leq x)$ and $G(y) = P(Y \leq y)$. Then C is a special bivariate function, known as a copula, which characterizes the dependence structure between X and Y. A famous result in Sklar (1959) ensures that C is one of the main components of the cumulative distribution function of (X, Y), say $F(x, y)$, and we have

$$H(x, y) = P(X \leq x, Y \leq y) = C(F(x), G(y)).$$

Or, equivalently, $H(x, y) = C(u, v)$, where $u = F(x) \in [0, 1]$ and $v = G(y) \in [0, 1]$. A copula C must satisfy the following properties:

P1 : C has uniform margins such that $C(u, 1) = u$ and $C(1, v) = v$ for any $u, v \in [0, 1]$,

P2 : C is grounded such that $C(u, 0) = C(0, v) = 0$ for any $u, v \in [0, 1]$,

P3 : C is a 2-increasing function meaning that, for any $u_1 \in [0, 1]$, $u_2 \in [0, 1]$, $v_1 \in [0, 1]$ and $v_2 \in [0, 1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$, we have

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

The most classical example is the independence copula defined by $C(u, v) = uv = \Pi(u, v)$. Thus, C is a function that determines the stochastic dependence between X and Y. It may depend on one or several tuning parameters. In order to evaluate the dependence between X and Y, several copula measures can be considered. Among them, we may focus on the Kendall tau indicated as

$$\tau = 4 \int\int_{[0,1]^2} C(u, v) dC(u, v) - 1.$$

There are different types of copulas, each with different characteristics suitable for different scenarios. Archimedean copulas, characterized by their generating function, provide flexibility in capturing positive and negative dependences. Elliptic copulas, such as Gaussian and Student (T), are suitable for modeling symmetric dependence and are often used in multivariate distributions. Perturbed independence copulas innovate by considering a sum scheme that aims to modify the independence copula by adding different functions, thus offering flexible dependence models. In addition, vine copulas provide a versatile framework for constructing complex dependence structures by decomposing joint distributions into simpler bivariate components.These different types of copulas allow statisticians and researchers to accurately model complex relationships between variables in a wide range of applications. An overview of these copula families can be found in Durante and Sempi (2016), Joe (1997) and Nelsen (2006).

For the purposes of this article, we will first highlight a special family of copulas that involve piecewise functions, i.e., functions defined on a sequence of intervals such as minimum, maximum, and change point functions. Copulas of this type offer advantages in dealing with dependence structures with substructures that can be distinguished according to certain square zones. The first example is the complete dependence copula defined by $C(u,v) = \min(u,v)$ (see Nelsen (2006)). There are also the Cuadras and Augé copula given as $C(u,v) = [\min(u,v)]^{\theta}(uv)^{1-\theta}$ with $\theta \in [0,1)$ (see Cuadras and Augé (1981)), and the Fréchet copula specified by $C(u,v) = a\min(u,v)+(1-a-b)uv+b\max(u+v-1,0)$, for $a \in [0,1]$, $b \in [0,1]$ such that $a+b \leq 1$ (see Fréchet (1958)). There are general approaches developed in Durante et al. (2007), which consider copulas of the following forms: $C(u,v) = \varphi^{-1}\{\varphi[\min(u,v)] + \psi[\max(u,v)]\}$ and $C(u,v) = \varphi^{-1}\{\varphi[\min(u,v)]\psi[\max(u,v)]\}$, where φ and ψ denote functions with rigorous assumptions (see Durante et al. (2007) for the technical details). We may mention the Marshall-Olkin copula family proposed in Marshall and Olkin (1967); a copula of this family can be expressed as

$$C(u,v) = \begin{cases} u^{1-\alpha}v & \text{if } u^{\alpha} \geq v^{\beta}, \\ uv^{1-\beta} & \text{if } u^{\alpha} < v^{\beta}, \end{cases}$$

where $\alpha \in [0,1]$ and $\beta \in [0,1]$. There are also the copulas associated with order statistics established in Schmitz (2004). We mention the following one:

$$C(u,v) = \begin{cases} v - [(1-u)^{1/m} + v^{1/m} - 1]^m & \text{if } 1-(1-u)^{1/m} < v^{1/m}, \\ v & \text{if } 1-(1-u)^{1/m} \geq v^{1/m}, \end{cases}$$

where $m \geq 1$ is an integer. Still with a piecewise component, there is the Raftery copula developed in Raftery (1984), and indicated as

$$C(u,v) = \begin{cases} u - \frac{1-\alpha}{1+\alpha}u^{1/(1-\alpha)}\left(v^{-\alpha/(1-\alpha)} - v^{1/(1-\alpha)}\right) & \text{if } u \leq v, \\ v - \frac{1-\alpha}{1+\alpha}v^{1/(1-\alpha)}\left(u^{-\alpha/(1-\alpha)} - u^{1/(1-\alpha)}\right) & \text{if } u > v. \end{cases}$$

As a final illustrative example, there is the Shih and Louis copula proposed in Shih and Louis (1995), and specified as

$$C(u,v) = \begin{cases} (1-\nu)uv + \nu\min(u,v) & \text{if } \nu > 0, \\ (1+\nu)uv + \nu(u-1+v)I(u+v-1) & \text{if } \nu \leq 0, \end{cases}$$

where $I(a) = 1$ if $a \geq 0$ and $I(a) = 0$ if $a < 0$. All of the above copulas have been used as dependence models in many applications, exploiting their piecewise nature to analyze appropriate real data.

In parallel with this family, the family of perturbed independence copulas has seen significant developments. Obviously, the independence copula cannot model complex dependence structures observed in real-world data. To address this limitation, researchers have developed various methods to perturb the independence copula, introducing additional parameters or functions that allow for more flexible modeling of dependence. Perturbing the independence copula involves modifying it by adding a perturbation function $H(u,v)$, resulting in a new copula of the form: $C_H(u,v) = \Pi(u,v)+$

$H(u,v)$. The function $H(u,v)$ must be carefully chosen to ensure that $C_H(u,v)$ remains a valid copula, i.e., it satisfies the necessary conditions of being 2-increasing, grounded, and having uniform margins. In this topic, we focus on the general methodology described in Rodriguez Lallena (1992), which has the characteristic of relying on a univariate generator function, denoted as ϕ. Such a function is defined on a unit interval, and the associated copula can be expressed as follows:

$$C_\phi(u,v) = \Pi(u,v) + \theta\phi(u)\phi(v), \tag{1.1}$$

with $\theta \in [-1,1]$ (and we recall that $\Pi(u,v) = uv$). The conditions on ϕ ensuring that C_θ is a valid copula are explicit in Ambrard and Girard (2002). They are given in Theorem 1. It is important to note that more general perturbation methods have been developed to extend the model defined in Equation (1.1). For instance, Mesiar et al. (2015) proposed a general framework for perturbing the independence copula by considering perturbation functions of the following product form:

$$H(u,v) = \lambda f(u)g(v),$$

where f and g are functions defined on the unit interval, and λ is a parameter that controls the strength of the perturbation. The functions f and g must satisfy certain conditions (e.g., $f(0) = g(0) = f(1) = g(1) = 0$ and Lipschitz continuity) to ensure that the resulting function $C_H(u,v)$ is a valid copula. This approach allows for an asymmetric version of the family in modeling dependence structures.

Theorem 1. *If ϕ meets the following conditions, it can be considered a valid generator function of the parametric family of copulas as defined by Equation* (1.1)*:*

1. $\phi(0) = 0$ *and* $\phi(1) = 0$.
2. ϕ *is a 1-Lipschitz function, i.e., for any* $t_1 \in [0,1]$ *and* $t_2 \in [0,1]$, *we have* $|\phi(t_1) - \phi(t_2)| \leq L|t_1 - t_2|$ *with* $L = 1$.

More precisely, the copula properties P1 and P2 hold if and only if $\phi(0) = \phi(1) = 0$. For the property P3 (the 2-increasing condition), the volume of the copula function simplifies to:

$$\Delta(u_1,u_2,v_1,v_2) = (u_2 - u_1)(v_2 - v_1) + \theta(\phi(u_2) - \phi(u_1))(\phi(v_2) - \phi(v_1)).$$

To ensure $\Delta \geq 0$, the Lipschitz condition on ϕ is required, i.e.,

$$|\phi(x) - \phi(y)| \leq |x - y|.$$

For more details, see Ambrard and Girard (2002).

Thus, in the setting of Theorem 1, the function ϕ plays a determinant role in generating a semiparametric family of copulas. Each copula C_ϕ is described by the function ϕ. Moreover, ϕ modulates the Kendall tau; the following simple formula holds:

$$\tau = \tau_{\phi} = 8\theta\left(\int_{[0,1]} \phi(t)dt\right)^2. \tag{1.2}$$

We refer to Ambrard and Girard (2002). Thus, researchers have made attempts to create new semiparametric families of copulas through the concept of generator function. Recent developments on the perturbed independence type copula family and some extensions can be found in Shubina and Lee (2004), Durante et al. (2013) and Saminger-Platz et al. (2021).

Some examples of perturbed copulas of the independence type defined with piecewise functions have recently been considered in Saminger-Platz et al. (2021). In particular, in Saminger-Platz et al. (2021) a sophisticated piecewise perturbed function is given as $\min(\alpha u, 1-u)\min(v,(1-v)/\alpha)$ with $\alpha > 0$. In this article, we follow the same spirit, but consider a simpler and more tunable approach, closer to the form in Equation (1.1), with the following min function as a candidate generator function:

$$\phi_{\alpha,\beta}(t) = \min\left[\beta\frac{1-\alpha}{\alpha}t, \beta(1-t)\right],\ t \in [0,1], \tag{1.3}$$

with $\alpha \geq 0$ and $\beta \geq 0$ (to be refined later), that is

$$\phi_{\alpha,\beta}(t) = \begin{cases} \beta\frac{1-\alpha}{\alpha}t & t \in [0,\alpha], \\ \beta(1-t) & t \in [\alpha,1]. \end{cases}$$

We can notice that $\phi_{\alpha,\beta}(t)$ is continuous in $t=\alpha$, since $\lim_{t\to\alpha^-}\phi_{\alpha,\beta}(t) = \lim_{t\to\alpha^+}\phi_{\alpha,\beta}(t) = \beta(1-\alpha)$. Also, the corresponding potential copula based on Equation (1.1) is defined as

$$C(u,v) = \Pi(u,v) + \theta\phi_{\alpha,\beta}(u)\phi_{\alpha,\beta}(v),$$

that is

$$C(u,v) = \begin{cases} uv + \theta\left(\beta\frac{1-\alpha}{\alpha}\right)^2 uv, & u \in [0,\alpha],\ v \in [0,\alpha], \\ uv + \theta\beta^2\frac{1-\alpha}{\alpha}u(1-v), & u \in [0,\alpha],\ v \in [\alpha,1], \\ uv + \theta\beta^2\frac{1-\alpha}{\alpha}(1-u)v, & u \in [\alpha,1],\ v \in [0,\alpha], \\ uv + \theta\beta^2(1-u)(1-v), & u \in [\alpha,1],\ v \in [\alpha,1], \end{cases}$$

where the exact range of values for α and β must be determined. This scheme allows us to provide a dependence model with theoretical guarantees, and to handle dependence structures with distinguishable substructures according to certain square zones. This article will discuss these details.

The following sections are briefly described: Section 2 presents the main theoretical properties of the proposed copula. Some related dependence measures are discussed in Section 3. A real data example is given in Section 4. A conclusion is provided in Section 5.

2 Theoretical Properties

The next result is about the value ranges for α and β, which makes the function in Equation (1.3) a valid generator function in the copula context of Equation (1.1).

Proposition 1. *The function $\phi_{\alpha,\beta}(t)$, $t \in [0,1]$, given in Equation* (1.3) *is valid generator function if the following conditions hold:*

1. $\beta \in [0,1]$ *and* $\alpha \in (0,1]$,
2. $\beta \frac{1-\alpha}{\alpha} \leq 1$.

Proof. It is immediate that $\phi_{\alpha,\beta}(0) = \phi_{\alpha,\beta}(1) = 0$. Now, let us examine the 1-Lipschitz condition, i.e., if, for any $t_1, t_2 \in [0,1]$, we have

$$|\phi_{\alpha,\beta}(t_1) - \phi_{\alpha,\beta}(t_2)| \leq |t_1 - t_2|.$$

by distinguishing between $t_1 \in [0,\alpha]$ and $t_1 \in [\alpha,1]$, and $t_2 \in [0,\alpha]$ and $t_2 \in [\alpha,1]$ in parallel.

- For $t_1 \in [0,\alpha]$ and $t_2 \in [0,\alpha]$, we have

$$\phi_{\alpha,\beta}(t_1) - \phi_{\alpha,\beta}(t_2) = \beta\frac{1-\alpha}{\alpha}(t_1 - t_2).$$

  Thus, under the assumptions $\beta \in [0,1]$, $\alpha \in (0,1]$ and $\beta(1-\alpha)/\alpha \leq 1$, it is immediate that $|\phi_{\alpha,\beta}(t_1) - \phi_{\alpha,\beta}(t_2)| \leq [\beta(1-\alpha)/\alpha]|t_1 - t_2| \leq |t_1 - t_2|$.

- For $t_1 \in [0,\alpha]$ and $t_2 \in [\alpha,1]$, so that $t_2 \geq t_1$, we have

$$\begin{aligned}\phi_{\alpha,\beta}(t_1) - \phi_{\alpha,\beta}(t_2) &= \beta\frac{1-\alpha}{\alpha}t_1 - \beta(1-t_2)\\ &= \beta\left[\left(\frac{1}{\alpha} - 1\right)t_1 - 1 + t_2\right]\\ &= \beta\left[\left(\frac{t_1}{\alpha} - 1\right) + (t_2 - t_1)\right].\end{aligned}$$

  Since $t_1 \leq \alpha$, it is clear that $t_1/\alpha - 1 \leq 0$, so

$$\phi_{\alpha,\beta}(t_1) - \phi_{\alpha,\beta}(t_2) \leq \beta(t_2 - t_1) = \beta|t_1 - t_2|. \tag{2.1}$$

On the other hand, we can also write

$$\begin{aligned}\phi_{\alpha,\beta}(t_1) - \phi_{\alpha,\beta}(t_2) &= \beta\frac{1-\alpha}{\alpha}t_1 - \beta(1-t_2)\\ &= \beta\left[\frac{1-\alpha}{\alpha}(t_1-t_2) + \frac{1-\alpha}{\alpha}t_2 - (1-t_2)\right]\\ &= \beta\left[\frac{1-\alpha}{\alpha}(t_1-t_2) + \left(1+\frac{1-\alpha}{\alpha}\right)t_2 - 1\right].\end{aligned}$$

Since $t_2 \geq \alpha$ and $1 + (1-\alpha)/\alpha \geq 0$, we have

$$\left(1+\frac{1-\alpha}{\alpha}\right)t_2 - 1 \geq \left(1+\frac{1-\alpha}{\alpha}\right)\alpha - 1 = 0.$$

As a result, we have

$$\phi_{\alpha,\beta}(t_1) - \phi_{\alpha,\beta}(t_2) \geq \beta\frac{1-\alpha}{\alpha}(t_1-t_2) = -\beta\frac{1-\alpha}{\alpha}|t_1-t_2|. \tag{2.2}$$

It follows from Equations (2.1) and (2.2), and $\beta \in [0,1]$, $\alpha \in (0,1]$ and $\beta(1-\alpha)/\alpha \leq 1$, that

$$|\phi_{\alpha,\beta}(t_1) - \phi_{\alpha,\beta}(t_2)| \leq \max\left(\beta, \beta\frac{1-\alpha}{\alpha}\right)|t_1-t_2| \leq |t_1-t_2|.$$

- For $t_1 \in [1,\alpha]$ and $t_2 \in [0,\alpha]$, the proof is identical to the above proof; it is enough to exchange the roles of t_1 and t_2.
- For $t_1 \in [\alpha,1]$ and $t_2 \in [\alpha,1]$, we have

$$\phi_{\alpha,\beta}(t_1) - \phi_{\alpha,\beta}(t_2) = \beta(t_2 - t_1).$$

Thus, under the assumptions $\beta \in [0,1]$, we obviously have $|\phi_{\alpha,\beta}(t_1) - \phi_{\alpha,\beta}(t_2)| = \beta|t_1-t_2| \leq |t_1-t_2|$.

□

Based on Proposition 1, it is worth noting that $\beta \in [0,1]$, $\alpha \in (0,1]$, and $\beta(1-\alpha)/\alpha \leq 1$, imply that $\alpha \in [1/2,1]$. These parameter constraints need to be taken into account for practical analysis.

Hence, under the related parameter assumptions, we define a new copula by

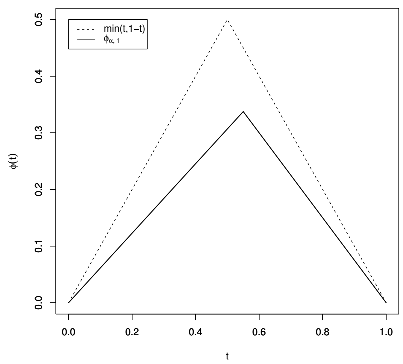
$$C_N(u,v) = \Pi(u,v) + \theta\phi_{\alpha,\beta}(u)\phi_{\alpha,\beta}(v), \ \ \theta \in [-1,1]. \tag{2.3}$$

In what follows, it will be sometimes denoted as $C_N = C_{\phi_{\alpha,\beta}}$ when the parameter needs to be explicit.

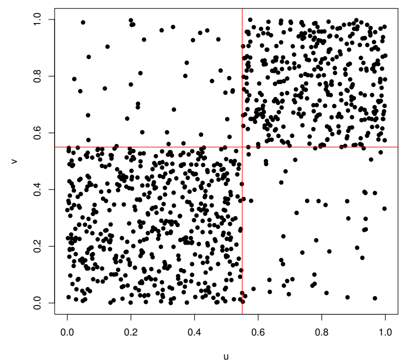
The created copula can also be written as

$$C_N(u,v) = \Pi(u,v) + \gamma\phi_{\alpha,1}(u)\phi_{\alpha,1}(v), \ \ \gamma \in [-1,1]. \tag{2.4}$$

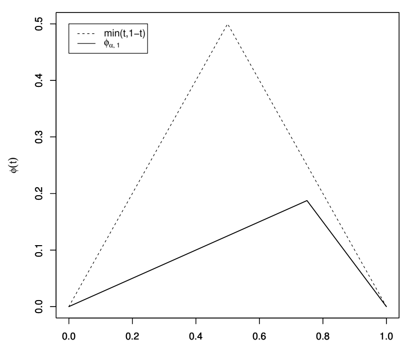
where $\gamma = \theta\beta^2$. To the best of our knowledge, it is new in the literature, totally practicable, and offers interesting perspectives on the analysis of data that show dependence structures in squared zones with different substructures. We complete this claim by a graphical analysis of the considered generator function in Figure 1, along with the generation of values represented by the associated scatterplots.



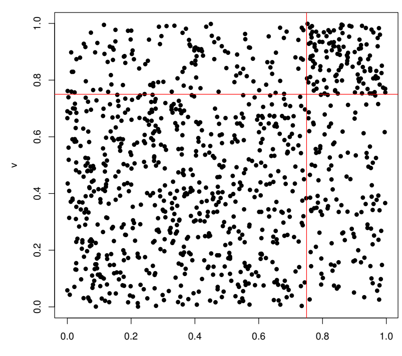
(a) Graph of $\phi_{\alpha,\beta}$ with $\alpha = 0.55, \beta = 1, \theta = 1$



(b) Random samples u, v from the $C_{\phi_{\alpha,\beta}}$ with $\alpha = 0.55, \beta = 1, \theta = 1$



(c) Graph of $\phi_{\alpha,\beta}$ with $\alpha = 0.75, \beta = 1, \theta = 1$



(d) Random samples u, v from the $C_{\phi_{\alpha,\beta}}$ with $\alpha = 0.75, \beta = 1, \theta = 1$

Figure 1: Generator function and random sample for the copula $C_N = C_{\phi_{\alpha,\beta}}$

In this figure, we clearly see the dense square zone of substructures of dependence. The proposed copula is designed to handle such a practical data scenario.

3 Dependence Measures

In this section, we investigate the dependence framework of the proposed copula. The next result determines the exact expression of the Kendall tau of the proposed copula.

It is centered on the general formula in Equation (1.2).

Proposition 2. *The Kendall tau associated to the copula in Equation* (2.4) *is given by*

$$\tau = \tau_{\phi_{\alpha,\beta}} = 2(1-\alpha)^2\beta^2\theta.$$

Proof. Based on the general formula in Equation (1.2), we have

$$\tau = 8\theta\left(\int_{[0,1]}\phi_{\alpha,\beta}(t)dt\right)^2 = 8\theta\Big(\int_{[0,\alpha]}\beta\frac{1-\alpha}{\alpha}tdt + \int_{[\alpha,1]}\beta(1-t)dt\Big)^2$$
$$= 8\theta\Big(\beta\frac{1-\alpha}{\alpha}\times\frac{\alpha^2}{2} + \beta\frac{(1-\alpha)^2}{2}\Big)^2 = 2(1-\alpha)^2\beta^2\theta.$$

The desired result is established. □

Assuming $\alpha \in [1/2,1]$ and $\beta \in [0,1]$, we find that $\tau \in [-1/2,1/2]$, which means that the proposed copula is able to model positive and negative dependence, with a moderate range of values. Not surprisingly, the case of independence, i.e., $\tau = 0$, is obtained by taking $\alpha = 1$ or $\theta = 0$. The Kendall tau varies linearly with θ, and depends on the values of α and β. For example, if $\alpha = 0.5$ and $\beta = 1$, then $\tau = 0.5\theta$, which ranges from -0.5 to 0.5 as θ varies from -1 to 1. Equivalently, with the re-parameterization $\gamma = \theta\beta^2$, we get $\tau = 2(1-\alpha)^2\gamma$.

Other dependence measures can be defined. For instance, the medial correlation as suggested in Nelsen (2006) is given by

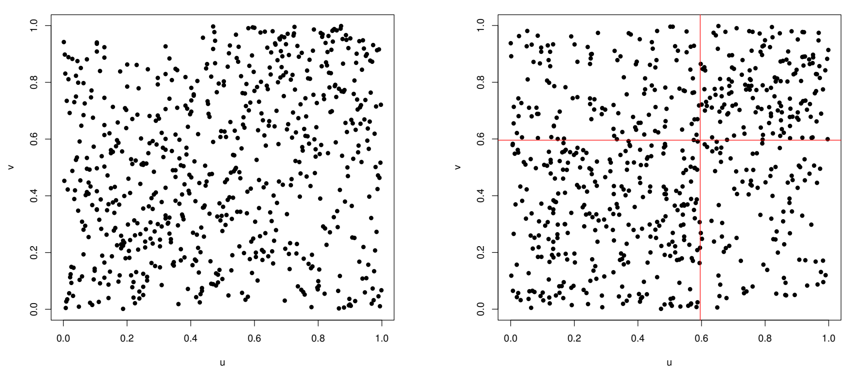
$$M = 4C_N\left(\frac{1}{2},\frac{1}{2}\right) - 1 = \begin{cases} \theta\Big(\beta\frac{1-\alpha}{\alpha}\Big)^2, & \frac{1}{2}\in[0,\alpha],\ \frac{1}{2}\in[0,\alpha], \\ \theta\beta^2\frac{1-\alpha}{\alpha}, & \frac{1}{2}\in[0,\alpha],\ \frac{1}{2}\in[\alpha,1], \\ \theta\beta^2\frac{1-\alpha}{\alpha}, & \frac{1}{2}\in[\alpha,1],\ \frac{1}{2}\in[0,\alpha], \\ \theta\beta^2, & \frac{1}{2}\in[\alpha,1],\ \frac{1}{2}\in[\alpha,1]. \end{cases}$$

This formula shows the complexity of the dependence, which depends mainly on the value of α.

Other dependence measures can be investigated in a similar way, including various local measures of dependence (see Huang (2018), among others). The article by Ambrard and Girard (2002) provides a detailed analysis of the symmetry and dependence properties of a semiparametric family of bivariate copulas. The results are particularly useful for understanding how the choice of the generating function ϕ influences the dependence structure of the copula. The conditions for radial symmetry, positive quadrant dependence, total positivity, and concordance ordering are clearly established, making this family of copulas a valuable tool for modeling moderate degrees of dependence in bivariate data. For further details, see the original article by Ambrard and Girard (2002).

4 Real Data Example

In this section, we compare the proposed copula with the most commonly used copulas, including the Frank, Clayton, Gumbel, Normal, Plackett and FGM copulas. See Nelsen (2006) for more information on these copulas. The data set used in this section is derived from 655 water samples with log concentrations of two chemical elements, where X is the log concentration of uranium (U) and Y is the log concentration of titanium (Ti). This data set, called the "uranium data set", can be found in the R package "copula" (see Hofert et al. (2020)).



(a) Real data set (b) Generated data set

Figure 2: Scatter plots of generated data sets

We use the goodness-of-fit test as suggested in Genest et al. (2006) to reveal the dependence structure between variables. Its definition is based on the copula-based Cramér-von-Mises distance, indicated as follows:

$$CvM = \int_{[0,1]^2} n(C_n(u,v) - C_N(u,v))^2 dC_n(u,v),$$

where C_n represents the empirical copula based on the data. We can thus compare the distances between the null hypothesis copula and the empirical copula using the CvM test statistic. In addition, the hypothesis that the data belong to the parametric family of copulas is assessed by the p-value (P-Val) derived from the bootstrap process. Furthermore, the minimum distance estimator is used to estimate the parameters of the copulas by minimizing the CvM.

In statistical modeling, particularly in copula selection, it is crucial to balance model complexity with goodness-of-fit. Two commonly used criteria for model selection are the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). These criteria help in identifying the most appropriate copula function by penalizing excessive model complexity.

The AIC and BIC are respectively defined as follows:

$$\text{AIC} = -2 \log L + 2k,$$
$$\text{BIC} = -2 \log L + k \log n,$$

where L is the maximum likelihood of the estimated copula model, k is the number of parameters in this model, and n is the sample size. The model with the lowest AIC or BIC value is preferred, as lower values indicate a better balance between goodness-of-fit and model simplicity. Both criteria suggest the same model, it is likely to be the optimal choice.

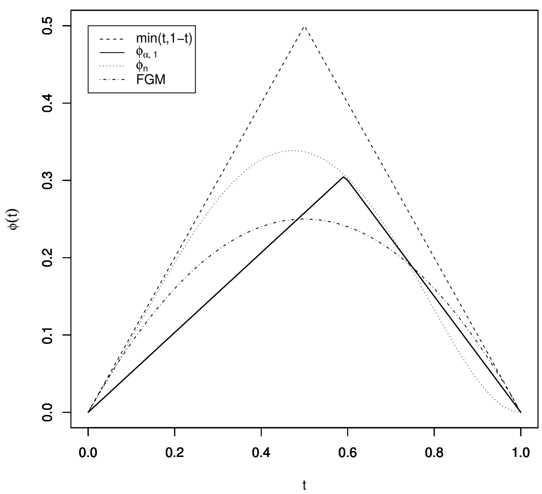


Figure 3: Generator functions based on the considered data sets

Table 1: Goodness-of-fit results for the copulas based on the considered data set

Copula	$\hat{\theta}$	$\hat{\alpha}$	CvM	P-Val	AIC	BIC
Normal	0.2097	—	0.0768	0.0004	-12.5647	-8.0801
Plackett	1.8262	—	0.0663	0.0004	-24.3768	-21.1921
Gumbel	1.1554	—	0.0601	0.0015	-13.1129	-8.6283
Clayton	0.3108	—	0.1359	0.0004	-1.1266	3.3579
Frank	1.2288	—	0.0657	0.0004	-24.9144	-20.4298
FGM	0.6053	—	0.0668	0.0004	-20.1731	-15.6885
$C_{\phi_{\alpha,1}}$	0.4193	0.5923	0.0361	0.0495	-25.3827	-21.5147

Table 1 presents the estimated parameters and goodness-of-fit results for the considered copula models at the hypotheses under the following generic forms: "$H_0 : C \in C_\theta$

versus $H_1 : C \notin C_\theta$". Table 1 indicates that $C_{\phi_{\alpha,1}}$ is the best-fitting copula model because it has the lowest CvM (0.0361) and highest p-value (0.0495). It is worth noting that $0.0495 \approx 0.05$, so we can conceptually accept H_1, and the adequate fit can be reasonably validated. When selecting among different copula models, AIC and BIC values are also computed for each candidate model. The model with the lowest both AIC and BIC is also $C_{\phi_{\alpha,1}}$.

We create random samples to evaluate the graphical goodness-of-fit of the data set for $C_{\phi_{\alpha,1}}$. The scatter plots of the data set and the simulated random sample from the $C_{\phi_{\alpha,1}}$ are shown in Figure 2, respectively. The suggested copulas $C_{\phi_{\alpha,1}}$ appear to fit the data set fairly well, as this figure demonstrates. Furthermore, the estimated generator functions associated with the copulas defined in Equation (1.1) are shown in Figure 3.

5 Conclusion

In conclusion, this article has highlighted the interest in a special copula of the perturbed independence type, defined with a simple piecewise generator function. It is designed to capture specific types of dependence patterns; it provides an advantage in dealing with complex dependence structures in data with distinct substructures in square zones. Theoretical properties of the proposed copula are presented, showing the possible ranges of parameter values and determining the exact expression of the Kendall tau. This theoretical work is followed by a practical study. In particular, an example of real data is used to illustrate the application. Comparisons with existing models are made using famous statistical criteria and convincing results are obtained. Through innovative approaches such as piecewise functions and perturbed independence models, this article helps to advance the understanding and use of copula theory in statistical modeling and analysis.

References

Amblard, C. and Girard, S. (2002). Symmetry and dependence properties within a semi-parametric family of bivariate copulas. *Journal of Non-parametric Statistics*, **14**(6), 715-727.

Amblard, C. and Girard, S. (2005). Estimation procedures for a semiparametric family of bivariate copulas, *Journal of Computational and Graphical Statistics*, **14**(2), 363-377,

Cuadras, C. M. and Augé, J. (1981). A continuous general multivariate distribution and its properties. *Communications in Statistics - Theory and Methods*, **10**, 339-353.

De Baets, B., De Meyer, H. and Úbeda-Flores, M. (2009). Opposite diagonal sections of quasi-copulas and copulas. *Internat. J. Uncertain. Fuzz.*, **17**(4), 481-490.

Domma, F. and Giordano, S. (2013). A copula-based approach to account for dependence in stress-strength models. *Statistical Papers*, **54**(3), 807-826.

Durante, F., Quesada-Molina, J.J. and Sempi, C. (2007). A generalization of the Archimedean class of bivariate copulas. *Annals of the Institute of Statistical Mathematics*, **59**, 487-498.

Durante, F., Fernández-Sánchez, J. and Úbeda-Flores, M. (2013). Bivariate copulas generated by perturbations. *Fuzzy Set. Syst.*, **228**, 137-144.

Durante, F. and Sempi, C. (2016). *Principles of Copula Theory*. CRS Press, Boca Raton FL.

Fernández-Sánchez, J. and Úbeda-Flores, M. (2018). Constructions of copulas with given diagonal (and opposite diagonal) sections and some generalizations, *Dependence Modeling*, **6**(1), 139-155.

Fréchet, M. (1958). Remarques au sujet de la note précédente. *Comptes Rendus de l'Académie des Sciences Paris*, **246**, 2719-2720.

Genest, C., Quessy, J. F. and Rémillard, B. (2006). Goodness-of-fit procedures for copula models based on the integral probability transformation. *Scandinavian Journal of Statistics*, **33**(2), 337-366.

Genest C., Molina J. and Lallena, J. (1995). De l'impossibilité de construire des lois á marges multidimensionnelles données á partir de copules, *Comptes rendus de lAcadmie des Sciences*, **320**, 723-726.

Hofert, M., Kojadinovic, I., Maechler, M. and Yan, J. (2020). Copula: Multivariate dependence with copulas. https://CRAN.R-project.org/package=copula

Huang, Z., (January 4, 2018) Copula-based Local Dependence Framework. Available at SSRN: https://ssrn.com/abstract=3641066 or http://dx.doi.org/10.2139/ssrn.3641066

Hudaverdi, B. and Susam, S. O. (2023) On the copula-based reliability of stress-strength model under bivariate stress. *International Journal of General Systems*, **52**(7), 842-863,.

Janssen, P., Swanepoel, J. and Veraverbeke, N. (2016). Bernstein estimation for a copula derivative with application to conditional distribution and regression functionals. *TEST*, **25**, 351374.

Joe, H. (1997). Multivariate models and dependence concepts. *Monographs on statistics and applied probability*, **73**, Chapman & Hall.

Marshall, A. W. and Olkin, I (1967). A generalized bivariate exponential distribution. *Journal of Applied Probability*, **4**, 291-302.

Mesiar, R. Magda Komorníková, M. and Komorník, J. (2015). Perturbation of bivariate copulas. *Fuzzy Sets and Systems*, **268**, 127-140.

Morgenstern, D. (1956), Einfache Beispiele Zweidimensionaler Verteilungen, *Mitteilingsblatt für Matematische Statistik*, **8**, 234235.

Nelsen, R. B. (2006). *An introduction to copulas*. New York, USA: Springer.

Quesada-Molina, J. J. and Rodríguez-Lallena, J. A. (1995). Bivariate copulas with quadratic sections, *Nonparametric Statistics*, **5**, 323-337.

Raftery, A. E. (1984). A continuous multivariate exponential distribution. *Communications in Statistics - Theory and Methods*, **13**, 947-965.

Rodriguez Lallena, J. A. (1992). Estudio de la comparability y dise no de nuevas familias en la teoria de copulas. Aplicaciones. Tesis doctoral, Universidad de Granada,

Saminger-Platz, S., Kolesárová, A., Šeliga, A., Mesiar, R. and Klement Erich, P. (2021), New results on perturbation-based copulas, *Dependence Modeling*, **9**(1), 347-373

Sancetta, A. and Satchell, S. (2004). The Bernstein copula and its applications to modeling and approximations of multivariate distributions, *Econometric Theory*, **20**(3), 535-562.

Schmitz, V. (2004). Revealing the dependence structure between x(1) and x(n). *Journal of Statistical Planning and Inference*, **123**, 41-47.

Shih, J. H. and Louis, T. A. (1995). Inferences on the association parameter in copula models for bivariate survival data. *Biometrics*, **51**, 1384-1399.

Shubina, M. and Lee, M. -L. T. (2004). On maximum attainable correlation and other measures of dependence for the Sarmanov family of bivariate distributions. *Comm. Statist. Theory Methods*, **33**(5), 1031-1052.

Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges, *Publications de l'Institut de Statistique de l'Université de Paris*, **8**, 229-231.

Susam, S. O. (2022). A multi-parameter Generalized Farlie-Gumbel-Morgenstern bivariate copula family via Bernstein polynomial. *Hacettepe Journal of Mathematics & Statistics*, **51**(2), 618-631.

Susam, S. O. and Hudaverdi, B. (2023). A goodness-of fit improvement based on τ-preserving transformation for semiparametric family of copulas, *Communications in Statistics - Theory and Methods*, **52**(21), 7699-7708.