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# On A Model For Bivariate Left Censored Data - Characterizations and its Applications

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**Abstract.** In this paper, we propose a class of bivariate distributions as a general solution to a functional equation. This general class of distributions proposed includes many well studied bivariate distributions. It also enjoys a proportional reversed hazards model for the distribution of the component-wise maxima. Characterizations of this general class based on a functional equation, conditional mean and conditional variance are studied. The simulation algorithm to generate bivariate pairs from the members of this general class is provided. It is shown that these properties find applications in developing simple univariate procedures in lieu of complicated bivariate goodness of fit procedures for members of the proposed class. The univariate goodness of fit procedure for the American Football dataset of the National Football League has been illustrated.

**Keywords.** Bivariate Distributions, Bivariate Proportional Reversed Hazard Model (BPRHM), Characterization, Functional Equations, Proportional Reversed Hazard Rate (PRH rate).

**MSC:** 62E10.

## 1 Introduction

Left censoring refers to a situation where a random variable is observable only when it is greater than or equal to a censoring variable $C_l$, otherwise $C_l$ is observed. Some left-censored data are represented by a pair of random variables $(Y,\delta)$ where $Y = \max\{T,C_l\}$ and $\delta = 1$ if $C_l \leq T$ and 0 if $C_l > T$. Left-censored data are extensively used in reliability and survival studies with occurrences in life-test scenarios where an item fails before its initial inspection. They are also

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prevalent in environmental and bio-monitoring research where observations may fall below a threshold known as the limit of detection (LOD).

Discarding the non-detected values for estimating parameters in a left-censored dataset is a naive approach and thus many alternative techniques for handling left-censored data have been proposed by many authors such as $\beta$-substitution method (Ganser and Hewett (2010)), model based imputation techniques (Krishnamoorthy et al. (2009)), and reverse algorithm of the Kaplan-Meier (Ware and Demets (1976)). But the conventional way is to estimate the parameters using the maximum likelihood estimation technique where the observed data contributes to the likelihood function through the probability density function (PDF) and the left-censored data contributes through the cumulative distribution function (CDF).

Another popular tool used to analyse the left-censored datasets is reversed hazard rate which was proposed by Barlow et al. (1963) as a dual to the hazard rate. The reversed hazard rate has been studied extensively by Ware and Demets (1976), Keilson and Sumita (1982) and Block et al. (1998). Nanda et al. (2003) highlighted the application of the reversed hazard rate function in Forensic Science, where the precise time of failure (such as death in the case of humans) is crucial. Additionally, the reversed hazard rate is used in the analysis of datasets in economics and financial studies, where determining the exact time of an event or outcome is also important.

Let $Y$ be an absolutely continuous random variable with $a = \inf\{y \mid F(y) > 0\}$ and $b = \sup\{y \mid F(y) < 1\}$. Let $I_{[a,b]}$ where $I_{[a,b]} = -\infty \leq a < b \leq \infty$ is the interval of support of $Y$ with the distribution function $F(y)$. The reversed hazard rate of $Y$, denoted as $r(y)$, is defined for $y > a$ as,

$$r(y) = \frac{f(y)}{F(y)} = \frac{d}{dy} \ln F(y),$$

where $f(y) = dF(y)$.

Gupta et al. (1998) proposed the proportional reversed hazard (PRH) model which is expressed as,

$$r(y) = \theta r_0(y), \tag{1}$$

where $\theta > 0$ and $r_0$ is the baseline reversed hazard rate with the corresponding distribution function as

$$F(y) = [F_0(y)]^{\theta},$$

where $F_0(y)$ is the baseline distribution function. The PRH model in (1) is helpful in the analysis of left-censored or right-truncated data. The PRH model has some extremely interesting properties, one of which is the importance of the parameter '$\theta$' in maintaining the structural properties of the baseline distribution and managing the skewness of the distribution. Additionally, the PRH model has been extensively studied for its ageing and relative ageing properties, as seen in Di Crescenzo (2000). For recent works on this model, see Balakrishnan et al. (2021) and Popović et al. (2021).

When univariate ideas are extended to the bivariate setup, there could be more than one extension since we need to capture the inherent dependence between the random variables. Hence there are more than one extension of the PRH model in the higher dimensions. A popular bivariate extension of PRH models is the one proposed by Kundu and Gupta (2010) where the marginals follow univariate PRH model and is specified by

$$F(y_1, y_2) = \begin{cases} (F_0(y_1))^{\alpha_1}(F_0(y_2))^{\alpha_2+\alpha_3} & \text{if } y_1 > y_2, \\ (F_0(y_1))^{\alpha_1+\alpha_3}(F_0(y_2))^{\alpha_2} & \text{if } y_1 < y_2, \\ (F_0(y))^{\alpha_1+\alpha_2+\alpha_3} & \text{if } y_1 = y_2 = y, \end{cases} \tag{2}$$

for $\alpha_i > 0$; i=1,2,3. We refer (2) as Bivariate Proportional Reversed Hazard Model 1 and denote as $BPRHM1(F_0, \alpha_1, \alpha_2, \alpha_3)$. Vasudevan and Asha (2023) extended the notion of PRH model to capture the lifetime behaviour of a two-component system incorporating the dependence they enjoy when the observations are left-censored. If $(Y_1, Y_2)$ forms a bivariate left-censored data, the model is specified by,

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \theta_1'\theta_2 f_0(y_1) f_0(y_2) [F_0(y_1)]^{\theta_1'-1} [F_0(y_2)]^{\theta_1+\theta_2-\theta_1'-1}, & a < y_1 < y_2 < b, \\ \theta_1\theta_2' f_0(y_1) f_0(y_2) [F_0(y_1)]^{\theta_1+\theta_2-\theta_2'-1} [F_0(y_2)]^{\theta_2'-1}, & a < y_2 < y_1 < b, \end{cases} \tag{3}$$

for $\theta_i, \theta_i' > 0$; i=1,2. We refer (3) as Bivariate Proportional Reversed Hazard Model 2 and denote as $BPRHM2(F_0, \theta_1, \theta_2, \theta_1', \theta_2')$. The models in (2) and (3) have a univariate PRH model for $Y = \max(Y_1, Y_2)$. For other extensions, see Roy (2002), Sankaran and Gleeja (2008), Dolati et al. (2014).

Despite their widespread applications, existing methods for analyzing left-censored data often rely on complex bivariate models, which can be computationally demanding and difficult to implement in practice. Motivated by this challenge, in this paper we propose a novel class of bivariate distributions as a general solution to a functional equation. This class not only encompasses several well-studied bivariate distributions, including (2) and (3), but also provides a unified framework for modeling left-censored data effectively. We develop key characterizations of this class based on the underlying functional equation and explore its properties through moments. Importantly, these properties facilitate the development of simplified univariate procedures, offering a practical alternative to traditionally complex bivariate goodness-of-fit tests. To demonstrate the applicability of our approach, we analyze the American Football dataset from the National Football League, which includes data from matches played over three consecutive weekends in 1986 (Csörgő and Welsh, 1989).The rest of the paper is organised as follows.

In Section 2, we propose a class of distributions based on a general functional equation. This class of distributions enjoys a PRH model for the distribution of component-wise maxima. Characterizations of this bivariate class of distributions based on functional equations, conditional mean and conditional variance are also derived in this section. In Section 3, we illustrate a simulation procedure for members of the proposed class of distributions based on the characterizations developed in Section 2. In Section 4, we illustrate the univariate goodness of fit procedure for the American Football dataset (Csörgő and Welsh (1989)) and show that the results are consistent with that in Kundu and Gupta (2010). The final Section 5 gives a discussion of the work done in this paper.

## 2 Proposed Model and Characterization Properties

Consider a bivariate random pair $\mathbf{Y} = (Y_1, Y_2)$ where $Y = \max\{Y_1, Y_2\}$ has a univariate proportional reversed hazard rate with proportionality parameter $\theta$ as in (1). Suppose that $(Y_1, Y_2)$ has the bivariate CDF $F(y_1, y_2)$ with support $I_{[a,b]} \times I_{[a,b]}$. Also assume that $F(y_1, y_2)$ can be written in the form $F(y_1, y_2) = G(F_0(y_1), F_0(y_2))$, where $G$ and $F_0$, are such that,

$$\begin{aligned} F(y_1, \infty) =& G(F_0(y_1), F_0(\infty)) = G(F_0(y_1), 1) = F_{Y_1}(y_1), \\ F(\infty, y_2) =& G(F_0(\infty), F_0(y_2)) = G(1, F_0(y_2)) = F_{Y_2}(y_2), \\ F(\infty, \infty) =& G(F_0(\infty), F_0(\infty)) = G(1, 1) = 1. \end{aligned} \tag{4}$$

Note that the function $G$ is a valid bivariate distribution of $(F_0(Y_1), F_0(Y_2))$ on $([0,1]\mathrm{x}[0,1])$ but is not a copula since $G(1, F_0) = G(F_0, 1) \neq F_0$.

**Theorem 2.1.** *Under the conditions in* (4)*, the distribution function* $F(y_1, y_2) = G(F_0(y_1), F_0(y_2))$ *satisfies the functional equation,*

$$G(F_0(y)F_0(y_1), F_0(y)F_0(y_2)) = G(F_0(y_1), F_0(y_2))G(F_0(y), F_0(y)), \tag{5}$$

*for all* $0 < F_0(y), F_0(y_1), F_0(y_2) < 1$*, if and only if it is of the form,*

$$G(F_0(y_1), F_0(y_2)) = F(y_1, y_2) = \begin{cases} [F_0(y_1)]^\theta G\Big(1, \frac{F_0(y_2)}{F_0(y_1)}\Big), & \textit{if } \ y_1 \geq y_2, \\ [F_0(y_2)]^\theta G\Big(\frac{F_0(y_1)}{F_0(y_2)}, 1\Big), & \textit{if } \ y_1 \leq y_2, \end{cases} \tag{6}$$

*for some y,* $\theta > 0$*.*

*Proof.* From the conditions in (4) and $y_1 = y_2 = y$

$$G(F_0(y), F_0(y)) = F(y, y) = [F_0(y)]^\theta,$$

for some $\theta > 0$. Therefore, (5) can be written as,

$$G(F_0(y)F_0(y_1), F_0(y)F_0(y_2)) = G(F_0(y_1), F_0(y_2))[F_0(y)]^\theta.$$

Hence, for $y_1 \geq y_2$,

$$\begin{aligned} G(F_0(y_1), F_0(y_2)) =& G\Big(F_0(y_1)F_0(\infty), F_0(y_1)\frac{F_0(y_2)}{F_0(y_1)}\Big) \\ =& [F_0(y_1)]^\theta G\Big(1, \frac{F_0(y_2)}{F_0(y_1)}\Big). \end{aligned}$$

Similarly proceeding for $y_1 \leq y_2$, we arrive at (6).

Conversely, from (6) for $y_1 = y_2 = y$, it follows that,

$$G(F_0(y), F_0(y)) = F(y, y) = [F_0(y)]^\theta.$$

The result is now directly established by substituting (6) in equation (5). □

We refer to this general class of distributions in (6) as Bivariate Proportional Reversed Hazard Model (*BPRHM*). The class (6) is a rich class of distributions generally. Many class of distributions existing in literature can be obtained by virtue of the marginals and baseline distributions. Note that the distribution function $F(y_1,y_2)$ in (6) of the *BPRHM* model can be written as,

$$F(y_1,y_2)=\begin{cases} [F_0(y_1)]^{\theta}F_{Y_2}\left[F_0^{-1}\left(\frac{F_0(y_2)}{F_0(y_1)}\right)\right], & \text{if } y_1\ge y_2,\\ [F_0(y_2)]^{\theta}F_{Y_1}\left[F_0^{-1}\left(\frac{F_0(y_1)}{F_0(y_2)}\right)\right], & \text{if } y_1\le y_2,\end{cases}$$

for $\theta>0$.

Assuming particular marginal distributions give rise to different classes of distributions as seen in the results below.

**Corollary 2.1.** *If the marginals $F_{Y_1}(y_1)$ and $F_{Y_2}(y_2)$ have the PRH model specified by,*

$$F_{Y_i}(y_i)=[F_0(y_i)]^{\alpha_i+\alpha_3}\ ;\ y_i>0,\ i=1,2, \tag{7}$$

*where $\alpha_i>0$; $i=1,2,3$ and rewriting $\theta=\alpha_1+\alpha_2+\alpha_3$,* (6) *reduces to* $BPRHM1(F_0,\alpha_1,\alpha_2,\alpha_3)$ *(Kundu and Gupta (2010)) in* (2) *as,*

$$F(y_1,y_2)=\begin{cases}[F_0(y_1)]^{\alpha_1}[F_0(y_2)]^{\alpha_2+\alpha_3}, & y_1\ge y_2,\\ [F_0(y_1)]^{\alpha_1+\alpha_3}[F_0(y_2)]^{\alpha_2}, & y_1\le y_2.\end{cases} \tag{8}$$

**Corollary 2.2.** *If the marginals $F_{Y_1}(y_1)$ and $F_{Y_2}(y_2)$ take the mixture form,*

$$F_{Y_i}(y_i)=\begin{cases}\frac{\theta_{3-i}}{\theta_i+\theta_{3-i}-\theta_i'}[F_0(y_i)]^{\theta_i'}+\frac{\theta_i-\theta_i'}{\theta_i+\theta_{3-i}-\theta_i'}[F_0(y_i)]^{\theta_i+\theta_{3-i}}, & \theta_i+\theta_{3-i}\neq\theta_i',\\ [F_0(y_i)]^{\theta_1+\theta_2}[1-\theta_{3-i}\ln F_0(y_i)], & \theta_i+\theta_{3-i}=\theta_i',\end{cases} \tag{9}$$

*for $a<y_i<b$; $i=1,2$ where $\theta_i,\theta_i'>0$; $i=1,2$ and rewriting $\theta=\theta_1+\theta_2$,* (6) *reduces to* $BPRHM2(F_0,\theta_1,\theta_2,\theta_1',\theta_2')$ *(Vasudevan and Asha (2023)) in* (3) *specified by,*

$$F(y_1,y_2)=\begin{cases}[F_0(y_i)]^{\theta_i+\theta_{3-i}}+\frac{\theta_{3-i}[F_0(y_i)]^{\theta_i'}}{\theta_i+\theta_{3-i}-\theta_i'}.[(F_0(y_{3-i}))^{\theta_i+\theta_{3-i}-\theta_i'}-(F_0(y_i))^{\theta_i+\theta_{3-i}-\theta_i'}], & y_i<y_{3-i},\\ & \theta_i+\theta_{3-i}\neq\theta_i',\\ [F_0(y_i)]^{\theta_i+\theta_{3-i}}[1+\theta_{3-i}(\ln F_0(y_{3-i})-\ln F_0(y_i))], & y_i<y_{3-i},\\ & \theta_i+\theta_{3-i}=\theta_i'.\end{cases} \tag{10}$$

Few interesting members of this class can be obtained by considering particular baseline distributions as seen in Corollary 2.3. Further examples are provided in Table 1.

**Corollary 2.3.** *If the baseline distribution is $F_0(y)=e^{c(y-b)}$; $c>0,\ a<y\le b,\ b<\infty$, then* (6) *becomes*

$$F(y_1,y_2)=\begin{cases} e^{c\theta(y_1-b)}F_{y_2}(b-(y_1-y_2)), & \textit{if } y_1\ge y_2,\\ e^{c\theta(y_2-b)}F_{y_1}(b-(y_2-y_1)), & \textit{if } y_1\le y_2,\end{cases}$$

*for $c > 0$, which is the general form of distribution satisfying the bivariate reversed lack of memory property (BRLMP),*

$$F(y_1, y_2)F(y, y) = F(0, 0)F(y_1 + y, y_2 + y),$$

*for all $y_i$ and $y$ such that $a < y_i - y \leq y_i \leq b,\ i = 1, 2$ (Asha and John (2007)).*

Table 1: Functional equations satisfied by different bivariate distributions

| **Baseline distribution** | **Functional equation** |
|---|---|
| Reflected Weibull (RW($c$))<br>$F_0(y) = e^{-cy^2};\ c > 0,\ y < 0$ | $F(y_1,y_2)F(y,y) = F(0,0)F(\sqrt{y_1^2+y^2},\ \sqrt{y_2^2+y^2})$ |
| Power function (PF($c$))<br>$F_0(y) = \left(\frac{y}{b}\right)^c;\ c > 0,\ 0 \leq y < b,\ b < \infty$ | $F(y_1,y_2)F(y,y) = F(1,1)F\left(\frac{yy_1}{b}, \frac{yy_2}{b}\right)$ |
| Inverse Exponential (IE($\lambda$))<br>$F_0(y) = e^{-\frac{\lambda}{y}};\ \ y > 0, \lambda > 0$ | $F(y_1,y_2)F(y,y) = F\left(\frac{y_1 y}{y_1+y}, \frac{y_2 y}{y_2+y}\right)$ |
| Inverse Weibull (IW($\alpha$))<br>$F_0(y) = e^{-y^{-\alpha}};\ \ y > 0,\ \alpha > 0$ | $F(y_1,y_2)F(y,y) =$<br>$F\left((y_1^{-\alpha}+y^{-\alpha})^{-\frac{1}{\alpha}}, (y_2^{-\alpha}+y^{-\alpha})^{-\frac{1}{\alpha}}\right)$ |
| Weibull (W($\lambda,\beta$))<br>$F_0(y) = 1 - e^{-\lambda y^\beta};\ \ y > 0, \lambda > 0, \beta > 0$ | $F(y_1,y_2)F(y,y)$<br>$= F\left(\left[\ln(e^{-\lambda y^2} + e^{-\lambda y_1^2} - e^{-\lambda(y_1^2+y^2)})^{-\frac{1}{\lambda}}\right]^{\frac{1}{\beta}},\right.$<br>$\left.\left[\ln(e^{-\lambda y^2} + e^{-\lambda y_2^2} - e^{-\lambda(y_2^2+y^2)})^{-\frac{1}{\lambda}}\right]^{\frac{1}{\beta}}\right)$ |
| Linear failure rate (LFR($\lambda,\beta$))<br>$F_0(y) = 1 - e^{-(\lambda y+\beta y^2)};\ \ y > 0, \lambda > 0, \beta > 0$ | $F(y_1,y_2)F(y,y)$<br>$= F\left(-\frac{\lambda}{2\beta} + \frac{1}{2\beta}\sqrt{\lambda^2 - 4\beta\ln(e^{-(\lambda y+\beta y^2)} + e^{-(\lambda y_1+\beta y_1^2)} - e^{-\lambda(y+y_1)-\beta(y^2+y_1^2)})},\right.$<br>$\left.-\frac{\lambda}{2\beta} + \frac{1}{2\beta}\sqrt{\lambda^2 - 4\beta\ln(e^{-(\lambda y+\beta y^2)} + e^{-(\lambda y_2+\beta y_2^2)} - e^{-\lambda(y+y_2)-\beta(y^2+y_2^2)})}\right),$ |

An interesting characterization of class of distributions in (6) based on the reversed hazard rate and moments of the distribution of $(Y_1, Y_2)$ are studied next. As mentioned earlier, the concept of reversed hazard rate when extended to the higher dimensions takes into consideration various inherent dependence enjoyed between the components. This resulted in many extensions of this concept. See Gürler (1996), Roy (2002).

Roy (2002) defined the bivariate reversed hazard rate of $\boldsymbol{Y} = (Y_1, Y_2)$ as a two component vector,

$$\lambda_{\mathbf{Y}}(y_1, y_2) = \big(\lambda_1(y_1|Y_2 \leq y_2), \lambda_2(y_2|Y_1 \leq y_1)\big), \tag{11}$$

where $\lambda_i(y_i|Y_{3-i} \leq y_{3-i}) = \lim_{\Delta y_i \to 0} \frac{P(y_i - \Delta y_i < Y_i \leq y_i | Y_1 \leq y_1, Y_2 \leq y_2)}{\Delta y_i} = \frac{\partial lnF(y_1, y_2)}{\partial y_i};\ i = 1, 2$. The definition in (11) uniquely determine the underlying distribution through

$$F(y_1, y_2) = exp\big[-\int_{y_1}^{b_1} \lambda_1(u|Y_2 \leq b_2)du - \int_{y_2}^{b_2} \lambda_2(v|Y_1 \leq y_1)dv\big], \tag{12}$$

or

$$F(y_1,y_2) = exp\big[-\int_{y_1}^{b_1}\lambda_1(u|Y_2\le y_2)du - \int_{y_2}^{b_2}\lambda_2(v|Y_1\le b_1)dv\big]. \tag{13}$$

Here, $\lambda_1(y_1|Y_2\le b_2)$ and $\lambda_2(y_2|Y_1\le b_1)$ are the marginal reversed hazard rates of $Y_1$ and $Y_2$, respectively.

Vasudevan and Asha (2023) proposed a definition of the bivariate reversed hazard rate of $\mathbf{Y}=(Y_1,Y_2)$ as a dual of the conditional hazard rate of Cox (1972) as a three component vector,

$$r_{\mathbf{Y}}(y_1,y_2) = \big(r_{\mathbf{Y}:10}(y)+r_{\mathbf{Y}:20}(y), r_{\mathbf{Y}:12}(y_1|y_2), r_{\mathbf{Y}:21}(y_2|y_1)\big), \tag{14}$$

where $r_Y(y)dy = (r_{\mathbf{Y}:10}(y)+r_{\mathbf{Y}:20}(y))dy$ with

$$r_{\mathbf{Y}:i0}(y) = \lim_{\Delta y\to 0^+}\frac{P(y-\Delta y<Y_i\le y|Y_1\le y, Y_2\le y)}{\Delta y}; y_i=y,\ i=1,2,$$

and

$$r_{\mathbf{Y}:i3-i}(y_i|y_{3-i}) = \lim_{\Delta y_i\to 0^+}\frac{P(y_i-\Delta y_i<Y_i\le y_i|Y_i\le y_i, Y_{3-i}=y_{3-i})}{\Delta y_i};\ y_i<y_{3-i},\ i=1,2.$$

If $Y_1$ and $Y_2$ are independently distributed, then $r_{\mathbf{Y}:i3-i}(y_i|y_{3-i}) = r_{Y_i}(y_i)$ and $r_{\mathbf{Y}:i0}(y) = r_{Y_i}(y)$; $i=1,2$. The bivariate reversed hazard rate $r_{\mathbf{Y}}(y_1,y_2)$ in (14) uniquely determine the underlying distribution through the equation,

$$f_{Y_1,Y_2}(y_1,y_2) = r_{\mathbf{Y}:20}(y_2)r_{\mathbf{Y}:12}(y_1|y_2)exp\Big[-\int_{y_1}^{y_2}r_{\mathbf{Y}:12}(u|y_2)du - \int_{y_2}^{b}\{r_{\mathbf{Y}:10}(u)+r_{\mathbf{Y}:20}(u)\}du\Big],$$

for $a<y_1<y_2<b$ with an analogous expression for $a<y_2<y_1<b$.

Few characterizations of the *BPRHM1* and *BPRHM2* models based on the reversed hazard rates (11) and (14) using the following definitions are given next. Let $\boldsymbol{X}=(X_1,X_2)$, $Y=\max\{Y_1,Y_2\}$ and $X=\max\{X_1,X_2\}$. Define the following functions for all $y$, $y_1$ and $y_2$ such that $F_X(y)>0$ and $F_{X_i|X_j}(y_i|y_j)>0$; $i\ne j=1,2$ as,

$$\begin{aligned}
A(y) &= -\ln F_X(y),\\
A_{ij}(y_i|y_j) &= -\ln F_{X_i|X_j=y_j}(y_i|y_j);\ y_i<y_j;\ i\le j=1,2,\\
a_{Y;n}(y) &= E[A^n(Y)|Y_1<y, Y_2<y] = E[A^n(Y)|Y<y],\\
a_{\mathbf{Y};n}(y_i|y_j) &= E[A^n_{ij}(Y_i|Y_j)|Y_i\le y_i, Y_j=y_j];\ y_i<y_j;\ i\ne j=1,2,
\end{aligned}$$

where $n$ is a positive integer and $F_X(y)$ and $F_{\mathbf{X}}(y_1,y_2)$ are the distribution functions of $X$ and $\mathbf{X}$, respectively. Now,

$$A^n(y) = (-\ln F_X(y))^n \text{ and } \frac{d}{dy}A^n(y) = -nr_X(y)A^{n-1}(y).$$

Similarly,

$$\frac{d}{dy_i}A^n_{ij}(y_i|y_j) = -nA^{n-1}_{ij}(y_i|y_j)r_{\mathbf{X}:ij}(y_i|y_j);\ i\ne j=1,2.$$

Further, define the functions for all $y_1$ and $y_2$ such that $F_{\mathbf{X}}(y_1,y_2) > 0$,

$$B(y_1,y_2) = -\ln F_{\mathbf{X}}(y_1,y_2),$$
$$b_{Y_i;n}(y_1,y_2) = E_{Y_i}[B^n(Y_1,Y_2)|Y_1 \le y_1, Y_2 \le y_2];\ i = 1,2,$$

where $n$ is a positive integer. Now,

$$\frac{d}{dy_i}B^n(y_1,y_2) = -nB^{n-1}(y_1,y_2)\lambda_{X_i}(y_i|X_2 \le y_2);\ i = 1,2.$$

Moreover, note that $A(Y)$, $A_{ij}(Y_i|Y_j)$; $i \neq j = 1,2$, and $B(Y_1,Y_2)$ are functions of the random variables $Y, Y_i|Y_j$; $i \neq j = 1,2$ and $(Y_1,Y_2)$, respectively, making themselves again random variables. Based on the above definitions, we propose few characterizations of the $BPRHM1(F_0,\alpha_1,\alpha_2,\alpha_3)$ and $BPRHM2(F_0,\theta_1,\theta_2,\theta_1',\theta_2')$ models.

**Theorem 2.2.** *The distribution function $F(y_1,y_2)$ of $(Y_1,Y_2)$ is distributed as BPRHM2 as in* (10) *if and only if,*

1. *for marginals specified as in* (9)*, $F(y_1,y_2)$ satisfies the functional equation* (5)*.*

2. *the bivariate reversed hazard rate $r_{\mathbf{Y}}(y_1,y_2)$ is of the form in* (14) *given by,*

$$r_{\mathbf{Y}}(y_1,y_2) = ((\theta_1+\theta_2)r_0(y), \theta_1' r_0(y_1), \theta_2' r_0(y_2)), \tag{15}$$

   *where $\theta_i, \theta_i' > 0$; i=1,2 and $r_0(y)$ is the reversed hazard rate corresponding to the baseline distribution.*

3. *(a)* *i.* $a_{Y;n}(y) = A^n(y) + \left(\frac{n}{\theta_1+\theta_2}\right) a_{Y;n-1}(y)$.

   *ii.* $V[A(Y)|Y < y] = \frac{1}{(\theta_1+\theta_2)^2}$.

   *(b) For any real numbers $y, y_i$ and $y_j$ such that $y_i < y_j$ and $i,j = 1,2$; $i \neq j$.*

   *i.* $a_{\mathbf{Y};n}(y_i|y_j) = A_{ij}^n(y_i|y_j) + \frac{n}{\theta_i'} a_{\mathbf{Y};n-1}(y_i|y_j)$; $i,j = 1,2,\ i \neq j$.

   *ii.* $V[A_{ij}(Y_i|Y_j)|Y_i \le y_i, Y_j = y_j] = \frac{1}{(\theta_i')^2}$; $i,j = 1,2,\ i \neq j$.

*Proof.* Note that,

1. for marginals specified as in (9), the *BPRHM2* is the only solution to the functional equation which follows from Corollary 2.2.

2. If $r_{\mathbf{Y}}(y_1,y_2)$ is of the form (15), then the underlying bivariate density function of $(Y_1,Y_2)$ is given as in (3) which leads to the bivariate distribution function in (10). The converse is direct.

3. (a) Observe that for $Y = \max\{Y_1,Y_2\}$, $F_Y(y) = (F_0(y))^{\theta_1+\theta_2}$, a univariate PRH model with proportionality parameter $\theta_1+\theta_2$. It now follows directly from Kundu and Gupta (2004) that $a_{Y;n}(y) = E[A^n(Y)|Y < y] = A^n(y) + \left(\frac{n}{\theta_1+\theta_2}\right)E[A^{n-1}(Y)|Y < y]$ and $V[A(Y)|Y < y] = \frac{1}{(\theta_1+\theta_2)^2}$.

(b) Assume that for $i, j = 1,2;\ i \neq j$ and $y_i < y_j$, $r_{\mathbf{Y}:ij}(y_i|y_j) = \theta_i' r_{\mathbf{X}:ij}(y_i|y_j);\ \theta_i' > 0$. Then,

$$
\begin{aligned}
a_{\mathbf{Y};n}(y_i|y_j) =& E[A_{ij}^n(Y_i|Y_j)|Y_i \leq y_i, Y_j = y_j] \\
=& \frac{1}{F_{Y_i|Y_j=y_j}(y_i|y_j)} \int_a^{y_i} A_{ij}^n(t_i|y_j) f_{Y_i|Y_j=y_j}(t_i|y_j) dt_i \\
=& \Big[\frac{A_{ij}^n(t_i|y_j) F_{Y_i|Y_j=y_j}(t_i|y_j)}{F_{Y_i|Y_j=y_j}(y_i|y_j)}\Big]_a^{y_i} + \\
& \int_a^{y_i} \frac{n A_{ij}^{n-1}(t_i|y_j) r_{\mathbf{X}:ij}(t_i|y_j) F_{Y_i|Y_j=y_j}(t_i|y_j)}{F_{Y_i|Y_j=y_j}(y_i|y_j)} dt_i \\
=& A_{ij}^n(y_i|y_j) + \frac{n}{\theta_i' F_{Y_i|Y_j=y_j}(y_i|y_j)} \int_a^{y_i} A_{ij}^{n-1}(t_i|y_j) f_{Y_i|Y_j=y_j}(t_i|y_j) dt_i \\
=& A_{ij}^n(y_i|y_j) + \Big(\frac{n}{\theta_i'}\Big) a_{\mathbf{Y};n-1}(y_i|y_j).
\end{aligned}
$$

Then, we have,

$$
a_{\mathbf{Y}:1}(y_i|y_j) = A_{ij}(y_i|y_j) + \frac{1}{\theta_i'},
$$

and

$$
\begin{aligned}
a_{\mathbf{Y};2}(y_i|y_j) =& A_{ij}^2(y_i|y_j) + \frac{2}{\theta_i'} a_{\underline{Y}:1}(y_i|y_j) \\
=& \Big(A_{ij}(y_i|y_j) + \frac{1}{\theta_i'}\Big)^2 + \frac{1}{(\theta_i')^2} \\
=& \big(a_{\mathbf{Y}:1}(y_i|y_j)\big)^2 + \frac{1}{(\theta_i')^2}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
V[A_{ij}(Y_i|Y_j)|Y_i \leq y_i, Y_j = y_j] =& E[A_{ij}^2(Y_i|Y_j)|Y_i \leq y_i, Y_j = y_j] \\
& - \big[E[A_{ij}(Y_i|Y_j)|Y_i \leq y_i, Y_j = y_j]\big]^2 \\
=& a_{\mathbf{Y};2}(y_i|y_j) - \big(a_{\mathbf{Y}:1}(y_i|y_j)\big)^2 \\
=& \frac{1}{(\theta_i')^2}.
\end{aligned}
$$

Conversely, suppose that

$$
a_{\mathbf{Y};n}(y_i|y_j) = A_{ij}^n(y_i|y_j) + \Big(\frac{n}{\theta_i'}\Big) a_{\mathbf{Y};n-1}(y_i|y_j), \tag{16}
$$

and

$$
V[A_{ij}(Y_i|Y_j)|Y_i \leq y_i, Y_j = y_j] = \frac{1}{(\theta_i')^2}, \tag{17}
$$

is true for $i,j=1,2;\ i\neq j$. Then (16) implies

$$\int_a^{y_i} A_{ij}^n(t_i|y_j)f_{Y_i|Y_j=y_j}(t_i|y_j)dt_i = A_{ij}^n(y_i|y_j)F_{Y_i|Y_j=y_j}(y_i|y_j)+ \Big(\frac{n}{\theta_i'}\Big)\int_a^{y_i} A_{ij}^{n-1}(t_i|y_j)f_{Y_i|Y_j=y_j}(t_i|y_j)dt_i. \tag{18}$$

Differentiating both sides of (18) w.r.t $t_i$ yields that,

$$r_{\mathbf{X}:ij}(y_i|y_j)A_{ij}^{n-1}(y_i|y_j)F_{Y_i|Y_j=y_j}(y_i|y_j) = \Big(\frac{1}{\theta_i'}\Big)A_{ij}^{n-1}(y_i|y_j)f_{Y_i|Y_j=y_j}(y_i|y_j),$$

which implies that

$$A_{ij}^{n-1}(y_i|y_j)\Big[r_{\mathbf{X}:ij}(y_i|y_j)F_{Y_i|Y_j=y_j}(y_i|y_j)-\Big(\frac{1}{\theta_i'}\Big)f_{Y_i|Y_j=y_j}(y_i|y_j)\Big]=0.$$

Since $A_{ij}^{n-1}(y_i|y_j)\neq 0$, we have, $r_{\mathbf{Y}:ij}(y_i|y_j)=\theta_i' r_{\mathbf{X}:ij}(y_i|y_j)$.

Now, (17) implies

$$\frac{1}{F_{Y_i|Y_j=y_j}(y_i|y_j)}\int_a^{y_i} A_{ij}^2(t_i|y_j)f_{Y_i|Y_j=y_j}(t_i|y_j)dt_i-\Big(\frac{1}{F_{Y_i|Y_j=y_j}(y_i|y_j)}\Big)^2 \times\Big[\int_a^{y_i} A_{ij}(t_i|y_j)f_{Y_i|Y_j=y_j}(t_i|y_j)dt_i\Big]^2=\frac{1}{(\theta_i')^2},$$

hence

$$F_{Y_i|Y_j=y_j}(y_i|y_j)\int_a^{y_i} A_{ij}^2(t_i|y_j)f_{Y_i|Y_j=y_j}(t_i|y_j)dt_i-\Big[\int_a^{y_i} A_{ij}(t_i|y_j)f_{Y_i|Y_j=y_j}(t_i|y_j)dt_i\Big]^2 =\Big(\frac{F_{Y_i|Y_j=y_j}(y_i|y_j)}{\theta_i'}\Big)^2. \tag{19}$$

Differentiating both sides of (19) w.r.t $y_i$ yields that,

$$A_{ij}^2(y_i|y_j)F_{Y_i|Y_j=y_j}(y_i|y_j)+\int_a^{y_i} A_{ij}^2(t_i|y_j)f_{Y_i|Y_j=y_j}(t_i|y_j)dt_i- 2A_{ij}(y_i|y_j)\int_a^{y_i} A_{ij}(t_i|y_j)f_{Y_i|Y_j=y_j}(t_i|y_j)dt_i =\frac{2F_{Y_i|Y_j=y_j}(y_i|y_j)}{(\theta_i')^2}. \tag{20}$$

Differentiating both sides of (20) w.r.t $y_i$ yields that,

$$A_{ij}(y_i|y_j)r_{\mathbf{X}:ij}(y_i|y_j)\frac{d}{dy_i}F_{Y_i|Y_j=y_j}(y_i|y_j)-r_{\mathbf{X}:ij}(y_i|y_j)\int_a^{y_i} A_{ij}(t_i|y_j)f_{Y_i|Y_j=y_j}(t_i|y_j)dt_i =-\frac{f_{Y_i|Y_j=y_j}(y_i|y_j)}{(\theta_i')^2},$$

which gives

$$A_{ij}(y_i|y_j)F_{Y_i|Y_j=y_j}(y_i|y_j) - \int_a^{y_i} A_{ij}(t_i|y_j)f_{Y_i|Y_j=y_j}(t_i|y_j)dt_i = -\frac{f_{Y_i|Y_j=y_j}(y_i|y_j)}{(\theta_i')^2 r_{\mathbf{X}:ij}(y_i|y_j)}. \tag{21}$$

Differentiating both sides of (21) w.r.t $y_i$ yields that,

$$(\theta_i')^2 F_{Y_i|Y_j=y_j}(y_i|y_j) = \frac{1}{r_{\mathbf{X}:ij}(y_i|y_j)} \frac{d}{dy_i}\Big[\frac{f_{Y_i|Y_j=y_j}(y_i|y_j)}{r_{\mathbf{X}:ij}(y_i|y_j)}\Big]. \tag{22}$$

Now, (22) can be written in the form of a second order homogeneous differential equation as,

$$\frac{d^2}{dy_i^2}\Big[F_{Y_i|Y_j=y_j}(y_i|y_j)\Big] - \frac{\frac{d}{dy_i} r_{\mathbf{X}:ij}(y_i|y_j)}{r_{\mathbf{X}:ij}(y_i|y_j)} \frac{d}{dy_i}\Big[F_{Y_i|Y_j=y_j}(y_i|y_j)\Big] - (\theta_i')^2 r^2_{\mathbf{X}:ij}(y_i|y_j)\Big[F_{Y_i|Y_j=y_j}(y_i|y_j)\Big] = 0.$$

The general solution of this homogeneous problem is,

$$\begin{aligned} F_{Y_i|Y_j=y_j}(y_i|y_j) &= c_1 e^{\theta_i' A_{ij}(y_i|y_j)} + c_2 e^{-\theta_i' A_{ij}(y_i|y_j)} \\ &= c_1 [F_{X_i|X_j=y_j}(y_i|y_j)]^{-\theta_i'} + c_2 [F_{X_i|X_j=y_j}(y_i|y_j)]^{\theta_i'}, \end{aligned} \tag{23}$$

where $c_1$ and $c_2$ are two arbitrary constants. Also, as $y_i \to a$ it follows that

$$F_{Y_i|Y_j=y_j}(y_i|y_j) \to 0, \text{ and } F_{X_i|X_j=y_j}(y_i|y_j) \to 0.$$

Similarly for $y_i \to b$, we have

$$F_{Y_i|Y_j=y_j}(y_i|y_j) \to 1, \text{ and } F_{X_i|X_j=y_j}(y_i|y_j) \to 1.$$

Using these conditions we obtain $c_1 = 0$ and $c_2 = 1$ in (23). Hence, we have,

$$F_{Y_i|Y_j=y_j}(y_i|y_j) = [F_{X_i|X_j=y_j}(y_i|y_j)]^{\theta_i'} \implies r_{\mathbf{Y}:ij}(y_i|y_j) = \theta_i' r_{\mathbf{X}:ij}(y_i|y_j).$$

If $X_1$ and $X_2$ are independently distributed, then

$$r_{\mathbf{X}:ij}(y_i|y_j) = r_{X_i}(y_i) \text{ for } i,j = 1,2;\ i \neq j.$$

Hence, for any real numbers $y_i$ and $y_j$ such that $y_i < y_j$ and $i,j = 1,2;\ i \neq j$,

$$r_{\mathbf{Y}:ij}(y_i|y_j) = \theta_i' r_{X_i}(y_i);\ \theta_i' > 0,$$

if and only if either of the following

i. $a_{\mathbf{Y};n}(y_i|y_j) = A^n_{ij}(y_i|y_j) + \left(\frac{n}{\theta_i'}\right) a_{\mathbf{Y};n-1}(y_i|y_j)$,
or
$E[A^n_{ij}(Y_i|Y_j)|Y_i \le y_i, Y_j = y_j] = A^n_{ij}(y_i|y_j) + \frac{n}{\theta_i'} E[A^{n-1}_{ij}(Y_i|Y_j)|Y_i \le y_i, Y_j = y_j]$.

ii. $V[A_{ij}(Y_i|Y_j)|Y_i \leq y_i, Y_j = y_j] = \frac{1}{\theta_i'^2};\ i,j = 1,2,\ i \neq j,$

where $r_{\mathbf{Y}:ij}(y_i|y_j)$ is the reversed hazard rate of $Y_i|Y_j$, is true. Hence the proof is completed.

□

**Theorem 2.3.** *The distribution function* $F(y_1,y_2)$ *of* $(Y_1,Y_2)$ *is distributed as BPRHM1 as in* (8) *if and only if,*

1. *for marginals specified as in* (7)*,* $F(y_1,y_2)$ *satisfies the functional equation* (5)*.*

2. $(Y_1,Y_2)$ *has the bivariate reversed hazard rate as defined in* (11) *given by,*

$$\lambda_{Y_i}(y_i|Y_{3-i} \leq y_{3-i}) = \begin{cases} (\alpha_i+\alpha_3)r_0(y_i), & y_i < y_{3-i}, \\ \alpha_i r_0(y_i), & y_i > y_{3-i}, \\ (\alpha_1+\alpha_2+\alpha_3)r_0(y), & y_i = y_{3-i} = y, \end{cases}$$

*for* $i = 1,2$*, where* $\alpha_i > 0;\ i = 1,2,3$ *and* $r_0(y)$ *is the reversed hazard rate corresponding to the baseline distribution.*

3. *(a)* *i.* $a_{Y;n}(y) = A^n(y) + \left(\frac{n}{\alpha_1+\alpha_2+\alpha_3}\right)a_{Y;n-1}(y).$

   *ii.* $V[A(Y)|Y < y] = \frac{1}{(\alpha_1+\alpha_2+\alpha_3)^2}.$

   *(b) For any real numbers* $y, y_i$ *and* $y_j$ *such that* $y_i < y_j$ *and* $i,j = 1,2;\ i \neq j$

   *i.* $b_{Y_1;n}(y_1,y_2) = B^n(y_1,y_2) + \frac{n}{\alpha_1+(j-1)\alpha_3}b_{Y_1;n-1}(y_1,y_2),$
   $b_{Y_2;n}(y_1,y_2) = B^n(y_1,y_2) + \frac{n}{\alpha_2+(i-1)\alpha_3}b_{Y_2;n-1}(y_1,y_2).$

   *ii.* $V_{Y_1}[B(Y_1,Y_2)|Y_1 \leq y_1, Y_2 \leq y_2] = \frac{1}{(\alpha_1+(j-1)\alpha_3)^2},$
   $V_{Y_2}[B(Y_1,Y_2)|Y_1 \leq y_1, Y_2 \leq y_2] = \frac{1}{(\alpha_1+(i-1)\alpha_3)^2}.$

*Proof.* The equivalency of statements 1 and 2 of Theorem 2.3 can be proved using the unique representation of distribution function by Roy (2002) given in (12) or (13). Also, part (a) of statement 3 of Theorem 2.3 follows similarly as the part (a) of statement 3 in Theorem 2.2. Now, assume that for $i = 1,2$,

$$\lambda_{Y_i}(y_i|Y_{3-i} \leq y_{3-i}) = \begin{cases} (\alpha_i+\alpha_3)\lambda_{X_i}(y_i|X_{3-i} \leq y_{3-i}), & y_i < y_{3-i}, \\ \alpha_i\lambda_{X_i}(y_i|X_{3-i} \leq y_{3-i}), & y_i > y_{3-i}. \end{cases}$$

Then, the results in part (b) of statement 3 follows proceeding similarly as in the proof of part (b) of statement 3 in Theorem 2.2. □

## 3 Simulations

The distribution function $F(y_1,y_2)$ in (6) of the *BPRHM* model can be written as,

$$F(y_1,y_2) = \begin{cases} [F_0(y_1)]^{\theta} F_{Y_2}\left[F_0^{-1}\left(\frac{F_0(y_2)}{F_0(y_1)}\right)\right], & \text{if } y_1 \geq y_2, \\ [F_0(y_2)]^{\theta} F_{Y_1}\left[F_0^{-1}\left(\frac{F_0(y_1)}{F_0(y_2)}\right)\right], & \text{if } y_1 \leq y_2, \end{cases} \tag{24}$$

for some y, $\theta > 0$. Note that $Y = \max\{Y_1,Y_2\}$ is a PRH model specified by $(F_0(y))^{\theta}$. Also, in (24), $F(y_1,y_2)$ is written as the product of two independent univariate random variables $Y$ and $Z_i|Y_i > Y_{3-i}$ where $Z_i = F_0^{-1}\left(\frac{F_0(y_{3-i})}{F_0(y_i)}\right)$; $i = 1,2$. Hence we can generate a bivariate pair $(Y_1,Y_2)$ by simulating $Y$ and $Z_i$; $i = 1,2$ and using the probability $P(Y_i > Y_{3-i})$; $i = 1,2$ for a given bivariate distribution belonging to the BPRHM class. Making use of these results, the following algorithm can be used for generating a bivariate random sample of size $n$ from the *BPRHM* model. The samples generated are then verified using the bivariate Kolmogorov-Smirnov test (Justel et al. (1997)).

Algorithm:

- Generate three independent uniform $(0,1)$ random variables $U_i$, $i = 1,2,3$.
- For a prefixed censoring percentage $p$ in the population, generate the censoring times $c_1$ and $c_2$ such that the random variable $c_i \sim U(0,z_i)$; $i = 1,2$ where $z_1$ and $z_2$ are derived by solving $P[0 \leq Y_i \leq c_i, 0 \leq c_i \leq z_i] = p$; $i = 1,2$.
- For a given baseline $F_0$, if $U_1 \geq P(Y_1 > Y_2)$, set
$$t_1 = F_0^{-1}(U_2^{\frac{1}{\theta}}) \text{ and } t_2 = F_0^{-1}[F_0(t_1)F_0(F_{Y_2}^{-1}(U_3))].$$
- If $U_1 < P(Y_1 > Y_2)$, set
$$t_2 = F_0^{-1}(U_2^{\frac{1}{\theta}}) \text{ and } t_1 = F_0^{-1}[F_0(t_2)F_0(F_{Y_1}^{-1}(U_3))].$$
- Set $y_1 = \max\{t_1,c_1\}$ and $y_2 = \max\{t_2,c_2\}$.
- Repeat $n$ times for a sample $(y_1,y_2)$ of size $n$ from the *BPRHM* model.

Samples of size $n = 100$ from the $BPRHM1(W(1.5,1.2),1.3,1.2,1.0)$ model with Weibull $(W(\lambda,\beta))$ as baseline distribution whose CDF is $F(y;\lambda,\beta) = 1 - e^{-\lambda y^{\beta}}$; $y > 0, \lambda > 0, \beta > 0$ were generated. The bivariate goodness of fit test (Justel et al. (1997)) is obtained as $p > 0.10$ confirming the sample. Alternatively, univariate Kolmogorov-Smirnov tests for goodness of fit for $Y = \max\{Y_1,Y_2\}$, the marginals, $Y_1$ and $Y_2$ gave consistent results in tune with Corollary 2.1. The results are presented in Table 2. Figure 1 gives the plots of theoretical and empirical distribution functions of *BPRHM1* model.

Table 2: Goodness of fit test for *BPRHM1* model with different baseline distributions

| Model | Baseline Distribution | Bivariate Kolmogorov-Smirnov Statistic | P value | Variable | Kolmogorov-Smirnov Statistic | P value |
|---|---|---|---|---|---|---|
| BPRHM1 (W(1.5,1.2),1.3,1.2,1.0) | Weibull | 0.1557 | $> 0.1000$ | Max$\{Y_1, Y_2\}$ | 0.0484 | 0.9730 |
| | | | | Marginal of $Y_1$ | 0.1096 | 0.1812 |
| | | | | Marginal of $Y_2$ | 0.1051 | 0.2195 |
| BPRHM1 (E(1.2),1.3,1.2,1.0) | Exponential | 0.2259 | $< 0.0010$ | Max$\{Y_1, Y_2\}$ | 0.1803 | 0.0030 |
| | | | | Marginal of $Y_1$ | 0.1895 | 0.0015 |
| | | | | Marginal of $Y_2$ | 0.1551 | 0.0162 |
| BPRHM1 (R(1.2),1.3,1.2,1.0) | Rayleigh | 0.2500 | $< 0.0010$ | Max$\{Y_1, Y_2\}$ | 0.1452 | 0.0294 |
| | | | | Marginal of $Y_1$ | 0.0878 | 0.4242 |
| | | | | Marginal of $Y_2$ | 0.1598 | 0.0121 |

Bivariate samples from $BPRHM2(IW(1.2), 1.2, 1.4, 1.6, 1.8)$ model with Inverse Weibull $(IW\ (\beta))$ as baseline distribution whose CDF is given by, $F(y;\beta) = e^{-y^{-\beta}};\ y > 0, \beta > 0$ were generated and illustrated similarly. The results are provided in Table 3 for uncensored samples and in Tables 4 and 5 under 20% censoring and 40% censoring, respectively. Figures 2 and 3 give the plots of theoretical and empirical distribution functions of *BPRHM2* model in the uncensored and 20% censored respective cases.

From Tables 2 and 3, the *BPRHM1* model with Weibull baseline and the *BPRHM2* model with Inverse Weibull baseline provide excellent fits to the simulated data, as confirmed by both bivariate and univariate Kolmogrov-Smirnov tests. Alternative baseline distributions (Exponential, Rayleigh, and Inverse Exponential) fail to fit the data, as indicated by their very low p-values. This supports the use of the characterization-based approach for goodness-of-fit testing, reinforcing Corollary 2.1. The results from tables 4 and 5 confirm that the Inverse Weibull distribution is the most appropriate baseline for the BPRHM2 model, maintaining goodness-of-fit even under 20% and 40% censoring. The Kolmogrov-Smirnov statistics for Inverse Weibull increase slightly when moving from 20% to 40% censoring, and the p-values decrease but remain above 0.05. This indicates that although higher censoring introduces some loss of information, Inverse Weibull still provides a valid fit, making it the most reliable choice for censored data.

Table 3: Goodness of fit test for *BPRHM2* model with different baseline distributions

| Model | Baseline Distribution | Bivariate Kolmogorov-Smirnov Statistic | P value | Variable | Kolmogorov-Smirnov Statistic | P value |
|---|---|---|---|---|---|---|
| BPRHM2 (IW(1.2),1.2,1.4,1.6,1.8) | Inverse Weibull | 0.1531 | $> 0.1000$ | Max$\{Y_1, Y_2\}$ | 0.0553 | 0.9196 |
| | | | | Marginal of $Y_1$ | 0.0826 | 0.5020 |
| | | | | Marginal of $Y_2$ | 0.1177 | 0.1254 |
| BPRHM2 (E(2.5),1.2,1.4,1.6,1.8) | Exponential | 0.8660 | $< 0.0010$ | Max$\{Y_1, Y_2\}$ | 0.7576 | $< 2.2 \times 10^{-16}$ |
| | | | | Marginal of $Y_1$ | 0.7420 | $< 2.2 \times 10^{-16}$ |
| | | | | Marginal of $Y_2$ | 0.6935 | $< 2.2 \times 10^{-16}$ |
| BPRHM2 (IE(2.0),1.2,1.4,1.6,1.8) | Inverse Exponential | 0.3613 | $< 0.0010$ | Max$\{Y_1, Y_2\}$ | 0.3668 | $4.106 \times 10^{-12}$ |
| | | | | Marginal of $Y_1$ | 0.2779 | $3.923 \times 10^{-7}$ |
| | | | | Marginal of $Y_2$ | 0.4235 | $5.551 \times 10^{-16}$ |

Table 4: Goodness of fit test for *BPRHM2* model with different baseline distributions under 20% censoring

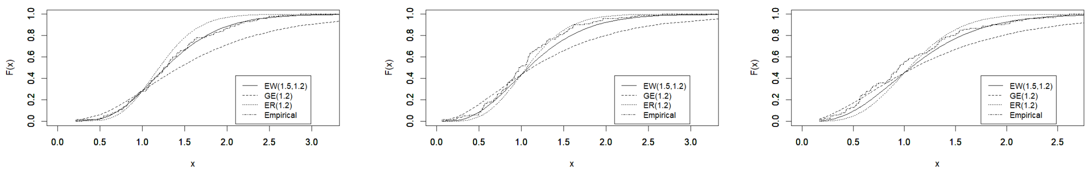
| Model | Baseline Distribution | Variable | Kolmogorov-Smirnov Statistic | P value |
|---|---|---|---|---|
| BPRHM2 (IW(2.1),1.5,1.6,2.0,1.8) | Inverse Weibull | Max$\{Y_1,Y_2\}$ | 0.0731 | 0.6596 |
| | | Marginal of $Y_1$ | 0.1292 | 0.0708 |
| | | Marginal of $Y_2$ | 0.1004 | 0.2661 |
| BPRHM2 (E(2.0),1.5,1.6,2.0,1.8) | Exponential | Max$\{Y_1,Y_2\}$ | 0.6865 | $< 2.2 \times 10^{-16}$ |
| | | Marginal of $Y_1$ | 0.7473 | $< 2.2 \times 10^{-16}$ |
| | | Marginal of $Y_2$ | 0.6652 | $< 2.2 \times 10^{-16}$ |
| BPRHM2 (IE(1.8),1.5,1.6,2.0,1.8) | Inverse Exponential | Max$\{Y_1,Y_2\}$ | 0.6475 | $< 2.2 \times 10^{-16}$ |
| | | Marginal of $Y_1$ | 0.5353 | $< 2.2 \times 10^{-16}$ |
| | | Marginal of $Y_2$ | 0.5825 | $< 2.2 \times 10^{-16}$ |

Table 5: Goodness of fit test for *BPRHM2* model with different baseline distributions under 40% censoring

| Model | Baseline Distribution | Variable | Kolmogorov-Smirnov Statistic | P value |
|---|---|---|---|---|
| BPRHM2 (IW(2.1),1.8,1.2,2.2,2.3) | Inverse Weibull | Max$\{Y_1,Y_2\}$ | 0.1106 | 0.1731 |
| | | Marginal of $Y_1$ | 0.1142 | 0.1471 |
| | | Marginal of $Y_2$ | 0.0904 | 0.3870 |
| BPRHM2 (E(1.8),1.8,1.2,2.2,2.3) | Exponential | Max$\{Y_1,Y_2\}$ | 0.6433 | $< 2.2 \times 10^{-16}$ |
| | | Marginal of $Y_1$ | 0.6616 | $< 2.2 \times 10^{-16}$ |
| | | Marginal of $Y_2$ | 0.6838 | $< 2.2 \times 10^{-16}$ |
| BPRHM2 (IE(2.1),1.8,1.2,2.2,2.3) | Inverse Exponential | Max$\{Y_1,Y_2\}$ | 0.6923 | $< 2.2 \times 10^{-16}$ |
| | | Marginal of $Y_1$ | 0.6460 | $< 2.2 \times 10^{-16}$ |
| | | Marginal of $Y_2$ | 0.6140 | $< 2.2 \times 10^{-16}$ |

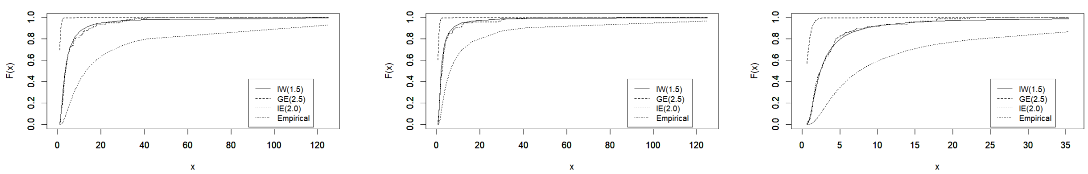
# 4 Application

In many situations, we may have to generate pairs of random variables from a continuous bivariate distribution. There exist specific simulation methods for certain bivariate distributions such as bivariate gamma, exponential and normal. However, in most scenarios we would run into multiple challenges while simulating from a bivariate distribution. The characterization result in Theorem 2.1 helps in simulating a pair of random variables whose distribution belong to the *BPRHM* class. Thus, we can test for the goodness of fit of a bivariate dataset with ease as it requires only the testing of univariate quantities. For illustrating this we use the American Football dataset of the National Football League which involves data from matches played on three consecutive weekends in 1986 (Csörgő and Welsh (1989)). Kundu and Gupta (2010) analyzed this data set and fitted a model that belongs to the *BPRHM* class given in (8) with Weibull as baseline distribution. We analyse it using the characterization result in Theorem 2.1 and this method is much simpler and aligns with the conclusion of Kundu and Gupta (2010).



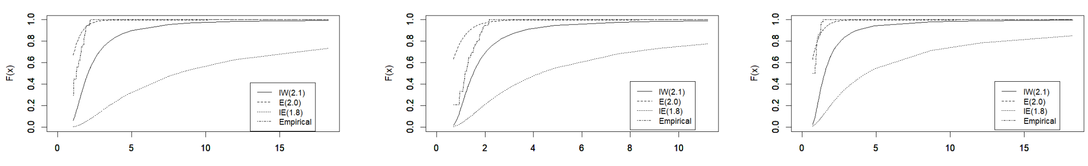

(a) Max$\{Y_1, Y_2\}$ (b) Marginal of $Y_1$ (c) Marginal of $Y_2$

Figure 1: Plots of theoretical and empirical distribution functions of *BPRHM1* model ($BPRHM1(F_0, 1.3, 1.2, 1.0)$)




(a) Max$\{Y_1, Y_2\}$ (b) Marginal of $Y_1$ (c) Marginal of $Y_2$

Figure 2: Plots of theoretical and empirical distribution functions of *BPRHM2* model ($BPRHM2(F_0, 1.2, 1.4, 1.6, 1.8)$)




(a) Max$\{Y_1, Y_2\}$ (b) Marginal of $Y_1$ (c) Marginal of $Y_2$

Figure 3: Plots of theoretical and empirical distribution functions of *BPRHM2* model under 20% censoring ($BPRHM2(F_0, 1.5, 1.6, 2.0, 1.8)$)

Here, $Y_1$ represents the game time to the first points scored by kicking the ball between goal posts and $Y_2$ represents the game time by moving the ball into the end zone. There are 42 independent pairs of observations assuming the matches within a weekend are independent and that the weekends are independent. The scoring times are given in minutes and seconds and they have been converted to decimal minutes similarly as in Csörgő and Welsh (1989) and are divided by 100 for computational purposes.

Kundu and Gupta (2010) analyzed this data set using *BPRHM1* model with the Weibull, Rayleigh, Exponential and Linear Failure Rate baseline distributions. The models with exponential and Rayleigh as baseline distributions had four parameters while the models with Weibull and linear failure rate as baseline distributions had five parameters. They estimated the parameters of the model in each case and based on the log-likelihood values, observed that among the four-parameter models (*BPRHM1* model with exponential and Rayleigh as baseline distributions), the model with Rayleigh distribution as the baseline is a better fit whereas among the five-parameter models (*BPRHM1* model with Weibull and linear failure rate as baseline distributions), the model with Weibull distribution as the baseline is a better fit. Hence, they carried out a hypothesis testing among the two better fitting models and concluded *BPRHM1* model with Weibull distribution as the preferred model.

Also notice that this distribution belongs to the *BPRHM1* class given in (8) with Weibull as baseline distribution. Analysing the fit of distribution of $Y = \max\{Y_1, Y_2\}$ and marginals of $Y_1$ and $Y_2$ using univariate Kolmogorov-Smirnov test, it is seen that $\max\{Y_1, Y_2\}$ has a Kolmogorov-Smirnov statistic value of 0.1137 with p-value 0.6093, the marginal of $Y_1$ has a Kolmogorov-Smirnov statistic value of 0.1584 with p-value 0.2426 and the marginal of $Y_2$ has a Kolmogorov-Smirnov statistic value of 0.1300 with p-value 0.4402. Hence it follows that $(Y_1, Y_2)$ has a *BPRHM1* distribution with Weibull as baseline. This is consistent with the findings of Kundu and Gupta (2010).

## 5 Discussion

A class of distributions referred as Bivariate Reversed Hazard Model (*BPRHM*) has been proposed as a general solution to the functional equation in (5) and it is specified by

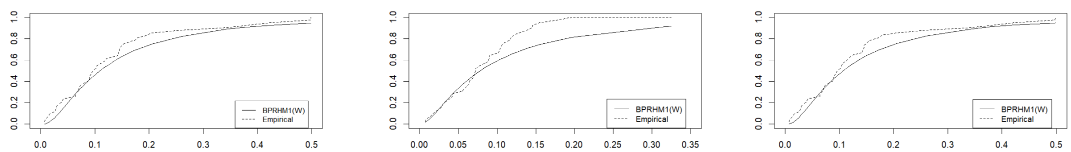
$$F(y_1, y_2) = \begin{cases} [F_0(y_1)]^{\theta} F_{Y_2}\left[F_0^{-1}\left(\frac{F_0(y_2)}{F_0(y_1)}\right)\right], & \text{if } y_1 \geq y_2, \\ [F_0(y_2)]^{\theta} F_{Y_1}\left[F_0^{-1}\left(\frac{F_0(y_1)}{F_0(y_2)}\right)\right], & \text{if } y_1 \leq y_2, \end{cases}$$

for some y, $\theta > 0$ where $F(y_1, y_2)$ has the support $I_{[a,b]} \times I_{[a,b]}$ and $F_0(\cdot)$ is the baseline distribution. This is a rich class of distributions which includes many well studied classes of distributions. Few examples are provided in Table 1. Two very well discussed classes are the *BPRHM1* (Kundu and Gupta (2010)) and *BPRHM2* classes (Vasudevan and Asha (2023)). Characterizations based on the moments of these particular classes have been also developed. The characterization Theorem 2.1 provides a characterization result that aids in the simulation of a pair of random variables whose distribution satisfy the functional equation (5). For a given baseline distribution $F_0(y_i)$, and the marginal distribution of $Y_i$, $i = 1, 2$, a random sample from a bivariate distribution belonging to the *BPRHM* class can be generated by simulating $Y = \max\{Y_1, Y_2\}$ and $Z = F_0^{-1}\left(\frac{F_0(y_{3-i})}{F_0(y_i)}\right)$; $i = 1, 2$ and using the probability $P(Y_i > Y_{3-i})$; $i = 1, 2$. The corresponding algorithm to execute the same is discussed in detail.

Further, this characterization can be effectively applied in testing goodness of fit of the member distributions belonging to the *BPRHM* class. The procedure is simplified by using univariate goodness of fit test instead of complicated bivariate procedures. We have illustrated these application using the well analysed American Football dataset of the National Football League (Csörgő and Welsh (1989)). This data was also analysed by Kundu and Gupta (2010). Based on log-likelihood criterion they concluded that the five parameter *BPRHM1* model with Weibull as baseline distribution specified by,

$$F(y_1, y_2) = \begin{cases} [1 - e^{-\lambda y_1^{\beta}}]^{\alpha_1} [1 - e^{-\lambda y_2^{\beta}}]^{\alpha_2 + \alpha_3}, & y_1 \geq y_2, \\ [1 - e^{-\lambda y_1^{\beta}}]^{\alpha_1 + \alpha_3} [1 - e^{-\lambda y_2^{\beta}}]^{\alpha_2}, & y_1 \leq y_2. \end{cases}$$

is the best fit. Instead of a bivariate Kolmogorov-Smirnov test, we can carry out univariate Kolmogorov-Smirnov tests of $Y = \max\{Y_1, Y_2\}$ and marginals of $Y_1$ and $Y_2$ in testing the goodness of fit of this model. Figure 4 shows the plots of empirical distribution functions and the distribution functions under *BPRHM1* model with Weibull as baseline of the dataset along with the corresponding Kolmogorov-Smirnov statistics and p-values.




(a) Max$\{Y_1, Y_2\}$ (Kolmogorov-Smirnov statistic: 0.1137 p-value: 0.6093)

(b) Marginal of $Y_1$ (Kolmogorov-Smirnov statistic: 0.1584 p-value: 0.2426)

(c) Marginal of $Y_2$ (Kolmogorov-Smirnov statistic: 0.1300 p-value: 0.4402)

Figure 4: Plots of empirical distribution functions and the distribution functions under *BPRHM1* model with Weibull as baseline for the American Football dataset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## Conflict of Interest Statement

The authors have no conflicts of interest to declare.

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