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Modeling Joint Lifetimes of Couples by Using Bivariate Phase-type Distributions

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Abstract. Many insurance products and pension plans provide benefits which are related to couples, and thus under influence of the survival status of two lives. Some studies show the future lifetime of couples is correlated. Three reasons are available to confirm this fact: (1) catastrophe events that affect both lives, (2) the impact of spousal death and (3) the long-term association due to common life style. Dependence between lifetimes of couples could have a financial impact on insurance companies and pension plans providers. In this paper, we use a health index called physiological age in a Markov process context by that we model aging process of joint and last survivor statuses. Under this model, future joint lifetime of couples follows a bivariate phase-type distribution. The model has physical interpretation and closed-form expressions for actuarial quantities and owns tractable computation for the other ones. We use the model to pricing products relevant to couples annuities and life insurances.

Keywords. Bivariate Phase-type Distribution, Physiological Age, Joint Lifetimes of Couples, Aging Process, Markov Processes.

MSC: 60J28.

1 Introduction and Motivation

It is common for life insurance and pension products that cash flow direction depends on status of joint-life and last-survivor statuses. In joint-life insurance a lump sum is

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paid out on the first death and in a last-survivor life annuity the amounts of benefit is paid as long as at least one of the couples survives. In reversionary annuities the benefit is paid to one of the couples if he/she survives after his/her partner. In some pension products the pension benefit is contingent on survival of a couple and even more members of families. In this paper, we focus on modelling future lifetimes of couples rather than a group of more than two.

In actuarial modelling, future lifetimes of couples are usually assumed to be independent which apparently are not. Couples due to many reasons share risks together. Common life style, depression after bereavement of one partner and common shock are the main reasons for dependence between the future lifetimes of couples. There are several investigations that confirm psychological and physical effects on the individuals. For instance, in Parkes and Brown (1972), based on structured interviews, it is observed that both widows and widowers have experienced disturbance of appetite and sleep, depression and restlessness after the bereavement. In Young et al. (1963), it is shown that the mortality rate of the survived couple increases by 40% during the first six months of bereavement and after decreases gradually to normal rate.

- Broken Heart: A Statistical Study of Increased Mortality among Widowers-parks 1969 : Reviewed 4,486 widowers of 55 years of age and older since the death of their wives for nine years. Their results showed that mortality rate of the widowers increase especially through the first six months.

Over the last few decades, several papers have been published which model future lifetimes of couples regarding to dependence, and pricing contingent coupled lives contracts. The papers show that there is a significant correlation between lifetimes of couples, causes substantial impact on relevant policies pricing. These models use variety of probabilistic tools in modelling the dependence between lifetimes. Frees et al. (1996), Carriere (2000), Luciano et al. (2008), model the dependence of the time of deaths of coupled lives based on copulas. Spreeuw and Owadally (2013) uses Markov model to modelling the short term dependence between two remaining lifetimes and applied to a life annuity portfolio. Although the assumption of dependence between future lifetimes of couples makes the calculations straightforward and easy, the consequences can be unfair either for insurers or policyholders. Under independence assumption, the probability of joint survival of a couple with ages x and y is given by product of individual survival probability, *i.e.* ${}_n p_{x,y} = {}_n p_x \times {}_n p_y$, where ${}_n p_x$ is probability of survival of a person aged x to age $x + n$ and ${}_n p_{x,y}$ is probability of joint survival of a couples aged x and y to age $x + n$ and $y + n$. The individual survival probabilities can be calculated easily from actuarial life table.

In order to include dependence between the future lifetimes of couples, multi-state models are widely used. A multi-state model is a model for a continuous-time stochastic process allowing the process to move among finite states. Multi-state models are widely used in demography, biostatistics and actuarial sciences. When the stochastic process of the multi-state is a Markov process, the calculations are tractable. Spreeuw and Owadally (2013) uses Markov model and Ji (2011) uses a semi-Markov model for

modelling joint-life mortality. Disability insurance is modelled in a multi-state context in Sverdrup (1965) and Hassan Zadeh et al. (2014). Multi-state models also are used in modelling aging-process Lin and Liu (2007). Ji et al. (2011) uses Markov and semi-Markov model to modelling dependence lifetimes for reverse mortgage terminations. For a thorough review of multi-state models and its applications in life insurance products see Dickson et al. (2013). See Zhang and Brockett (2020), Henshaw and Khalil (2023), and references therein for more references in area of mortality of couples.

Modelling joint-life mortality presents challenges because of possibility of moving from joint-life status for different reasons into three other states, *i.e.*, wife-dead husband-alive, husband-dead wife-alive and wife and husband both dead. Modelling a movement from an active joint-life status to the state when one of the couples has died needs special considerations. In this case, the broken-heart syndrome causes the rate of mortality of alive partner to move much higher than normal case. In this paper, a new method for modelling joint-life mortality is presented based on Markov chain. We also use a hypothetical health index called physiological age that representing the degree of aging in a human body. See Lin and Liu (2007) and referral inside it for additional information.

This paper is organized as following. In section 2 a brief introduction of phase-type distribution (*PH*) is given. Section 3 describes the model and section 4 includes the actuarial calculations. A numerical example is presented in section 5 and finally we conclude this paper with conclusions in 6.

2 Phase-type Distribution

Phase-type distributions recently have been received attentions by actuaries due to nice properties they own. Closed form expressions, interpretable parameters, being dense in all positive support distributions and ability to model complex systems make *PH* distributions very attractive and applicable in actuarial context. In Lin and Liu (2007), by imposing a physiological age process on the underlying Markov chain of the *PH*, mortality rates is successfully modelled. In order to model disability, recovery and death, the same technique as of Lin and Liu (2007) was used in Hassan Zadeh et al. (2014) and all actuarial expressions are obtained in closed-forms. In Drekić et al. (2004), in the Sparre Andersen renewal models with *PH* distributed claims, the distribution of deficit at ruin is found to be of *PH*. Asmussen (2000) applies *PH* distributions to risk theory. Credibility theory in context of *PH* distributions has been developed by Hassan Zadeh and Stanford (2016), Hassan Zadeh (2009) and Zang (2013). See Cai and Li (2005b), Cai and Li (2005a) and Hassan Zadeh and Bilodeau (2013) for applications of multivariate *PH* in actuarial science.

Consider $\{Z_t, t \geq 0\}$ a right continuous-time Markov process on the finite state space $\Gamma = \{0, 1, 2, \dots, m, \Delta\}$ with initial probability vector α and infinitesimal generator matrix Λ . We assume that Δ is the only absorbent state. In this case the matrix Λ can be written

as

$$\begin{pmatrix} \mathbf{Q} & \mathbf{q} \\ \mathbf{0}' & 0 \end{pmatrix}, \quad (2.1)$$

where the matrix $\mathbf{Q} = (q_{ij}), i, j = 0, 1, 2, \dots, m$ is a sub-intensity matrix (a square matrix $\mathbf{B} = (b_{ij}), i, j = 1, \dots, k$ is called a sub-intensity matrix if $b_{ii} \leq 0; b_{ij} \geq 0$ for $i \neq j$, and $\sum_{j=1}^k b_{ij} \leq 0$, with strick inequality for at least one i ; for $i, j = 1, \dots, k$) and \mathbf{q} is the exit vector to the absorbent state Δ which equals $-\mathbf{Q}\mathbf{1}$. Without loss of generality, we presume that $\alpha_{\Delta} = 0$, i.e. $P(Z_0 \in \{\Delta\}) = 0$. The initial probability vector over the transients elements is denoted by $\boldsymbol{\pi}$ such that $\boldsymbol{\alpha} = (\boldsymbol{\pi}, 0)$.

Let's define $T = \inf\{t; Z_t \in \Delta\}$. In this case T is said to follow a *PH* distribution with representation $(\boldsymbol{\pi}, \mathbf{Q})$ The probability density function $f_T(t)$ and survival function, $S_T(t)$ are given as follow

$$f_T(t) = \boldsymbol{\pi}e^{\mathbf{Q}t}\mathbf{q}, \quad (2.2)$$

$$S_T(t) = \boldsymbol{\pi}e^{\mathbf{Q}t}\mathbf{1}, \quad (2.3)$$

where exponential of a square matrix \mathbf{A} is defined as $e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$ and $\mathbf{1}$ is a column vector of 1s with a proper dimension. See Neuts (1981) for more details and proofs.

Bivariate phase type (*BPH*) distributions, in a version that we are interested, are defined by Assaf et al. (1984). Suppose that Γ_1 and Γ_2 are two nonempty stochastically closed subsets (E is said to be stochastically closed if once Z_t has entered E , it never leaves) of Γ such that $\Gamma_1 \cap \Gamma_2 = \{\Delta\}$. Random variable $\mathbf{T} = (T_1, T_2)$ is called a *BPH* if $T_i = \inf\{t; Z_t \in \Gamma_i\}, i = 1, 2$. It can be shown that the survival function of \mathbf{T} equals to

$$S(t_1, t_2) = \begin{cases} \boldsymbol{\pi}e^{\mathbf{Q}t_1}\mathbf{g}_1e^{\mathbf{Q}(t_2-t_1)}\mathbf{g}_2\mathbf{1}, & \text{if } t_2 \geq t_1 \geq 0, \\ \boldsymbol{\pi}e^{\mathbf{Q}t_2}\mathbf{g}_2e^{\mathbf{Q}(t_1-t_2)}\mathbf{g}_1\mathbf{1}, & \text{if } t_1 \geq t_2 \geq 0, \end{cases} \quad (2.4)$$

where $\mathbf{g}_k, k = 1, 2$, is an $(m+1) \times (m+1)$ diagonal matrix whose i th diagonal element is 1 if $i \in \Gamma_k^c, k = 1, 2$ and 0 otherwise. After simple calculations, the joint probability density function of \mathbf{T} of the absolutely continuous component is given by the following

$$f(t_1, t_2) = \begin{cases} \boldsymbol{\pi}e^{\mathbf{Q}t_1}\mathbf{G}_1e^{\mathbf{T}(t_2-t_1)}\mathbf{Q}\mathbf{g}_2\mathbf{1}, & \text{if } t_2 > t_1 > 0, \\ \boldsymbol{\pi}e^{\mathbf{Q}t_2}\mathbf{G}_2e^{\mathbf{Q}(t_1-t_2)}\mathbf{Q}\mathbf{g}_1\mathbf{1}, & \text{if } t_1 > t_2 > 0, \end{cases} \quad (2.5)$$

where $\mathbf{G}_i = \mathbf{Q}\mathbf{g}_i - \mathbf{g}_i\mathbf{Q}$, for $i = 1, 2$.

The singular component on $t_1 = t_2$, is very useful for our case of application. In Assaf et al. (1984) and Hassan Zadeh and Bilodeau (2013), the following quantity is given for this quantity

$$P(T_1 = T_2 > t) = \boldsymbol{\pi}e^{\mathbf{Q}t}\mathbf{Q}^{-1}\mathbf{g}_1\mathbf{g}_2\mathbf{Q}\mathbf{1}. \quad (2.6)$$

By setting $t = 0$ in (2.6), one can get the the following

$$P(T_1 = T_2) = \boldsymbol{\pi}\mathbf{Q}^{-1}\mathbf{g}_1\mathbf{g}_2\mathbf{Q}\mathbf{1}.$$

Define $\Gamma_0 = \Gamma - (\Gamma_1 \cup \Gamma_2)$. Based on the definition for the *BPH* by Assaf et al. (1984), the sub-intensity matrix \mathbf{Q} can be written as follows

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_0 & \mathbf{Q}_{01} & \mathbf{Q}_{02} \\ \mathbf{0} & \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_2 \end{bmatrix}. \tag{2.7}$$

In this case (2.5), (2.4) and (2.6) will be simplified as noted in Assaf et al. (1984). In the next section the proposed model in this paper is explained in details.

3 The Model

As it is mentioned in Section 1, there are three main reasons which significantly affect the remaining lifetimes of a couple. In this section, we will develop the classic comomos-shock model taking into account the dependence factors in a Markov chain environment. The model has four states, which explains survival status of a couple (see Figure 1).

Let $\{Z_t, t \geq 0\}$ be a continuous-time Markov chain on finite state space $E = \{0, 1, 2, \Delta\}$ to represent the survival status of a couple at time t in Figure 1, where the state Δ is absorbent and the rest are transient. In addition, we assume that in $t = 0$ the husband (with real age x) and the wife (with real age y) are both alive, *i.e.* $Z_0 = 0$. Suppose T_x and T_y to be two random variables that indicate the remaining lifetimes of the couple. We define

$$\begin{aligned} T_x &= \inf\{t \geq 0; Z_t \in \Gamma_2\}, \\ T_y &= \inf\{t \geq 0; Z_t \in \Gamma_1\}, \end{aligned} \tag{3.1}$$

where $\Gamma_1 = \{1, \Delta\}$ and $\Gamma_2 = \{2, \Delta\}$ are two stochastically closed subsets of E . Under this definition and the definition of *BPH*, the random vector (T_x, T_y) follows a *BPH* distribution.

In order to reflect the causes of dependence in our model, we need to impose some conditions that reflect both common life style and broken-heart syndrome effects in the model. To this end, in our model, each state is decomposed into sub-states. Decomposition of the state 0 is presented as follows:

$$\{(i, j), (i + 1, j + 1), \dots, (n + i - \max(i, j), n + j - \max(i, j))\},$$

with a cardinality of $d_0 := n - \max(i, j) + 1$, where i and j denote the physiological age of the husband and the wife at issue of the insurance contract and n is the maximum physiological age that a new born own. See Figure 2. (we need more explanation and references about physiological age here). In Lin and Liu (2007) it is shown that with $n = 200$, a good fit for mortality rates of a newborn can be obtained. The direct transition from the state 0 to the state Δ models the common shock effect.

If the couple starts with the physiological age (i, j) , the process Z_t either will run into the next physiological age $(i + 1, j + 1)$ with rate λ or the process will move to the states 1, 2 or Δ , as it is shown in Figure 2.

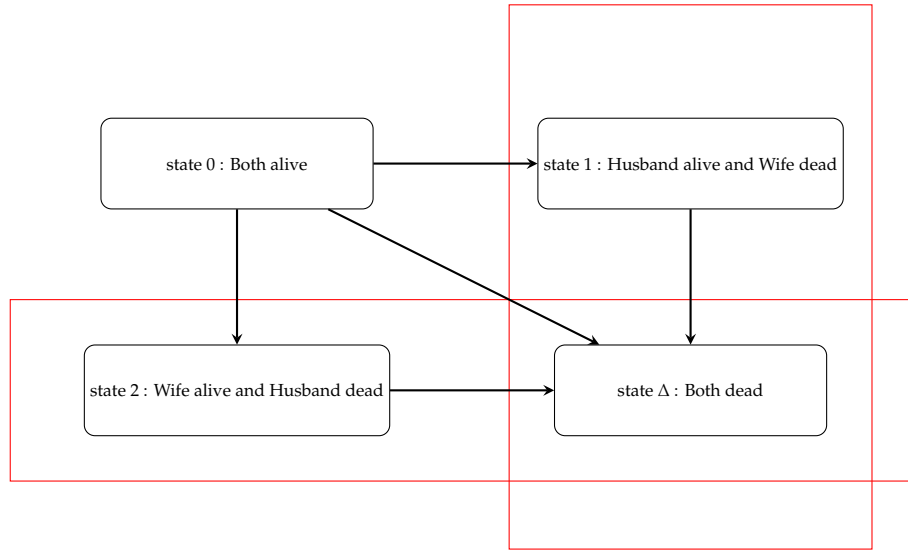


Figure 1: Multi-state model for joint and last survivor statuses .

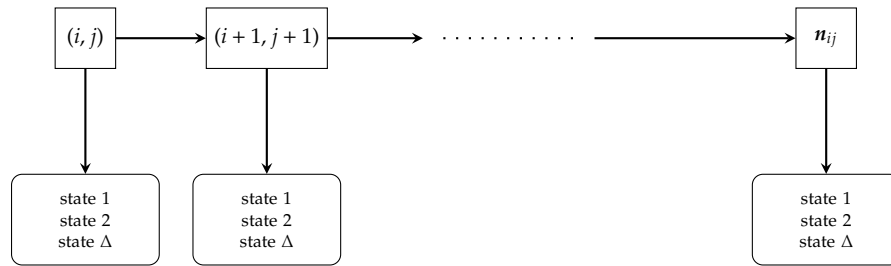


Figure 2: Decomposition of state 0 in common shock model

In order to decompose the states 1 and 2 to appropriate sub-states, we have to consider the bereavement effect on mortality. It's important to note that spousal death has a short-term effect on the survived partner.

Thus, we use two sets of sub-states in the states 1 and 2 to reflect the bereavement effect. The first set includes sub-states that indicate the survived individual's physiological ages after spousal death. The second set represents the states after broken-heart syndrome effect period when the survived partner is in normal condition from mortality point of view.

In the first phase of mortality, under the influence of bereavement effect, the rate of mortality is higher than the second one. We use subscripts wm (wf) and m (f) for physiological ages in bereavement (first phase) and after bereavement (second phase) of the husband (wife). In other words, symbols k_{wm} (k_{wf}) and k_m (k_f) represent the husband (wife) physiological age k under the influence of spousal death and after

the broken-heart effect vanished, respectively. The result can be seen in Figure 3 for decomposition of the state 2. As we see in Figure 3, the state 2 can be decomposed into $2d_2$, where $d_2 := n - j + 1$ states, as below:

$$\{j_{wf}, (j + 1)_{wf}, \dots, n_{wf}, j_f, (j + 1)_f, \dots, n_f\}.$$

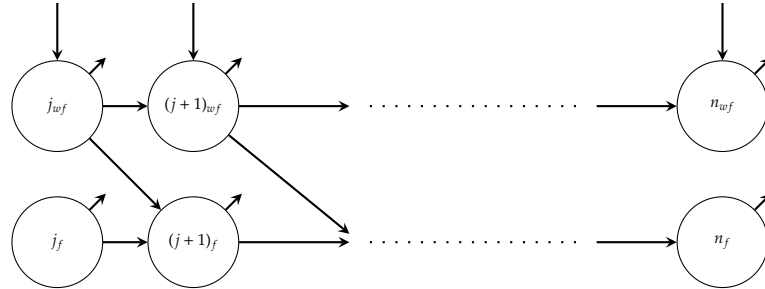


Figure 3: Decomposition of state 2 in common shock model

Where symbols \circlearrowright and \circlearrowleft indicate the process meets the Δ state and transition from state 0 to state 2, respectively. As it can be seen in Figure 3, after husband death, the process enters to state 2 through the first phase of sub-states with wf subscript, *i.e.* $Z_t \in \{j_{wf}, (j + 1)_{wf}, \dots, n_{wf}\}$. In first phase the wife has further force of mortality, the cause of the broken-heart effect. After a while, provided that the wife is survived, the process moves to the second phase of sub-states with f subscript, *i.e.* $Z_t \in \{j_f, (j + 1)_f, \dots, n_f\}$.

We use the second phase of states to show the wife aging process after recovery from the broken-heart syndrome. Decomposition of state 1 is done in the same way, as for state 2. See Figure 4. This time with $2d_1$, where $d_1 := n - i + 1$, states as follow:

$$\{i_{wm}, (i + 1)_{wm}, \dots, n_{wm}, i_m, (i + 1)_m, \dots, n_m\}.$$

The entire model is depicted in Figure 5.

Next step, after we have determined the structure of the model, we will explain the paramteres of the sub-intensity matrix Q in the underlying Markov chain $\{Z_t, t \geq 0\}$.

In the following we will specify the elements of the sub-intensity matrix Q in (2.7) by determining the elements of each sub-intensity matrix separately. We will use the rates given in Lin and Liu (2007) with a slight difference. In the joint status (state 0), the rate of death for the joint status in physiological age (i, j) is given by

$$a_m + b_m i^{c_m} + a_f + b_f j^{c_f}. \tag{3.2}$$

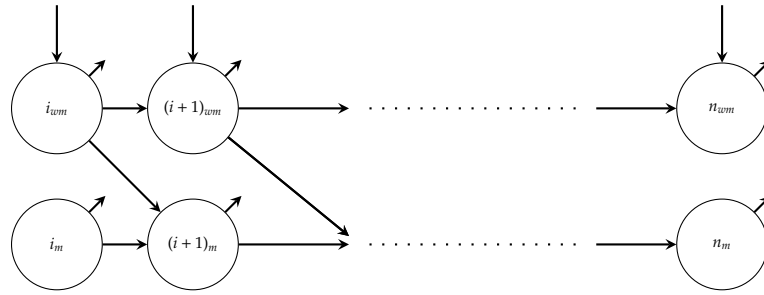


Figure 4: Decomposition of state 1 in common shock model

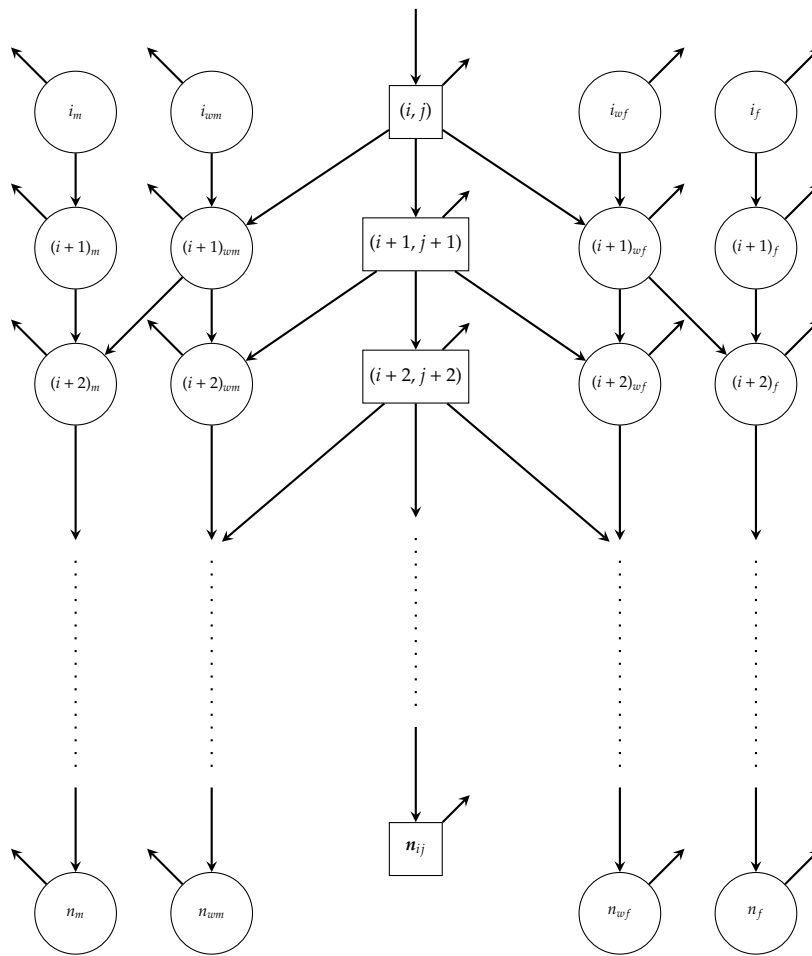


Figure 5: The BPH model at a glance

The joint status fails as soon as the first member dies and this explains the choice

of the rate of mortality of joint status in (3.2). The subscript m (f) for the parameter in (3.2) is abbreviation for male (female). The form and interpretation of these two intensities are as defined in Lin and Liu (2007), but here the parameters are different for the husband and wife cases. Therefore the matrix \mathbf{Q}_0 can be written as below.

$$\begin{cases} Q_0(l, l) = -[\lambda_c + a_f + b_f(j + l - 1)^{c_f} + a_m + b_m(i + l - 1)^{c_m} + \lambda], & l < d_0, \\ Q_0(l, l) = -[\lambda_c + a_f + b_f(j + d_0 - 1)^{c_f}], & l = d_0, i < j, \\ Q_0(l, l) = -[\lambda_c + a_m + b_m(i + d_0 - 1)^{c_m}], & l = d_0, i > j, \\ Q_0(l, l + 1) = \lambda, & l < d_0. \end{cases} \quad (3.3)$$

Other elements of \mathbf{Q}_0 are zero. The parameter λ is used for rate of progression of aging of joint status of the couple. In other words, we use λ for transition rate from one pair physiological age to the next one. The parameter λ_c is used for common shock event. The matrices \mathbf{Q}_{01} and \mathbf{Q}_{02} , in (2.7) are $d_0 \times 2d_1$ and $d_0 \times 2d_2$ matrices which contain the rates of a single death and $2d_1$ and $2d_2$ are dimensions of the matrices \mathbf{Q}_1 and \mathbf{Q}_2 , respectively. The Non-zero elements of the matrix \mathbf{Q}_{01} and \mathbf{Q}_{02} can be written as follows.

$$\begin{cases} Q_{01}(l, l + 1) = a_f + b_f(j + l - 1)^{c_f}, & l < d_0, \\ Q_{01}(l, l + 1) = a_f + b_f(j + l - 1)^{c_f}, & l = d_0, j > i, \end{cases} \quad (3.4)$$

$$\begin{cases} Q_{02}(l, l + 1) = a_m + b_m(i + l - 1)^{c_m}, & l < d_0, \\ Q_{02}(l, l + 1) = a_m + b_m(i + l - 1)^{c_m}, & l = d_0, i > j, \end{cases} \quad (3.5)$$

Now it's enough to determine the elements of sub-matrices \mathbf{Q}_1 and \mathbf{Q}_2 . As it is mentioned before, we will use two different sets of mortality rates to reflect the effect of the broken-heart syndrome after bereavement. The first set is for the short term high mortality rates after bereavement and the second set for after recovery. Therefore the matrix \mathbf{Q}_1 , is formulated as follows

$$\mathbf{Q}_1 = \begin{bmatrix} \mathbf{Q}_{m1} & \mathbf{Q}_{m12} \\ \mathbf{0} & \mathbf{Q}_{m2} \end{bmatrix}. \quad (3.6)$$

In order to magnify rates of mortality after breavement, we employ two parameters λ_{wf} and λ_{wm} as multipliers of mortlaity rates of the wife and the husband, respectively. The effect of bereavement lasts for a while and then vanishes. The elements of the matrix \mathbf{Q}_1 is given in the following.

$$\begin{cases} Q_{m1}(l, l + 1) = \lambda_{in}, & l < d_1, \\ Q_{m1}(l, l) = -[\lambda_{rm} + \lambda_{in} + \lambda_{wm}(a_m + b_m(i + l - 1)^{c_m})], & l < d_1, \\ Q_{m1}(l, l) = -\lambda_{wm}(a_m + b_m(i + l - 1)^{c_m}), & l = d_1, \end{cases} \quad (3.7)$$

$$Q_{m12}(l, l + 1) = \lambda_{rm}, \quad l < d_1, \quad (3.8)$$

and

$$\begin{cases} Q_{m2}(l, l + 1) = \lambda_{in}, & l < d_1, \\ Q_{m2}(l, l) = -[\lambda_{in} + a_m + b_m(i + l - 1)^{c_m}], & l < d_1, \\ Q_{m2}(l, l) = -[a_m + b_m(i + l - 1)^{c_m}], & l = d_1, \end{cases} \quad (3.9)$$

where λ_{in} is used for integrated measure of the deteriorating intensity of aging for a widower. The matrix Q_2 has the same structure as Q_1 , but the parameters $a_m, b_m, c_m, \lambda_{rm}, \lambda_{wm}$ are replaced with $a_f, b_f, c_f, \lambda_{rf}$ and λ_{wf} , respectively.

Hence, the structure of the proposed model completely determined by the intensity matrix. In the next section, we will use the results obtained in this section to derive some essential actuarial quantities under the proposed model.

4 Actuarial Present Value (APV) Calculations

We assume that the process $\{Z_t, t \geq 0\}$ begin from state 0 with probability 1, i.e. $Pr[Z_0 \in \Gamma_0] = 1$. Therefore the initial probability vector can be written as $\pi = (\pi_0, \mathbf{0}, \mathbf{0})$, where $\pi_0 = (1, 0, \dots, 0)$. Hence, the lifetimes of the couple (T_x, T_y) follow *BPH* distribution with presentation (π, Q) . As mentioned in Assaf et al. (1984) the individual lifetimes will follow *PH* distribution with representation (π_x, Q_x) and (π_y, Q_y) , respectively, where

$$\pi_x = (\pi_0, \mathbf{0}), \quad \pi_y = (\pi_0, \mathbf{0}),$$

and

$$Q_x = \begin{bmatrix} Q_0 & Q_{01} \\ \mathbf{0} & Q_1 \end{bmatrix}, \quad Q_y = \begin{bmatrix} Q_0 & Q_{02} \\ \mathbf{0} & Q_2 \end{bmatrix}.$$

Also, it's easy to note that $\min(T_x, T_y)$ follows univariate *PH* with representation (π_0, Q_0) . Now we will take advantages of the properties of *PH*(*BPH*) distribution's and of the underlying Markov process, $\{Z_t, t \geq 0\}$, to derive essential actuarial quantities. As we will see all the quantities have closed-form expressions. Some of these quantities are defined as follows.

- ${}_t p_{x:y}^{00} = Pr[(x) \text{ and } (y) \text{ are both alive in } t \text{ years.}]$
- ${}_t p_x = Pr[(x) \text{ is alive in } t \text{ years.}]$

- ${}_t p_y = Pr[(y) \text{ is alive in } t \text{ years.}]$
- ${}_t p_{x:y}^{01} = Pr[(y) \text{ dies within } t \text{ years and } (x) \text{ is alive at } t.]$
- ${}_t p_{x:y}^{02} = Pr[(x) \text{ dies within } t \text{ years and } (y) \text{ is alive at } t.]$

The ${}_t p_{x:y}^{00}$ can be derived as following:

$$\begin{aligned} {}_t p_{x:y}^{00} &= Pr(J_t \in \Gamma_0 | J_0 \in \Gamma_0) = Pr(J_t \in \Gamma_0) \\ &= Pr(T_x > t, T_y > t) \\ &= Pr[\min(T_x, T_y) > t] \\ &= \pi_0 \exp\{Q_0 t\} e. \end{aligned}$$

Also, by using definition of ${}_t p_x$ we have:

$${}_t p_x = Pr(T_x > t) = \pi_x \exp\{Q_x t\} e.$$

where ${}_t p_y$ can be derived analogous to ${}_t p_x$. According to Dickson et al. (2013), the probabilities ${}_t p_{x:y}^{01}$ and ${}_t p_{x:y}^{02}$ are given by:

$${}_t p_{x:y}^{01} = {}_t p_x - {}_t p_{x:y}^{00}$$

$${}_t p_{x:y}^{02} = {}_t p_y - {}_t p_{x:y}^{00}$$

Now, we will determine the actuarial present values (APV) of annuities and life insurance products in joint and last-survivor contracts. The definition of these quantities can be found in Dickson et al. (2013).

For the joint life annuity, APV is given by:

$$\begin{aligned} \bar{a}_{xy} &= \int_0^{\infty} e^{-\delta t} {}_t p_{x:y}^{00} dt. \\ &= \int_0^{\infty} e^{-\delta t} \pi_0 \exp\{Q_0 t\} dt. \\ &= \pi_0 \int_0^{\infty} e^{-\delta t} \exp\{Q_0 t\} dt \\ &= \pi_0 \int_0^{\infty} \exp\{(Q_0 - \delta I)t\} dt = \pi_0 (Q_0 - \delta I)^{-1} e. \end{aligned}$$

where δ is the force of interest and I is an identity matrix with an appropriate dimension. APV of a single life annuity issued for (x) , can be derived as below.

$$\begin{aligned} \bar{a}_x &= \int_0^{\infty} e^{-\delta t} {}_t p_x dt = \int_0^{\infty} e^{-\delta t} \pi_x \exp\{Q_x t\} dt \\ &= \pi_x \int_0^{\infty} e^{-\delta t} \exp\{Q_x t\} dt \\ &= \pi_x \int_0^{\infty} \exp\{(Q_x - \delta I)t\} dt = \pi_x (Q_x - \delta I)^{-1} e. \end{aligned}$$

The APV of the last survivor annuity and the reversionary annuity can be determined by (see Dickson et al. (2013) for the definition):

$$\bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy}, \quad \bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}.$$

APVs of the life insurance contracts through the relationship between the APV of a life annuity and a life insurance contracts (see Dickson et al. (2013)).

$$\bar{A}_{xy} = 1 - \delta \bar{a}_{xy},$$

$$\bar{A}_x = 1 - \delta \bar{a}_x,$$

$$\bar{A}_y = 1 - \delta \bar{a}_y,$$

and

$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}.$$

Please note that when x and y are replaced by age values, the numbers in the subscript of a_{xy} will be separated by a colon. For instance, a_{3050} is written as $a_{30:50}$.

5 Numerical Illustrations

In this part two examples are presented. In the first example, we show that the model is actually able to capture all dependencies and desired properties. In the second example, we calibrate our model by a well-known model used in the industry and the literature. All the codes in this paper are written in MATLAB2023b The MathWorks (2023).

5.1 Example 1

As we have already seen, in the proposed model all the basic quantities that are attractive to actuaries have closed-form expressions. In this section, we will compute these quantities through the derived equations from the earlier section. The computations will show that the model reflects all the dependencies of lifetimes of a couple.

Example: In this example we use results in Lin and Liu (2007) for setting some of the model parameters. We have assumed that the rate of mortality of male and female in the state 0 are the same. The parameters are given in Table 1.

We assume that the couple are of real ages 42 (husband) and 35 (wife) in this example. The physiological age at issue, *i.e.* i and j are assumed to be the mean of conditional expected value of the physiological age given the real age. The conditional probability vector of physiological given real age x is given in the following:

$$Pr[\text{physiological age} | \text{real age} = x] = \frac{\pi e^{\mathbf{Q}x}}{\pi e^{\mathbf{Q}x} \mathbf{1}}.$$

a_f	b_f	c_f	a_m	b_m	c_m	λ_c
$9.0987e - 04$	$1.8872e - 15$	6	$9.0987e - 04$	$1.8872e - 15$	6.5	.0002
λ_{in}	λ	λ_{rf}	λ_{rm}	λ_{wf}	λ_{wm}	n
2.3707	2.2	5	10	4	6	200

Table 1: Parameters values in Example 1

Figure 6 shows some quantities of interest. As stated in sub-Figures 6a, 6d and 6e, these probabilities are decreasing functions of time. The remaining probabilities ${}_t p_{42:35}^{01}$ and ${}_t p_{42:35}^{02}$ in sub-Figures 6b and 6c increase at first and after around 38 years decrease to zero. The increase part is obvious as probability of death is an increasing function of age. The decrease part is interpretable as after 38 years the probability of death for the wife increases as well. The same interpretation is true for the 6c. However, since the age of the wife is less than the age of the husband, the probability has the same shape as in 6b but lesser than it.

Other important quantities that discussed in latter section were APVs of annuities and life insurances. We can find the computed values of three different annuity contracts with different interest rates in Table 2.

Interest rate	$\bar{a}_{42:35}$	$\bar{a}_{42:35}$	\bar{a}_{35}
5%	17.4444	14.2534	16.4525
10%	10.1519	9.1281	9.8199
15%	7.0833	6.6433	6.9370

Table 2: APVs of annuities contracts

As shown in Table 2, since the payments is stopped when the first one dies, $\bar{a}_{35:42}$ has the smallest amount compared to the last survivor and single one in Table 2. This direction is opposite in the case of the corresponding life insurance contracts (see Table 3).

Interest rate	$\bar{A}_{42:35}$	$\bar{A}_{42:35}$	\bar{A}_{35}
5%	0.1489	0.3046	0.1973
10%	0.0324	0.1300	0.0641
15%	0.0100	0.0715	0.0305

Table 3: APVs of life insurance contracts

In the next step, we use conditional force of mortality, *i.e.* $\mu_{T_x|T_y}$ and $\mu_{T_y|T_x}$ to evaluate the model in reflection of the broken-heart syndrome effect. In *BPH* distributions, conditional force of mortality has closed-form expression as mentioned in Assaf et al. (1984). $\mu_{T_x|T_y}(t_x|t_y)$ equals to

$$\frac{\pi_0 e^{Q_0 t_y} Q_{01} e^{Q_1(t_x-t_y)} Q_1 \mathbf{1}}{\pi_0 e^{Q_0 t_y} Q_{01} e^{Q_1(t_x-t_y)} \mathbf{1}} \tag{5.1}$$

A similar formula is available for $\mu_{T_y|T_x}$.

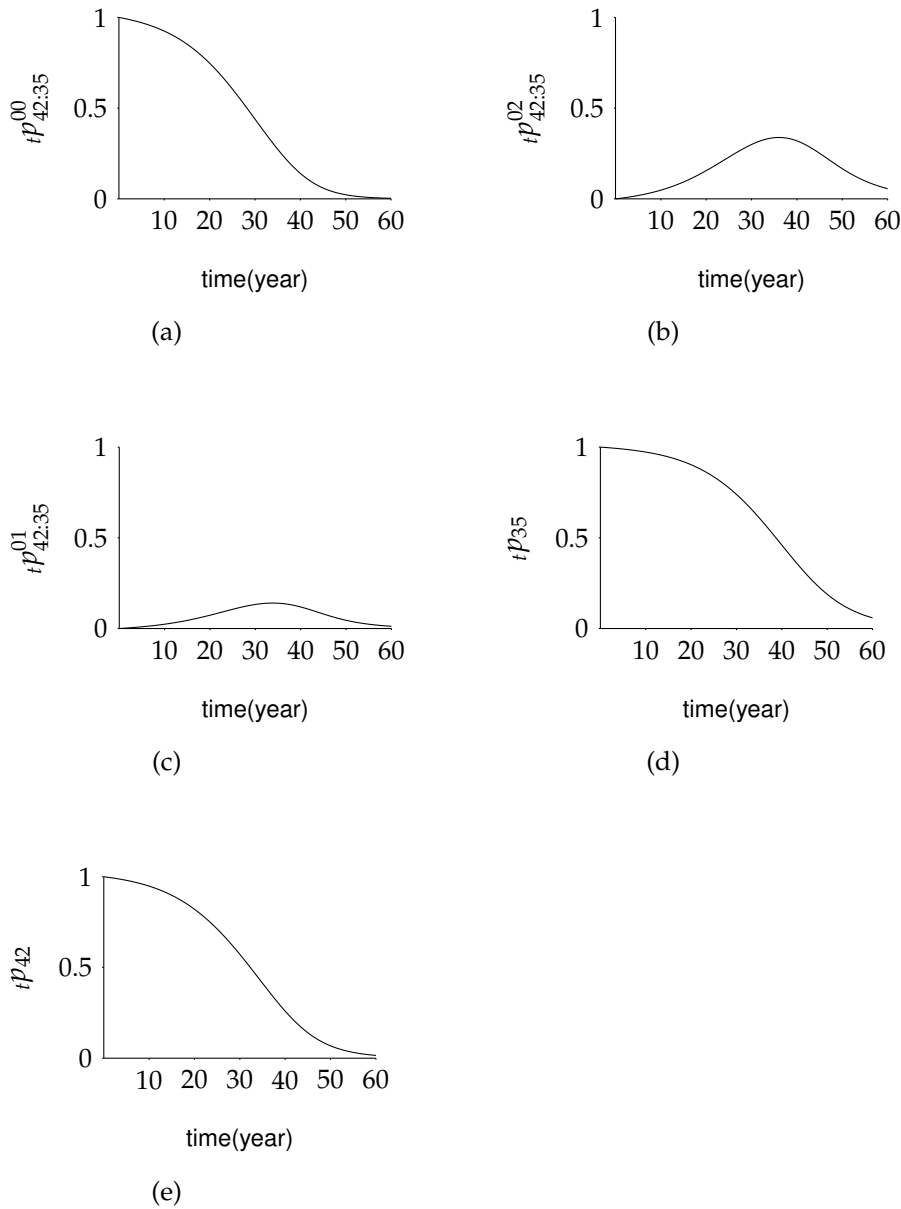


Figure 6: The probabilities plots for the husband and the wife with real ages 42 and 35, respectively.

As it is seen from figure 7, given that one of the couples dies after 20 years, the

rate of mortality for the survived couple is high after the death of his/her couple. The broken-heart syndrom effect for the husband is stronger than for the wife. Since age of the man is higher than the wife, the total rate of mortality of the husband is higher than wife's.

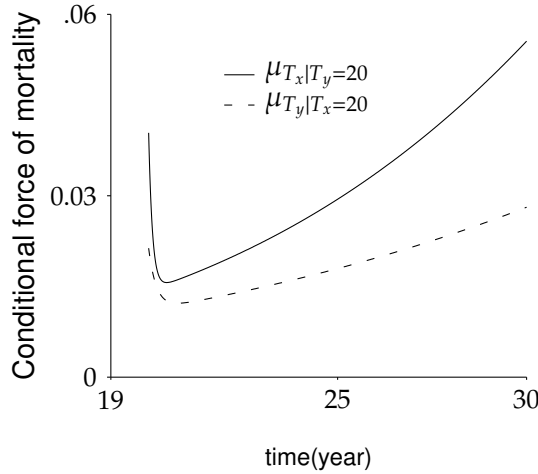


Figure 7: Conditional force of mortality

5.2 Example 2

A common model used in numerical illustrations is the model used in Dickson et al. (2013) in Example 10.9. (which we will call it *Model I*), with,

$$\mu_{x+t,y+t}^{01} = B_f c_f^{y+t}, \quad \mu_{x+t,y+t}^{02} = B_m c_m^{x+t}, \quad \mu_{x+t}^{13} = C_m d_m^{x+t}, \quad \mu_{y+t}^{23} = C_f d_f^{y+t},$$

where,

$$B_f = 9.741 \times 10^{-7}, c_f = 1.1331, B_m = 2.622 \times 10^{-5}, c_m = 1.0989, \\ C_m = 3.899 \times 10^{-4}, d_m = 1.0725, C_f = 2.638 \times 10^{-05}, d_f = 1.1020.$$

In order to estimate the parameters of our model, we have used the least squares method. We use a modified version of fminsearch (constrained version) function in Matlab to minimize the following function:

$$\sum_{t=1}^{60} \left[{}_tP_{55:50}^{00} - {}_tP_{55:50}^{00} \right]^2 + \left[{}_tP_{55:50}^{01} - {}_tP_{55:50}^{01} \right]^2 + \left[{}_tP_{55:50}^{02} - {}_tP_{55:50}^{02} \right]^2. \quad (5.2)$$

While we could add more terms to the optimization function (5.2), we realized adding more terms will not improve the accuracy of the results. Due to simplicity of the model, we can reduce the number of parameters in our model. Therefore, we revised our model as follows:

1. We consider a couple in with ages 55 (male) and the woman is 50 (female) to fit and comparison.
2. In Phase 0, we assumed $a = a_f = a_m$, $b = b_f = b_m$, and $c = c_f = c_m$. Basically, the Equation (3.2) changes to $a + bi^c + bj^c$.
3. Instead of using the same rate as the joint phase with a multiplication, we use different parameters for men and women, which we denote by a_{wm}, b_{wm}, c_{wm} for widowers and a_{wf}, b_{wf}, c_{wf} for widows.

Model I is properly fitted by our model. The estimated parameters are given in the Table 4. The graphs in 8 show the fitted and original probabilities of Model I for the most important probabilities in pricing.

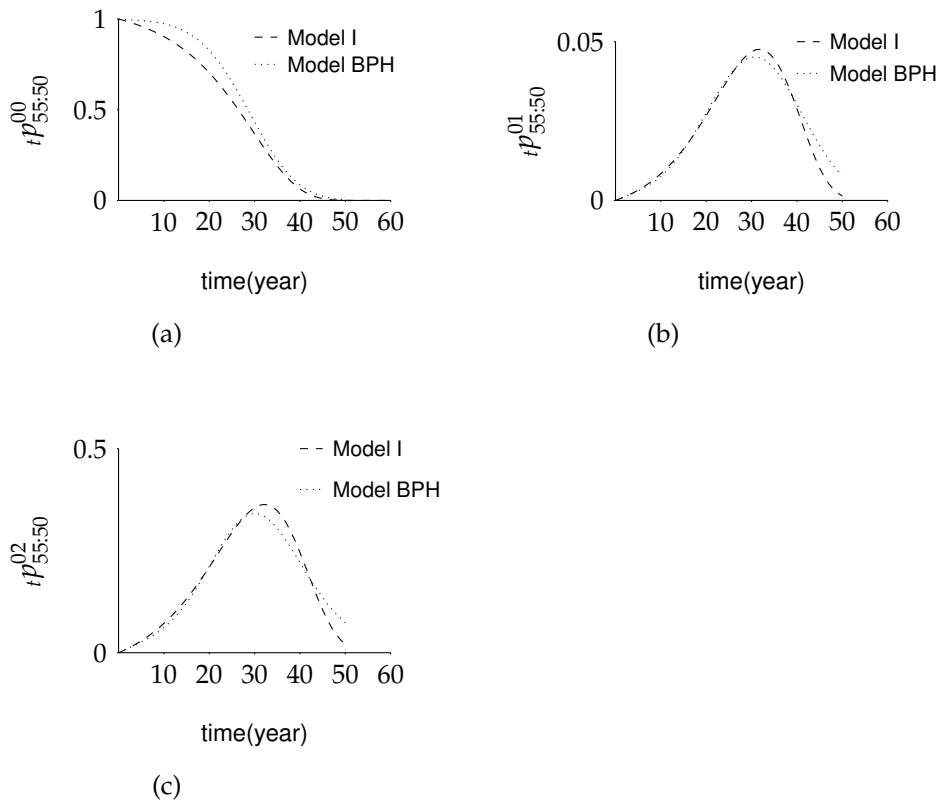


Figure 8: The probability plots for Example 2.

It is worth mentioning that pricing life insurance and annuities under *Model I* leads to integral forms, whereas using the BPH model results in closed-form solutions for pricing.

a	b	c	λ	λ_{ind}	λ_{rm}	λ_{rf}
$2.38e - 4$	$4.83e - 08$	3.31	2.16	2.74	3.087	2.68
a_{wm}	b_{wm}	c_{wm}	a_{wm}	b_{wm}	c_{wm}	
$1.71e - 05$	$1.24e - 03$	2.19	$3.92e - 03$	$2.49e - 03$	2.47	

Table 4: Parameters values in Example 2

6 Conclusions and Future of the Research

In this paper, we have presented a new model for modelling the future lifetime of a couple. Based on this model, the future lifetime follows a bivariate phase-type distribution. The model reflects dependence between future lifetimes of the wife and the husband. Some actuarial quantities are obtained. The work in this article can be developed by estimation of the parameters. The EM algorithm developed by (Asmussen 1996) is not appropriate as the E-step will be very slow. Another interesting topic can be studying dependence structure of the proposed model. Based of our simulation the correlation coefficient of random variable varies in a range of $[-\frac{1}{3}, 1]$. However, we could not prove it in theory.

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Appendix 1

The following function has written in MATLAB2023b to generate the maximum of minimum of the correlation of the bivariate phase type distribution introduced in Assaf et al. (1984).

```

function[corr_min, corr_max] = CorrSim(n_sim, m, n, k)
% n_sim : number of simulations
% m : size of matrix Q0
% n : size of matrix Q1
% k : size of matrix Q2
corr_vec = zeros(1, n_sim);
forr = 1 : n_sim
A0 = zeros(m, m); B1 = zeros(m, n); B2 = zeros(m, k); A1 = zeros(n, n); A2 = zeros(k, k);
for a = 1 : m, forb = 1 : m, A0(a, b) = rand(); end, end
for s = 1 : m; for d = 1 : n, B1(s, d) = rand(); end; end for s = 1 : m; for d = 1 :
k, B2(s, d) = rand(); end; end
for a = 1 : m; A0(a, a) = 0; A0(a, a) = -(sum(A0(a, :)) + sum(B1(a, :)) + sum(B2(a, :)) +
rand()); end
for a = 1 : n, forb = 1 : n, A1(a, b) = rand(); end, end
for a = 1 : k, forb = 1 : k, A2(a, b) = rand(); end, end
for a = 1 : n; A1(a, a) = 0; A1(a, a) = -(sum(A1(a, :)) + rand()); end for a = 1 : k; A2(a, a) =
0; A2(a, a) = -(sum(A2(a, :)) + rand()); end p = rand(1, m);
p0 = p/sum(p);
pi1 = [p0, zeros(1, n)];
pi2 = [p0, zeros(1, k)];
AX1 = [A0, B1; zeros(n, m), A1];
AX2 = [A0, B2; zeros(k, m), A2];
EX1 = -pi1 * (AX1 * ones(m + n, 1));
EX12 = 2 * pi1 * inv(AX1) * inv(AX1) * ones(m + n, 1);
VarX1 = EX12 - (EX1)^2;
EX2 = -pi2 * inv(AX2) * ones(m + k, 1);
EX22 = 2 * pi2 * inv(AX2) * inv(AX2) * ones(m + k, 1);
VarX2 = EX22 - (EX2)^2;

EX1X2 = p0 * inv(A0) * inv(A0) * (2 * ones(m, 1) - B1 * inv(A1) * ones(n, 1) - B2 * inv(A2) *
ones(k, 1));
corr_vec(r) = (EX1X2 - EX1 * EX2)/(sqrt(VarX1 * VarX2));
end
    
```

$$\begin{aligned} \text{corr}_{\min} &= \min(\text{corr}_{\text{vec}}); \\ \text{corr}_{\max} &= \max(\text{corr}_{\text{vec}}); \end{aligned}$$