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Reliability Inference for Inverse Power Maxwell Distribution under Progressive Type-II Censored Sample

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Abstract. This paper investigates the reliability and parametric inference for the inverse power Maxwell distribution under progressive Type-II censored sample. Under the frequentist approach, the maximum likelihood estimate, least square, and weighted least square methods are considered for estimating the model parameters and any parametric function involved in this model. Approximate confidence intervals for parameters and any of their functions are created via a variance-covariance matrix. Bayes estimates are obtained using Lindley's approximation and Markov chain Monte Carlo (MCMC) technique under squared error loss function. Additionally, the highest posterior density (HPD) credible intervals are constructed using MCMC approximation techniques. A comprehensive Monte Carlo simulation study is conducted to assess the efficiency of the proposed methodologies. Furthermore, three optimality criteria are presented to choose the most suitable progressive scheme from various sampling plans. The practical utility of the proposed methods is demonstrated using two real-world datasets: the failure times of mechanical components and the strength of glass fiber.

Keywords. Inverse Power Maxwell Distribution, Progressive Type-II Censoring Scheme, Lindley Approximation, Metropolis-Hasting Algorithm, Highest Posterior Density Credible Interval, Optimality.

MSC: 62F15, 65C05, 62F10, 62G05.

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1 Introduction

The Maxwell distribution is widely used to model the distribution of speeds or magnitudes, particularly where random motion or equilibrium is involved. Originating from physics, it's foundational in describing particle velocities in gases at thermal equilibrium, helping predict particle behaviour in various states. Beyond physics, it has applications in fields such as reliability engineering, where it models time-to-failure for systems under continuous wear, and environmental science, which describes energy distributions in turbulent wind patterns and assists in estimating wind energy potential. In medical imaging, particularly MRI, the Maxwell distribution helps characterize tissue types by modelling signal intensities, aiding diagnostic precision. It is also applied in biological contexts, modelling diffusion patterns in cellular processes. Its ability to represent data with natural variances in speed or energy makes it valuable across these diverse domains. A significant body of literature exists on statistical inference for the Maxwell distribution, including contributions by Tyagi and Bhattacharya (1989), Bekker and Roux (2005), Dey et al. (2013). The inverse Maxwell distribution (IMD), introduced by Singh and Srivastava (2014) and reviewed by Tomer and Panwar (2020), has applications in medical science. Saghir et al. (2023) applied the IMD in optimal design and adaptive EWMA monitoring schemes, and Sindhu et al. (2019) derived Bayesian estimates of its reliability function. Yadav et al (2023) further explored the survival function of the IMD under random censoring.

The two-parameter inverse power Maxwell (IPM) distribution, introduced by Al-Kzzaz and Abd El-Monsef (2022), features an upside-down bathtub-shaped hazard rate function, making it valuable for modelling lifetime data of systems with an upside-down bathtub-shaped hazard rate, where failure risk initially increases, peaks, and then decreases. This distribution applies well to electronic components that face initial "wear-in" issues, mechanical parts (such as bearings or engine parts) that wear out over time, and medical devices (e.g., cardiac pacemakers) that may encounter early complications followed by stabilization. It is also useful in industrial equipment maintenance, where systems like power plant turbines initially adjust, peak in failure risk, and later stabilize, and in quality control for consumer products, items have an early low-risk period, then peak in returns, followed by stable performance. Additionally, the IPM distribution can model patient survival times after specific treatments, where initial recovery, risk peaks, and eventual stabilization are observed. The IPM distribution's flexibility makes it valuable in reliability studies of positively skewed lifetime data across these fields.

Recently, Irfan and Sharma (2024) derived classical and Bayesian estimates of the parameters of IPM distribution under complete sampling.

The cumulative distribution function (CDF) and a probability distribution function (PDF) of IPM distribution, are receptively given by

$$F(x;\zeta,\delta) = \frac{2}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{2},\delta x^{-2\zeta}\right); x > 0, \zeta, \delta > 0, \tag{1.1}$$

and

$$f(x;\zeta,\delta) = \frac{4}{\sqrt{\pi}}\,\zeta\delta^{\frac{3}{2}}x^{-(3\zeta+1)}\,exp(-\delta x^{-2\zeta}); x > 0, \zeta, \delta > 0, \tag{1.2}$$

where δ is scale parameter and ζ is shape parameter of IPM distribution and $\Gamma(a,b) = \int_b^{\infty}\, t^{a-1}exp(-t)\,dt$ is called upper incomplete gamma function. The survival and hazard rate function of IPM distribution are given by

$$\begin{aligned} r(t;\zeta,\delta) &= 1 - \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}, \delta t^{-2\zeta}\right) \\ &= \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \delta t^{-2\zeta}\right); t > 0, \zeta, \delta > 0, \end{aligned} \tag{1.3}$$

and

$$h(t;\zeta,\delta) = \frac{2\zeta\delta^{\frac{3}{2}}\; t^{-3\zeta-1}\; exp(-\delta t^{-2\zeta})}{\gamma\left(\frac{3}{2}, \delta t^{-2\zeta}\right)}; t > 0, \zeta, \delta > 0, \tag{1.4}$$

where $\gamma(a,b) = \int_0^{b} u^{b-1}exp(-u)du$ is called lower incompLete gamma function. The hazard rate function of the IPM distribution is shaped like an upside-down bathtub, which is very useful in reliability theory and appears in most real-world cases. Figure 1 and 2 represents the plot of PDF and hazard rate function of IPM distribution for different value of parameters.

In survival analysis and reliability studies, censoring has become essential for managing experimental data efficiently. Censored data differs from complete data as it only provides the exact failure times for a subset of units during a lifetime experiment, helping to reduce both time and cost. With advancements in science and technology, the longevity and quality of products have significantly improved, making rapid and cost-effective reliability testing more challenging. Traditional censoring schemes, like Type-I and Type-II, are frequently used in such studies but have limitations; for instance, they don't allow removing units at any stage of the experiment. To address this issue, progressive Type-II censoring (PT-II CS) scheme is developed, allowing selective withdrawal of units over time, thereby enhancing flexibility and making testing more adaptable to modern experimental needs. For more on PT-II CS, see Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014). For recent development on inference under PT-II CS, see for example, Guo and Gui (2018), Kotb and Raqab (2019), Valiollahi et al. (2022), Dey and Elshahhat (2022), Bera and Jana (2023),Bazyar et al. (2023), Sharafi et al. (2024), Khalifa et al. (2024).

The progressive censoring scheme is flexible and efficient for life-testing and reliability studies, allowing partial data collection by removing test items at scheduled intervals. This approach saves time and resources, handles incomplete data, and models real-world scenarios with long lifetimes or limited resources without compromising statistical validity.

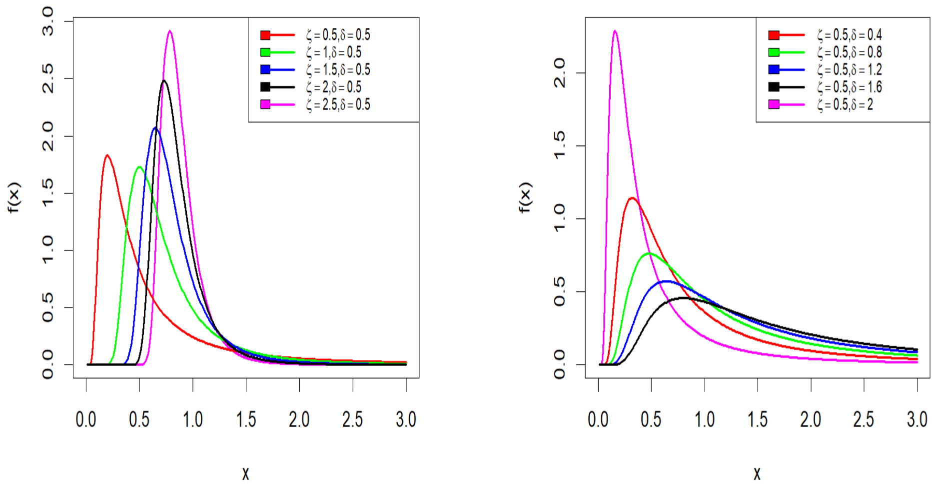


Figure 1: PDF of IPM distribution(ζ, δ).

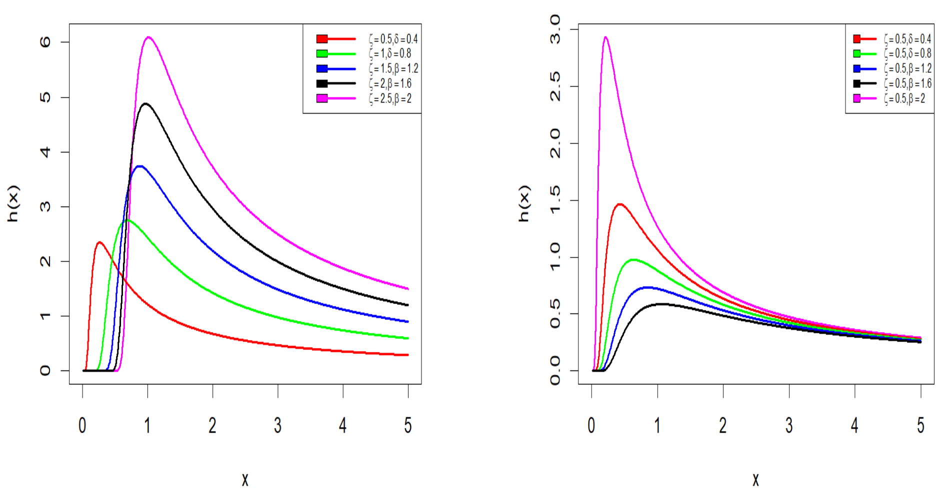


Figure 2: HRF of IPM distribution(ζ, δ).

This research addresses challenges in survival analysis with censored data using the two-parameter IPM distribution, which effectively models lifetime data with upside-down bathtub-shaped hazard rates and positively skewed distributions. Combining the progressive censoring scheme with the IPM distribution enhances reliability analysis by reducing testing time and cost, offering a robust framework for parameter estimation. Its success in fitting real datasets highlights its practical relevance. Maximum likelihood estimation (MLE) is used for parameter estimation due to its efficiency, consistency, and asymptotic normality. MLE provides unbiased estimates, even with censored data, making it well-suited for reliability studies. While least square (LS) and weighted least square (WLS) are simpler alternatives, they may not fully utilize censored data. MLE ensures precise and reliable results, especially for large datasets.

Briefly, progressive Type-II censoring scheme operates as follows: suppose n independent, identical units are placed under test, with a predetermined goal of observing m failures, where $m < n$. At the first failure (denoted $x_{1:m:n}$), R_1 units are randomly withdrawn from the $n-1$ remaining items. At the second failure (denoted $x_{2:m:n}$), R_2 of the remaining $n - R_1 - 2$ units are randomly selected and removed from the test. This process continues until the m^{th} failure (denoted $x_{m:m:n}$), at which point the remaining units, denoted $R_m = n - m - \sum_{i=1}^{m-1} R_i$, are removed. The values R_i $i = 1, 2, ..., m$ are fixed in advance, and it follows that $n = m + \sum_{i=1}^{m} R_i$. For simplicity, we refer to $x_{i:m:n}$ as x_i throughout the rest of the article. The likelihood function for this scheme can then be expressed as follows

$$l(\zeta, \delta) = c \prod_{i=1}^{m} f(x_i; \zeta, \delta|x)[1 - F(x_i; \zeta, \delta)]^{R_i}, \tag{1.5}$$

where $c = n(n - 1 - R_1) \cdots (n - \sum_{i=1}^{m-1} R_i - m + 1)$.

To the best of our knowledge, no research has been conducted on parameter estimation and parametric function inference for the IPM distribution under PT-II CS. To address this gap, the present study employs maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least squares estimation (WLSE), and Bayesian techniques to derive point and interval estimates of the unknown model parameters, as well as some reliability characteristics of the IPM distribution under PT-II CS. Due to the intractable nature of the likelihood equations involved in MLE, LSE, and WLSE, Newton's iterative method is utilized to solve these complex nonlinear equations. Approximate confidence intervals for the parameters and parametric functions are constructed using the normal approximation. Bayes estimates are computed via Lindley's approximation and Markov Chain Monte Carlo (MCMC) techniques under the squared error loss function. Given the complexity of the likelihood function, analytical computation of Bayes estimates is infeasible; hence, MCMC methods are employed to generate posterior samples from the posterior distributions. The efficacy of the proposed estimators is evaluated through a Monte Carlo simulation study, where performance metrics such as root mean square error (RMSE), relative absolute bias, and average confidence length are considered. Furthermore, various optimality criteria are proposed to identify the optimal censoring scheme. The applicability of the proposed methods is illustrated by analyzing two real datasets, highlighting their potential in practical scenarios.

The highlights of the findings of this study are given below:

- Firstly, two classical estimation techniques, maximum likelihood and least square methods, have been employed to estimate the parameters and reliability indices of the IPM distribution under PT-II CS. Further, approximate confidence intervals for the parameters and any function of them of IMP distribution under PT-II CS are also obtained.

- Secondly, Bayes estimates for the parameters have been derived using two methods such as Lindley approximation and MCMC M-H algorithm using SELF along with HPD credible intervals.

- Finally, we have obtained the optimal progressive censoring plan using three different optimality criteria.

In Section 2, the maximum likelihood estimates (MLEs), least square estimates (LSEs) and weighted least square estimates (WLSEs) of the unknown parameters and any parametric function with approximate confidence interval are obtained. Section 3 presents the derivation of Bayesian estimation along with the credible interval based on the highest posterior density (HPD). Section 4 is dedicated to the execution of a Monte Carlo simulation study. Section 5 covers the discussion of various optimality criteria for choosing an optimal progressive censoring plan. Section 6 delves into the analysis of a real-life datasets to confirm the practical applicability of the proposed techniques. Finally, the paper concludes in Section 7.

2 Frequentist Inference

This section will employ three ways to estimate the unknown model parameters and any parametric function of IPM distribution: maximum likelihood, least squares, and weighted least square estimation method. Approximate confidence for each of them is also constructed.

2.1 Maximum Likelihood Estimation

Let $x_{1:m:n}, x_{2:m:n}, ..., x_{m:m:n}$ be PT-II CS from the IPM distribution (1.2) with censoring scheme $R_1, R_2, ..., R_m$. For the shake of simplicity, we will use $x_1, x_2, ..., x_m$ instead of $x_{1:m:n}, x_{2:m:n}, ..., x_{m:m:n}$. Under the PT-II CS $x = x_1, x_2, ..., x_m$, the likelihood function for ζ and δ is given by

$$L(\zeta, \delta|x) = c \prod_{i=1}^{m} \left[f(x_i; \zeta, \delta)[1 - F(x_i; \zeta, \delta)]^{R_i} \right], \tag{2.1}$$

where $c = \prod_{i=1}^{m}(n - m + 1 - \sum_{i=1}^{m-1} R_i)$ (See Balakrishnan and Aggrawala, Balakrishnan and Aggarwala (2000)). It follows from the equations (1.1), (1.2) and (2.1), that

$$L(\zeta,\delta|x) = c\prod_{i=1}^{m}\left[\left(\frac{4}{\sqrt{\pi}}\right)\zeta\delta^{\frac{3}{2}}x_i^{-3\zeta-1}exp(-\delta x_i^{-2\zeta})\left\{1-\frac{2}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\right\}^{R_i}\right]$$
$$= \left(\frac{4}{\sqrt{\pi}}\right)^{m}\zeta^m\delta^{\frac{3m}{2}}\prod_{i=1}^{m}x_i^{(-3\zeta-1)}exp(-\delta\sum_{i=1}^{m}x_i^{-2\zeta})\prod_{i=1}^{m}\left\{\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\right\}^{R_i}. \quad (2.2)$$

Suppose that the log likelihood function of L is represented by l as

$$l = \frac{m}{2}\log\left(\frac{16}{\pi}\right)+m\log(\zeta)+\frac{3m}{2}\log(\delta)-(3\zeta+1)\sum_{i=1}^{m}\log(x_i)-\delta\sum_{i=1}^{m}x_i^{-2\zeta}+\sum_{i=1}^{m}R_i\log\left[\frac{2}{\sqrt{\pi}}\,\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\right]. \quad (2.3)$$

In order to derive the likelihood equations, we now differentiate equation (2.3) with regard to ζ and δ and and set the result equal to zero

$$\frac{\partial l}{\partial\zeta} = \frac{m}{\zeta}-3\sum_{i=1}^{m}\log(x_i)+2\delta\sum_{i=1}^{m}\left[x_i^{-2\zeta}\ \log(x_i)\right]+\sum_{i=1}^{m}R_i\frac{\gamma'_\zeta\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)}{\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)} = 0, \quad (2.4)$$

and

$$\frac{\partial l}{\partial\delta} = \frac{3n}{2\delta}-\sum_{i=1}^{n}x_i^{-2\zeta}+\sum_{i=1}^{m}R_i\frac{\gamma'_\delta\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)}{\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)} = 0, \quad (2.5)$$

where $\gamma'_\zeta\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)$ and $\gamma'_\delta\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)$ are derivative of $\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)$ with respect to ζ and δ respectively. The above two equations (2.4) and (2.5) are very complex to solve analytically. Therefore, Newton Rapson's iterative method via the "nleqslv" package in R software is implemented to obtain the MLEs of ζ and δ, respectively.

Let $\hat{\zeta}_{MLE}$ and $\hat{\delta}_{MLE}$ represent MLE's of ζ and δ respectively Then, from the invariance property of MLE, the MLE of $r(t)$, say $\hat{r}(t)_{MLE}$ and $h(t)$, say $\hat{h}(t)_{MLE}$ are respectively given by

$$\hat{r}(t)_{MLE} = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\hat{\delta}_{MLE}t^{-2\hat{\zeta}_{MLE}}\right), \quad (2.6)$$

$$\hat{h}(t)_{MLE} = \frac{2\hat{\zeta}_{MLE}\hat{\delta}_{MLE}^{\frac{3}{2}}\ t^{-3\hat{\zeta}_{MLE}-1}\ exp(-\hat{\delta}_{MLE}t^{-2\hat{\zeta}_{MLE}})}{\gamma\left(\frac{3}{2},\hat{\delta}_{MLE}t^{-2\hat{\zeta}_{MLE}}\right)};t>0. \quad (2.7)$$

2.2 Least Square Estimation Method

The least squares (LS) method is widely employed to estimate parameters in regression analysis and curve fitting. However, this method is very useful for estimating the unknown parameters of statistical models. The key idea behind LS is to find the values of parameters that minimize the sum of the squared differences between observed and predicted values. Assume that $X_1, X_2, ..., X_m$ PT-II CS drawn from the continuous distribution with CDF $F(x)$. Then according to Balakrishnan and Aggarwala (2000) we have:

$$\omega_{i:m} = E[F(X_i)] = 1 - \prod_{k=m-i+1}^{m} Q_k, \quad i = 1, 2, ..., m, \tag{2.8}$$

$$var[F(X_i)] = \left(\prod_{k=m-i+1}^{m} Q_k\right)\left(\prod_{k=m-i+1}^{m} W_k - \prod_{j=m-i+1}^{m} Q_k\right), \quad i = 1, 2, ..., m, \tag{2.9}$$

where, $k = 1, 2, ..., m$, $Q_k = \frac{B_k}{1+B_k}$, $C_k = \frac{1}{(1+B_k)(2+B_k)}$, $B_k = k + \sum_{j=m-k+1}^{m} R_j$ and $W_k = Q_k + C_k$.

Now, for IPM distribution with CDF (1.1), the least square estimates (LSEs) of ζ and δ can be computed by minimizing following equation

$$\xi(\zeta, \delta) = \sum_{i=1}^{m} [F(x_i) - w_{i:m}]^2 = \sum_{i=1}^{m} \left[\frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}, \delta x_i^{-2\zeta}\right) - \omega_{i:m}\right]^2, \tag{2.10}$$

with respect to ζ and δ. In order to find the minimum of $\xi(\zeta, \delta)$, taking the derivative of (2.10) with respect to ζ and δ and equating them to zero, the LSEs of ζ and δ is derived by solving the equations $\frac{\partial}{\partial\zeta} = 0$ and $\frac{\partial}{\partial\delta} = 0$ with respect ζ and δ. Let $\hat{\zeta}_{LSE}$ and $\hat{\delta}_{LSE}$ are the least square estimates of ζ and δ then the LSEs of $r(t)$ and $h(t)$ are given by

$$\hat{r}_{LSE}(t) = \frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \delta_{LSE} t^{-2\hat{\zeta}_{LSE}}\right), \tag{2.11}$$

and

$$\hat{h}_{LSE}(t) = \frac{2\hat{\zeta}_{LSE}\hat{\delta}_{LSE}^{\frac{3}{2}} \; t^{-3\hat{\zeta}_{LSE}-1} \; exp(-\hat{\delta}_{LSE} x^{-2\hat{\zeta}_{LSE}})}{\gamma\left(\frac{3}{2}, \hat{\delta}_{LSE} t^{-2\hat{\zeta}_{LSE}}\right)}, x > 0. \tag{2.12}$$

2.3 Weighted Least Square Estimation Method

Sometimes, the assumption of constant variance for all data points may not hold. Weighted least squares (WLS) consider this and allow for assigning different weights to different data points. The $F(x_i)$, different variance for various values of i. Like LS, WLS aims to find the values that minimize the weighted sum of squared residuals. The WLS can be obtained by minimizing:

$$\psi(\zeta,\delta)=\sum_{i=1}^{m}v_i\,[F(x_i)-\omega_{i:m}]^2=\sum_{i=1}^{m}v_i\left[\frac{2}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)-\omega_{i:m}\right]^2, \tag{2.13}$$

with respect to ζ and δ, where v_i is the weight obtained as $v_i = var[E(X_i)]^{-1}$. By solving the equations $\frac{\partial}{\partial\zeta}=0$ and $\frac{\partial}{\partial\delta}=0$ one can get the minimizing of equation (2.13). Let $\hat{\zeta}_{WLSE}$ and $\hat{\delta}_{WLSE}$ are the WLSEs of ζ and δ then the WLSEs of $r(t)$ and $h(t)$ are given by

$$\hat{r}_{WLSE}(t)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\delta_{WLSE}t^{-2\hat{\zeta}_{WLSE}}\right), \tag{2.14}$$

and

$$\hat{h}(t)_{WLSE}=\frac{2\hat{\zeta}_{WLSE}\hat{\delta}_{WLSE}^{\frac{3}{2}}\,t^{-3\hat{\zeta}_{WLSE}-1}\,exp(-\hat{\delta}_{WLSE}t^{-2\hat{\zeta}_{WLSE}})}{\gamma\left(\frac{3}{2},\hat{\delta}_{WLSE}t^{-2\hat{\zeta}_{WLSE}}\right)},t>0. \tag{2.15}$$

2.4 Approximate Confidence Interval

In order to obtain the frequentist confidence intervals for unknown parameters, the ζ and δ asymptotic properties of their MLEs based on large sample theory are considered. The $100(1-\gamma)\%$ approximate confidence intervals (ACIs) for the unknown parameters ζ and δ can be created using the MLEs $\hat{\zeta}_{MLE}$ and $\hat{\delta}_{MLE}$ from asymptotic normal distribution with mean (ζ, δ) and variance-covariance matrix $I^{-1}(\hat{\zeta}_{MLE},\hat{\delta}_{MLE})$ under the certain regularity conditions, that is

$$(\hat{\zeta},\hat{\delta})\sim N((\zeta,\delta),I^{-1}(\hat{\zeta},\hat{\delta})),$$

where $I^{-1}(\hat{\zeta}_{MLE},\hat{\delta}_{MLE})$ is the observed information matrix and defined as

$$I^{-1}(\hat{\zeta}_{MLE},\hat{\delta}_{MLE})=-\begin{bmatrix}\frac{\partial^2 l}{\partial\zeta^2} & \frac{\partial^2 l}{\partial\zeta\partial\delta}\\ \frac{\partial^2 l}{\partial\delta\partial\zeta} & \frac{\partial^2 l}{\partial^2\zeta^2}\end{bmatrix}^{-1}_{(\zeta=\hat{\zeta}_{MLE},\delta=\hat{\delta}_{MLE})}=\begin{bmatrix}var(\hat{\zeta}_{MLE}) & cov(\hat{\zeta}_{MLE},\hat{\delta}_{MLE})\\ cov(\hat{\zeta}_{MLE},\hat{\delta}_{MLE}) & var(\hat{\zeta}_{MLE})\end{bmatrix}, \tag{2.16}$$

where

$$\frac{\partial^2 l}{\partial\zeta^2}=-\frac{m}{\zeta^2}-4\delta\sum_{i=1}^{m}\left[x_i^{-2\zeta}(\log x_i)^2\right]+\sum_{i=1}^{m}R_i\frac{\gamma''_\zeta\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)-\left(\gamma'_\zeta\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\right)^2}{\left(\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\right)^2},$$

$$\frac{\partial^2 l}{\partial\delta^2}=-\frac{3m}{2\delta^2}+\sum_{i=1}^{m}R_i\frac{\gamma''_\delta\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)-\left(\gamma'_\delta\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\right)^2}{\left(\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\right)^2},$$

$$\frac{\partial^2 l}{\partial\zeta\partial\delta}=\frac{\partial^2 l}{\partial\delta\partial\zeta}=2\sum_{i=1}^{n}\left[x_i^{-2\zeta}\log(x_i)\right]+\sum_{i=1}^{m}R_i\frac{\gamma''_{\delta\zeta}\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)-\left(\gamma'_\delta\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\right)^2}{\left(\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\right)^2}.$$

Also, $\gamma''_\zeta\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)$, $\gamma''_\delta\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)$ and $\gamma''_{\delta\zeta}\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)$ are the second derivative of lower incomplete gamma function with respect to ζ, δ and $\delta\zeta$ respectively. Equation (2.16) can be expressed as a variance and covariance matrix, where the non-diagonal entries show the covariance between the $\hat{\zeta}_{MLE}$ and $\hat{\delta}_{MLE}$ and the diagonal elements show the variance of $\hat{\zeta}_{MLE}$ and $\hat{\delta}_{MLE}$.

Hence, the $100(1-\gamma)\%$ ACI of ζ and δ are derived as

$$\left(\hat{\zeta}_{MLE} - z_{\gamma/2}\sqrt{var(\hat{\zeta}_{MLE})}, \hat{\zeta}_{MLE} + z_{\gamma/2}\sqrt{var(\hat{\zeta}_{MLE})}\right),$$

and

$$\left(\hat{\delta}_{MLE} - z_{\gamma/2}\sqrt{var(\hat{\delta}_{MLE})}, \hat{\delta}_{MLE} + z_{\gamma/2}\sqrt{var(\hat{\delta}_{MLE})}\right),$$

where $z_{\frac{\tau}{2}}$ represents the upper $\frac{\tau}{2}$-th quantile for standard normal distribution.

To obtain the ACIs of $r(t)$ and $h(t)$, we need to obtain the variances of $\hat{r}(t)_{MLE}$ and $\hat{h}(t)_{MLE}$. In some circumstances, it may be challenging or computationally intensive to derive exact confidence intervals, leading to the use of approximate methods. The Delta technique is a valuable tool for approximating confidence intervals for functions of random variables, especially in cases where exact solutions are difficult to obtain. Therefore, using the delta technique, the variances of $\hat{r}(t)_{MLE}$ and $\hat{h}(t)_{MLE}$ can be roughly calculated as

$$var(\hat{r}(t)_{MLE}) = [\nabla r(t)]^T I^{-1}(\zeta,\delta)[\nabla r(t)]_{(\zeta=\hat{\zeta}_{MLE},\delta=\hat{\delta}_{MLE})}, \tag{2.17}$$

$$var(\hat{h}(t)_{MLE}) = [\nabla h(t)]^T I^{-1}(\zeta,\delta)[\nabla h(t)]_{(\zeta=\hat{\zeta}_{MLE},\delta=\hat{\delta}_{MLE})}, \tag{2.18}$$

where $\nabla r(t)$ and $\nabla h(t)$ are the gradient vector of the $r(t)$ and $h(t)$ with respect to ζ and δ calculated at their MLEs, respectively, as

$$[\nabla r(t)]^T = \left[\frac{\partial r(t)}{\partial \zeta}, \frac{\partial r(t)}{\partial \delta}\right]_{(\zeta=\hat{\zeta}_{MLE},\delta=\hat{\delta}_{MLE})} \text{ and } [\nabla h(t)]^T = \left[\frac{\partial h(t)}{\partial \zeta}, \frac{\partial h(t)}{\partial \delta}\right]_{(\zeta=\hat{\zeta}_{MLE},\delta=\hat{\delta}_{MLE})}. \tag{2.19}$$

Therefore, the $100(1-\gamma)\%$ ACI of $r(t)$ and $h(t)$, are given, respectively, by

$$\left(\hat{r}(t)_{MLE} - z_{\tau/2}\sqrt{var(\hat{r}(t)_{MLE})}, \hat{r}(t)_{MLE} + z_{\tau/2}\sqrt{var(\hat{r}(t)_{MLE})}\right)$$

and

$$\left(\hat{h}(t)_{MLE} - z_{\tau/2}\sqrt{var(\hat{h}(t)_{MLE})}, \hat{\delta}_{MLE} + z_{\tau/2}\sqrt{var(\hat{h}(t)_{MLE})}\right).$$

3 Bayesian Inference

This section addresses the Bayesian method for estimating unknown parameters and the corresponding HPD credible intervals.

3.1 Loss Function

In order to obtained Bayes estimates, we considered the symmetric loss function which is called squared error loss function (SELF) and is defined as follows

$$l_{SELF}(\tau, \hat{\tau}) = (\tau - \hat{\tau})^2, \tag{3.1}$$

where $\hat{\tau}$ is an estimated value of τ. The Bayes estimates of α under SELF can be written as follows

$$\hat{\tau}_{SElF} = E_\tau(\tau|x), \tag{3.2}$$

where $E_\tau(\tau|x)$ represents the posterior mean of τ.

3.2 Prior Information

The prior distribution is a crucial component of Bayesian estimation. If an appropriate prior is available for the unknown parameters, informative priors can be helpful for integrating the knowledge into the model. The gamma prior is chosen for the shape and scale parameters of the IPM distribution due to its flexibility, suitability for modeling positive parameters, and alignment with Bayesian analysis. It supports a range of prior beliefs, simplifies posterior inference, and aligns with established methodologies despite not being conjugate. Independent gamma priors are assumed to reduce complexity, and non-informative hyperparameters (shape = 0.001, scale = 0.001) ensure data-driven posterior inference, enhancing transparency and simplicity. Let $\zeta \sim Gamma(a_1, b_1)$ and $\delta \sim Gamma(a_2, b_2)$, respectively. Then the prior density for ζ and δ are expressed as follows

$$u_1(\zeta) = \frac{b_1^{a_1}}{\Gamma(a_1)} \zeta^{a_1-1} exp\,(-b_1\zeta)\,; \zeta \in (0, \infty), \tag{3.3}$$

and

$$u_2(\delta) = \frac{b_2^{a_2}}{\Gamma(a_2)} \delta^{a_2-1} exp\,(-b_2\delta)\,; \delta \in (0, \infty), \tag{3.4}$$

where a_i, $b_i > 0$, i = 1,2 are hyperparameters.

The following is an expression for the joint prior density of ζ and δ

$$u(\zeta, \delta) = \frac{b_1^{a_1}}{\Gamma(a_1)} \frac{b_2^{a_2}}{\Gamma(a_2)} \zeta^{a_1-1} \delta^{a_2-1} exp(-b_1\zeta - b_2\delta). \tag{3.5}$$

3.3 Posterior Analysis

The joint posterior density function of ζ and δ are computed as

$$\pi(\zeta,\delta|x) = \frac{l(x|\zeta,\delta)\pi(\zeta,\delta)}{\int_\zeta\int_\delta l(x|\zeta,\delta)\pi(\zeta,\delta)\,d\zeta d\delta}$$
$$= k^{-1}\zeta^{m+a_1-1}\delta^{\frac{3m}{2}+a_2-1}\prod_{i=1}^{m} x_i^{-3\zeta-1} exp(-\zeta b_1) exp\left(-\delta(b_2+\sum_{i=1}^{m} x_i^{-2\zeta})\right)\prod_{i=1}^{m}\left[\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\right]^{R_i}, \tag{3.6}$$

where
$k = \int_0^\infty\int_0^\infty \zeta^{m+a_1-1}\delta^{\frac{3m}{2}+a_2-1}\prod_{i=1}^{m} x_i^{-3\zeta-1} exp(-\zeta b_1) exp\left(-\delta(b_2+\sum_{i=1}^{n} x_i^{-2\zeta})\right)\prod_{i=1}^{m}\left[\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\right]^{R_i} d\zeta d\delta$
and is called the normalising constant. It is observed that joint posterior density is complex, and obtaining a close form of Bayes estimators seems tedious. So, to tackle this situation, two widely applicable approximation methods, Lindley approximation and Markov chain Monte Carlo method, are applied.

3.4 Lindley Approximation

Lindley's approximation is introduced by Lindley (1980), a method used in Bayesian estimation to approximate posterior expectations when closed-form solutions are intractable. It simplifies the computation of integrals in Bayes' theorem by expanding the log-posterior distribution as a second-order Taylor series around its mode. This approach is computationally efficient and provides accurate estimates for unimodal and relatively smooth posterior distributions. It is especially useful when full numerical integration or MCMC methods are computationally expensive.

Let $v(\zeta,\delta)$ be the function of ζ and δ; then, in general, the form of Bayes estimator for any loss function of the parameter (ζ,δ) is given as

$$E(v(\zeta,\delta)|x) = \frac{\int_\zeta\int_\delta v(\zeta,\delta)e^{l(\zeta,\delta|x)+\rho(\zeta,\delta)}d\zeta d\delta}{\int_\zeta\int_\delta e^{l(\zeta,\delta|x)+\rho(\zeta,\delta)}d\zeta d\delta}, \tag{3.7}$$

where $l(\zeta,\delta|x)$ logarithm of likelihood function and $\rho(\zeta,\delta)$ is the logarithm of joint prior distribution. Using the Lindley approximation method, $E(v(\zeta,\delta)|x)$ can be approximated as, which is required Bayes estimator

$$\begin{aligned} E(v(\zeta,\delta)|x) \approx{}& v(\hat\zeta,\hat\delta) + 0.5(\hat v_{\zeta\zeta}\hat\sigma_{\zeta\zeta} + \hat v_{\delta\delta}\hat\sigma_{\delta\delta}) + \hat v_{\zeta\delta}\hat\sigma_{\zeta\delta} + \hat v_\zeta(\hat\sigma_{\zeta\zeta}\hat\rho_\zeta + \hat\sigma_{\delta\zeta}\hat\rho_\delta) + \hat v_\delta(\hat\sigma_{\zeta\delta}\hat\rho_\zeta + \hat\sigma_{\delta\delta}\hat\rho_\delta) \\ &+ 0.5\hat l_{\zeta\zeta\zeta}(\hat v_\zeta\hat\sigma^2_{\zeta\zeta} + \hat v_\delta\hat\sigma_{\zeta\zeta}\hat\sigma_{\zeta\delta}) + 0.5\hat l_{\zeta\zeta\delta}\left(3\hat v_\zeta\hat\sigma_{\zeta\zeta}\hat\sigma_{\zeta\delta} + \hat v_\delta(\hat\sigma_{\zeta\zeta}\hat\sigma_{\delta\delta} + 2\hat\sigma^2_{\zeta\delta})\right) \\ &+ 0.5\hat l_{\zeta\delta\delta}\left(\hat v_\zeta(\hat\sigma_{\zeta\zeta}\hat\sigma_{\delta\delta} + 2\hat\sigma^2_{\zeta\delta}) + 3\hat v_\delta\hat\sigma_{\zeta\delta}\hat\sigma_{\delta\delta}\right) + 0.5\hat l_{\delta\delta\delta}(\hat v_\zeta\hat\sigma_{\zeta\delta}\hat\sigma_{\zeta\zeta} + \hat u_\delta\hat\sigma^2_{\delta\delta}), \end{aligned} \tag{3.8}$$

where $v_{\zeta\zeta}$ is the second derivative of the function $v(\zeta,\delta)$ with respect to ζ, which $\hat v_{\zeta\zeta}$ is the same expression as evaluated for $v(\hat\zeta,\hat\delta)$ and σ_{ij} is the $(i,j)^{th}$ element of matrix $[-\hat l_{ij}]^{-1}; i,j=\zeta,\delta$. Other notations are defined in similar way as

$$\hat v_\zeta = \frac{\partial v}{\partial\zeta},\quad \hat v_\delta = \frac{\partial u}{\partial\delta},\quad \hat v_{\delta\zeta} = \hat v_{\zeta\delta} = \frac{\partial^2 v}{\partial\zeta\partial\zeta},\quad \hat\rho_\zeta = \frac{\partial\log\ u(\zeta,\delta)}{\partial\zeta},\quad \hat\rho_\delta = \frac{\partial\log\ u(\zeta,\delta)}{\partial\delta}, \hat l_\delta = \frac{\partial l}{\partial\delta}, \hat l_\zeta = \frac{\partial l}{\partial\zeta},$$

$\hat{l}_{\zeta\delta} = \hat{l}_{\zeta\delta} = \frac{\partial^2 l}{\partial\zeta\delta}, \hat{l}_{\delta\delta} = \frac{\partial^2 l}{\partial\delta^2}, \hat{l}_{\zeta\zeta} = \frac{\partial^2 l}{\partial\zeta^2}, \hat{l}_{\delta\delta\delta} = \frac{\partial^3 l}{\partial\delta^3}, \hat{l}_{\zeta\zeta\zeta} = \frac{\partial^3 l}{\partial\zeta^3}, \hat{l}_{\delta\delta\zeta} = \frac{\partial^3 l}{\partial\delta^2\partial\zeta}, \hat{l}_{\delta\zeta\zeta} = \frac{\partial^3 l}{\partial\delta\partial\zeta^2}.$

Now, with respect to our problem, the second and third derivatives of $l(\zeta, \delta)$ can be obtained from the equations (2.4) and (2.5), respectively, using the "numDeriv" package in R software. If $v(\zeta, \delta) = \zeta$, then $v_\zeta = 1$, $v_{\delta\delta} = v_\delta = v_{\zeta\zeta} = v_{\delta\zeta} = v_{\zeta\delta} = 0$. Thus, the Bayes estimator of ζ under SELF is expressed as follows

$$\begin{aligned}\hat{\zeta}_{BS} &= \hat{\zeta}_{MLE} + (\hat{\rho}_\zeta\hat{\sigma}_{\zeta\zeta} + \hat{\rho}_\delta\hat{\sigma}_{\delta\zeta}) + 0.5\left[\hat{l}_{\zeta\zeta\zeta}\hat{\sigma}^2_{\zeta\zeta} + 3\hat{l}_{\zeta\zeta\delta}\hat{\sigma}_{\zeta\delta}\hat{\sigma}_{\zeta\zeta} + \hat{l}_{\zeta\zeta\delta}(\hat{\sigma}_{\delta\delta}\hat{\sigma}_{\zeta\zeta} + 2\hat{\sigma}^2_{\zeta\delta})\right.\\ &\quad \left.+\hat{l}_{\delta\delta\delta}\hat{\sigma}_{\zeta\delta}\hat{\sigma}_{\delta\delta}\right]. \end{aligned} \tag{3.9}$$

If $v(\zeta, \delta) = \delta$, then $v_\delta = 1$, $v_{\delta\delta} = v_\delta = v_{\zeta\zeta} = u_{\delta\zeta} = u_{\zeta\delta} = 0$. So, the Bayes estimator of δ under SELF is found as follows:

$$\begin{aligned}\hat{\delta}_{BS} &= \hat{\delta}_{MLE} + (\hat{\rho}_\delta\hat{\sigma}_{\delta\delta} + \hat{\rho}_\zeta\hat{\sigma}_{\zeta\delta}) + 0.5\left[\hat{l}_{\zeta\zeta\zeta}\hat{\sigma}_{\zeta\zeta}\hat{\sigma}_{\zeta\delta} + 3\hat{l}_{\zeta\delta\delta}\hat{\sigma}_{\delta\delta}\hat{\sigma}_{\zeta\delta} + \hat{l}_{\zeta\zeta\delta}(\hat{\sigma}_{\delta\delta}\hat{\sigma}_{\zeta\zeta} + 2\hat{\sigma}^2_{\zeta\delta})\right.\\ &\quad \left.+\hat{l}_{\delta\delta\delta}\hat{\sigma}^2_{\delta\delta}\right]. \end{aligned} \tag{3.10}$$

If $v(\zeta, \delta) = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \delta t^{-2\zeta}\right)$, then $v_\zeta = \frac{2}{\sqrt{\pi}}\gamma'_\zeta\left(\frac{3}{2}, \delta t^{-2\zeta}\right)$, $v_\delta = \frac{2}{\sqrt{\pi}}\gamma'_\delta\left(\frac{3}{2}, \delta t^{-2\zeta}\right)$, $v_{\delta\delta} = \frac{2}{\sqrt{\pi}}\gamma''_{\delta\delta}\left(\frac{3}{2}, \delta t^{-2\zeta}\right)$, $v_{\zeta\zeta} = \frac{2}{\sqrt{\pi}}\gamma''_{\zeta\zeta}\left(\frac{3}{2}, \delta t^{-2\zeta}\right)$, $v_{\delta\zeta} = v_{\zeta\delta} = \frac{2}{\sqrt{\pi}}\gamma''_{\zeta\delta}\left(\frac{3}{2}, \delta t^{-2\zeta}\right)$. Thus, the Bayes estimator of $r(t)$ under SELF is expressed as follows

$$\begin{aligned}\hat{r}(t)_{BS} &= \hat{r}(t)_{MLE} + 0.5(\hat{v}_{\zeta\zeta}\hat{\sigma}_{\zeta\zeta} + \hat{v}_{\delta\delta}\hat{\sigma}_{\delta\delta}) + \hat{v}_{\zeta\delta}\hat{\sigma}_{\zeta\delta} + \hat{v}_\zeta(\hat{\sigma}_{\zeta\zeta}\hat{\rho}_\zeta + \hat{\sigma}_{\delta\zeta}\hat{\rho}_\delta) + \hat{v}_\delta(\hat{\sigma}_{\zeta\delta}\hat{\rho}_\zeta + \hat{\sigma}_{\delta\delta}\hat{\rho}_\delta)\\ &+ 0.5\hat{l}_{\zeta\zeta\zeta}(\hat{v}_\zeta\hat{\sigma}^2_{\zeta\zeta} + \hat{v}_\delta\hat{\sigma}_{\zeta\zeta}\hat{\sigma}_{\zeta\delta}) + 0.5\hat{l}_{\zeta\zeta\delta}\left(3\hat{v}_\zeta\hat{\sigma}_{\zeta\zeta}\hat{\sigma}_{\zeta\delta} + \hat{v}_\delta(\hat{\sigma}_{\zeta\zeta}\hat{\sigma}_{\delta\delta} + 2\hat{\sigma}^2_{\zeta\delta})\right)\\ &+ 0.5\hat{l}_{\zeta\delta\delta}\left(\hat{v}_\zeta(\hat{\sigma}_{\zeta\zeta}\hat{\sigma}_{\delta\delta} + 2\hat{\sigma}^2_{\zeta\delta}) + 3\hat{v}_\delta\hat{\sigma}_{\zeta\delta}\hat{\sigma}_{\delta\delta}\right) + 0.5\hat{l}_{\delta\delta\delta}(\hat{v}_\zeta\hat{\sigma}_{\zeta\delta}\hat{\sigma}_{\zeta\zeta} + \hat{u}_\delta\hat{\sigma}^2_{\delta\delta}). \end{aligned} \tag{3.11}$$

If $v(\zeta, \delta) = \frac{2\zeta\delta^{\frac{3}{2}}\ t^{-3\zeta-1}\ exp(-\delta t^{-2\zeta})}{\gamma\left(\frac{3}{2}, \delta t^{-2\zeta}\right)}$. Since the partial derivatives of the function $v(\zeta, \delta)$ can not solve analytically. Therefore, we used "numDeriv" package in R software to solve the derivative of $v(\zeta, \delta)$. Thus, the Bayes estimator of $h(t)$ under SELF is found as follows:

$$\begin{aligned}\hat{h}(t)_{BS} &= \hat{h}(t)_{MLE} + 0.5(\hat{v}_{\zeta\zeta}\hat{\sigma}_{\zeta\zeta} + \hat{v}_{\delta\delta}\hat{\sigma}_{\delta\delta}) + \hat{v}_{\zeta\delta}\hat{\sigma}_{\zeta\delta} + \hat{v}_\zeta(\hat{\sigma}_{\zeta\zeta}\hat{\rho}_\zeta + \hat{\sigma}_{\delta\zeta}\hat{\rho}_\delta) + \hat{v}_\delta(\hat{\sigma}_{\zeta\delta}\hat{\rho}_\zeta + \hat{\sigma}_{\delta\delta}\hat{\rho}_\delta)\\ &+ 0.5\hat{l}_{\zeta\zeta\zeta}(\hat{v}_\zeta\hat{\sigma}^2_{\zeta\zeta} + \hat{v}_\delta\hat{\sigma}_{\zeta\zeta}\hat{\sigma}_{\zeta\delta}) + 0.5\hat{l}_{\zeta\zeta\delta}\left(3\hat{v}_\zeta\hat{\sigma}_{\zeta\zeta}\hat{\sigma}_{\zeta\delta} + \hat{v}_\delta(\hat{\sigma}_{\zeta\zeta}\hat{\sigma}_{\delta\delta} + 2\hat{\sigma}^2_{\zeta\delta})\right)\\ &+ 0.5\hat{l}_{\zeta\delta\delta}\left(\hat{v}_\zeta(\hat{\sigma}_{\zeta\zeta}\hat{\sigma}_{\delta\delta} + 2\hat{\sigma}^2_{\zeta\delta}) + 3\hat{v}_\delta\hat{\sigma}_{\zeta\delta}\hat{\sigma}_{\delta\delta}\right) + 0.5\hat{l}_{\delta\delta\delta}(\hat{v}_\zeta\hat{\sigma}_{\zeta\delta}\hat{\sigma}_{\zeta\zeta} + \hat{u}_\delta\hat{\sigma}^2_{\delta\delta}). \end{aligned} \tag{3.12}$$

3.5 Metropolis-Hastings (M-H) Algorithm

The M-H algorithm is a crucial Markov chain Monte Carlo (MCMC) method used in Bayesian statistics. It tackles the difficulty that frequently arises in Bayesian analysis when sampling from complex, high-dimensional posterior probability distributions. The algorithm creates a Markov chain of samples by repeatedly proposing, accepting, and rejecting samples according to a proposal distribution. This chain of samples eventually converges to resemble the target posterior distribution. Please refer to Metropolis et al. (1953) and Smith and Report (1993) for additional details.

The full conditional posterior density function of ζ and δ are obtained in the following expressions using the joint posterior density function ζ and δ which are derived in the equation (3.6)

$$\pi_1(\zeta|\delta,x) \propto \zeta^{m+a_1-1} exp\left[-\left\{\zeta b_1 + \delta\sum_{i=1}^{m} x_i^{-2\zeta} + 3\zeta\sum_{i=1}^{m}\log(x_i)\right\}\right]\prod_{i=1}^{m}\left[\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\right]^{R_i}, \tag{3.13}$$

and

$$\pi_2(\delta|\zeta,x) \propto \delta^{\frac{3m}{2}+a_2-1} exp\left[-\delta\left(b_2 + \sum_{i=1}^{m} x_i^{-2\zeta}\right)\right]\prod_{i=1}^{m}\left[\gamma\left(\frac{3}{2},\delta x_i^{-2\zeta}\right)\right]^{R_i}. \tag{3.14}$$

From (3.13) and (3.14), the conditional posterior densities of ζ and δ cannot be simplified into any standard distribution. To generate samples from these posterior densities, the Metropolis-Hastings (M-H) algorithm with a normal proposal distribution is employed. Follow the steps below to compute the Bayes MCMC estimates and the associated credible intervals for ζ, δ, or any function of these parameters using the M-H sampling process:

Step 1: Set $j = 1$ and start with initial guess $\zeta^{(0)} = \hat{\zeta}_{MLE}$ and $\delta^{(0)} = \hat{\delta}_{MLE}$.

Step 2: Generate ζ^* and δ^* from (3.13) and (3.14) with their normal proposal distributions, respectively, as

(i) Generate a candidate ζ^* from $N(\zeta^{(j-1)}, var(\zeta_{MLE}))$ and δ^* from $N(\delta^{(j-1)}, var(\delta_{MLE}))$.

(ii) Compute $\nu_\zeta = min\left(1, \frac{\pi_1(\zeta^*|\delta^{(j-1)},x)}{\pi_1(\zeta^{(j-1)}|\delta^{(j-1)},x)}\right)$ and $\nu_\delta = min\left(1, \frac{\pi_2(\delta^*|\zeta^{(j)},x)}{\pi_2(\delta^{(j-1)}|\zeta^{(j)},x)}\right)$.

(iii) Generate u_1 and u_2 from uniform distribution $U(0,1)$.

(iv) If $u_1 \leq \nu_\zeta$, set $\zeta^{(j)} = \zeta^*$, else set $\zeta^{(j)} = \zeta^{(j-1)}$. Similarly If $u_2 \leq \nu_\delta$, set $\delta^{(j)} = \delta^*$, else set $\delta^{(j)} = \delta^{(j-1)}$.

Step 3: On substituting the parameters ζ and δ of $r(t)$ and $h(t)$ as in (1.3) and (1.4) by their $\zeta^{(j)}$ and $\delta^{(j)}$, the MCMC samples of $r^{(j)}(t)$ and $h^{(j)}(t)$, for given $t > 0$, can be obtained as

$$r^{(j)}(t) = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \delta^{(j)} t^{-2\zeta^{(j)}}\right) \text{ and } h^{(j)}(t) = \frac{2\zeta^{(j)}(\delta^{(j)})^{\frac{3}{2}}\ t^{-3\zeta^{(j)}-1}\ exp(-\delta^{(j)} t^{-2\zeta^{(j)}})}{\gamma\left(\frac{3}{2}, \delta^{(j)} t^{-2\zeta^{(j)}}\right)}. \tag{3.15}$$

Step 4: Set j = j+1.

Step 5: Repeat the steps 2-4, N times and compute $\Delta^{(j)} = (\zeta^{(j)}, \delta^{(j)}, r^{(j)}(t), h^{(j)}(t))$ for $j = 1, 2, ..., N$.

Step 6: The following formula can be used to obtain the Bayes estimator of the parameters ζ, δ, $r(t)$, and $h(t)$ under SELF:

$$\hat{\Delta} = \frac{1}{(N-M)}\sum_{j=M+1}^{N}\Delta^{(j)}, \tag{3.16}$$

where, M represents the burn-in period of the Markov chain, a predefined constant set in advance to mitigate the influence of initial value selection and ensure the convergence of the algorithm.

3.6 HPD Credible Interval

The highest posterior density (HPD) interval is a Bayesian interval estimate providing the shortest interval containing a specified probability (e.g., 95%) of the posterior distribution, emphasizing the most probable parameter values. Unlike frequentist intervals, HPD intervals are derived from the posterior distribution and can be asymmetric.

In order to calculate the HPD credible intervals for ζ, first the MCMC sample $\zeta_1, \zeta_2, ..., \zeta_N$ is ordered as $\zeta_{(1)}, \zeta_{(2)}, ..., \zeta_{(N)}$ then for arbitrary $0 < \gamma < 1$, the $100(1-\gamma)\%$ credible intervals of ζ can be obtained as $(\hat{\zeta}^{[K]}, \hat{\zeta}^{[K+N-(\gamma N+1)})$ where $K = 1, 2, ..., [N\gamma]$ and [z] represents the greatest integer less or equal to z (see for more details Chen and Shao (1999)). Hence, the credible intervals is computed as

$$(\hat{\zeta}^{[K^*+N-(\gamma N+1)} - \hat{\zeta}^{[K^*]]}) = min_{k=1}^{N\gamma}(\hat{\zeta}^{[K+N-(\zeta N+1)} - \hat{\zeta}^{[K]]}).$$

Similarly, the HPD credible interval of δ, $r(t)$ and $h(t)$ can be also derived.

4 Simulation Study

In this section, we perform Monte Carlo simulations to investigate the behaviour of considered estimators of the parameters as well as associated reliability and hazard rate functions of IPM distribution based on progressive type-II censored sample (PT-II CS). According to the progressive generation algorithm introduced by Balakrishnan and Sandhu Balakrishnan and Sandhu (1995). we replicated 10^3 times PT-II CS from IPM distribution when the set of true parameter values of $(\zeta, \delta) = (0.5, 1.5)$ and $(1, 2)$ and the corresponding actual values of $r(t) = 0.8884, 0.7385$ and $h(t) = 0.6572, 1.1696$ at distinct time $t = 0.5, 1$, respectively. For different choices of $n(= 40, 80)$, the effective sample size are taken to be $m = 20, 30$ for $n = 40$ and $m = 40, 60$ for $n = 80$. Also, for each n and m, different censoring schemes, R, to withdraw the live items during the experiment are used, where $R = (4, 0, 0, 1)$ is denoted by $R = (4, 0^{*2}, 1)$ for brevity, such as:

Scheme I : $R_1 = (n - m), \quad R_i = 0 \;\text{ for }\; i \neq 1;$

Scheme II : $R_1 = R_m = \frac{(n-m)}{2}, \;\; R_i = 0 \;\text{ for }\; i \neq 1, m;$

Scheme III : $R_m = (n - m), \quad R_i = 0 \;\text{ for }\; i \neq m.$

In Bayesian analysis, to observe the effects of priors, two different sets of hyperparameters for ζ and δ are used: Prior-1 with $a_1 = a_2 = b_1 = b_2 = 0.001$, and Prior-2 with $(a_1, a_2) = (1, 3)$, $(b_1, b_2) = (2, 2)$ for $(\zeta, \delta) = (0.5, 1.5)$, and $(a_1, a_2) = (4, 10)$, $(b_1, b_2) = (4, 5)$ for $(\zeta, \delta) = (1, 2)$. This study uses informative and non-informative priors to perform Bayesian inference. A non-informative prior is applied when limited or no historical data is available. However, when prior information about the unknown parameters is unavailable, using the classical approach over Bayesian methods is often preferable due to the latter's higher computational cost. The hyperparameters of ζ and δ are selected such that the prior mean equals the expected value of the corresponding unknown parameter. Using 10^3 samples, the average classical estimates of ζ, δ, $r(t)$, and $h(t)$, along with their 95% asymptotic confidence intervals (ACIs), are computed. In Bayesian framework, the Lindley's approximation techniques is employed to obtain the Bayes estimates. For the Metropolis-Hastings (M-H) sampler, 11×10^3 MCMC samples are generated, with the first 10^3 observations discarded as "burn-in." Based on 10^4 MCMC samples, the average Bayes MCMC estimates, and 95% highest posterior density (HPD) credible intervals are calculated.

Furthermore, the average estimates (AEs), root mean square errors (RMSEs) and relative absolute biases (RABs) are used to evaluate the performance of the various estimates. The values of the AEs, RMSEs, and RABs are calculated for different censoring schemes using the following formulas:

$$\overline{\hat{\xi}_g} = \frac{\sum_{i=1}^{s} \hat{\xi}_g^{(i)}}{s}, \quad g = 1, 2, 3, 4, \tag{4.1}$$

$$\text{RMSE}(\hat{\xi}) = \sqrt{\frac{\sum_{i=1}^{s} (\hat{\xi}_g^{(i)} - \xi_g)^2}{s}}, \quad g = 1, 2, 3, 4, \tag{4.2}$$

and

$$\text{RAB}(\hat{\xi}_g) = \frac{1}{s} \sum_{i=1}^{s} \frac{|\hat{\xi}_g^{(i)} - \xi_g|}{\xi_g}, \quad g = 1, 2, 3, 4. \tag{4.3}$$

where $g = 1, 2, 3, 4$, $\hat{\xi}_g^{(i)}$, denotes the empirical estimate of the unknown parameter ξ_g at the i-th sample, where s represents the number of generated sample data. Specifically, $\hat{\xi}_1 = \hat{\zeta}$, $\hat{\xi}_2 = \hat{\delta}$, $\hat{\xi}_3 = \hat{r}(t)$, and $\hat{\xi}_4 = \hat{h}(t)$. Furthermore, the corresponding average credible lengths (ACLs) associated with the ACI/HPD credible intervals for the unknown parameter ξ_g, for $g = 1, 2, 3, 4$, are calculated using the following formula:

$$\text{ACL}_{\xi(g)} = \frac{1}{g} \sum_{i=1}^{g} \left(U_{\hat{\xi}_g^{(i)}} - L_{\hat{\xi}_g^{(i)}} \right),$$

where $L(\cdot)$ and $U(\cdot)$ denotes the lower and upper bounds, respectively, of $(1 - \tau)\%$ ACI/HPD credible interval of ξ_g. Comparison between different point estimates is conducted based on their RMSE and RAB values. Additionally, the performance of the 95% two-sided ACI/HPD credible interval estimates is evaluated in terms of their average confidence lengths (ACLs).

The root means square errors (RMSEs), relative absolute basis (RABs), and approximate confidence intervals (ACLs) of ζ, δ, $r(t)$, and $h(t)$ are calculated and reported in Tables 1-12. All numerical computations were performed using *R* statistical programming language software version 4.0.4 with some useful packages, namely 'zipfR', 'coda' package proposed by Plummer (2015), 'nleqslv' package proposed by Hasselman and Hasselman (2018), 'numDeriv', and 'MCMCpack' etc. From the simulation results, we can make the following observations:

Using various classical and Bayesian approaches for estimating the unknown parameters and related reliability characteristics, all estimates demonstrate satisfactory performance in terms of RMSEs, RABs, and ACIs. As the effective sample size m increases, the point estimates improve, and the lengths of confidence intervals decrease as expected. Therefore, increasing the effective sample size can lead to more accurate estimation results. Among classical approaches, the MLE method outperforms LSE and WLSE methods in terms of RMSE and RAB. When comparing classical and Bayesian methods, Bayes estimates under an informative prior exhibit superior performance in terms of RMSE, RABs, and ACLs. In contrast, Bayes estimates under a non-informative prior exhibit behavior similar to classical approaches. Hence, when prior information is unavailable, classical estimates are generally preferred over Bayesian estimates due to the latter's higher computational cost. Within the Bayesian framework, Lindley's approximation performs better than the MCMC Metropolis-Hastings (M-H) algorithm for estimating parameters ζ and δ in terms of RMSEs and RABs. However, for reliability characteristics $r(t)$

and $h(t)$, the MCMC M-H algorithm outperforms Lindley's approximation in terms of RMSEs, RABs, and ACIs. Regarding interval estimates, HPD credible intervals are shorter than ACIs, offering more precise interval estimates.

Table 1: Simulated RMSEs (First row) and RABs (Second row) of ζ. The true values of $(\zeta, \delta) = (0.5, 1.5)$.

						Lindley		MCMC	
n	m	Scheme	MLE	LSE	WLSE	Prior-I	Prior-II	Prior-I	Prior-II
		I	0.0973	0.1225	0.1124	0.0958	0.0892	0.0978	0.0914
			0.1435	0.1752	0.1599	0.1415	0.1332	0.1442	0.1361
	20	II	0.0924	0.1026	0.0920	0.0901	0.0849	0.0909	0.0861
			0.1405	0.1599	0.1445	0.1381	0.1311	0.1385	0.1321
		III	0.1061	0.1172	0.1084	0.1031	0.0970	0.1030	0.0973
			0.1592	0.1811	0.1670	0.1565	0.1480	0.1557	0.1477
40									
		I	0.0728	0.0917	0.0844	0.0710	0.0686	0.0725	0.0702
			0.1112	0.1397	0.1272	0.1092	0.1058	0.1108	0.1077
	30	II	0.0752	0.0859	0.0797	0.0731	0.0706	0.0745	0.0720
			0.1137	0.1333	0.1229	0.1115	0.1082	0.1131	0.1096
		III	0.0792	0.0853	0.0800	0.0768	0.0742	0.0779	0.0755
			0.1205	0.1349	0.1254	0.1174	0.1138	0.1188	0.1155
		I	0.0595	0.0755	0.0679	0.0588	0.0574	0.0599	0.0584
			0.0903	0.1184	0.1044	0.0895	0.0874	0.0907	0.0885
	40	II	0.0608	0.0721	0.0662	0.0598	0.0583	0.0604	0.0588
			0.0947	0.1136	0.1045	0.0935	0.0914	0.0943	0.0920
		III	0.0686	0.0790	0.0739	0.0674	0.0656	0.0676	0.0658
			0.1042	0.1247	0.1158	0.1030	0.1005	0.1031	0.1005
80									
		I	0.0509	0.0584	0.0531	0.0502	0.0494	0.0510	0.0501
			0.0788	0.0914	0.0831	0.0781	0.0770	0.0789	0.0777
	60	II	0.0524	0.0602	0.0554	0.0517	0.0510	0.0522	0.0515
			0.0819	0.0948	0.0876	0.0813	0.0802	0.0817	0.0807
		III	0.0536	0.0614	0.0566	0.0530	0.0523	0.0534	0.0526
			0.0834	0.0985	0.0912	0.0828	0.0817	0.0830	0.0819

Table 2: Simulated RMSEs (first row) and RABs (second row) of δ. The true values of $(\zeta, \delta) = (0.5, 1.5)$.

						Lindley		MCMC	
n	m	Scheme	MLE	LSE	WLSE	Prior-I	Prior-II	Prior-I	Prior-II
		I	0.2856	0.3379	0.3241	0.2818	0.2538	0.2832	0.2556
			0.1494	0.1600	0.1569	0.1476	0.1332	0.1481	0.1339
	20	II	0.2516	0.2662	0.2579	0.2481	0.2283	0.2480	0.2285
			0.1317	0.1300	0.1296	0.1305	0.1203	0.1299	0.1200
		III	0.2449	0.2355	0.2381	0.2408	0.2232	0.2401	0.2233
			0.1285	0.1236	0.1255	0.1266	0.1175	0.1258	0.1173
40									
		I	0.2392	0.2597	0.2498	0.2335	0.2171	0.2375	0.2204
			0.1269	0.1313	0.1281	0.1249	0.1163	0.1262	0.1175
	30	II	0.2338	0.2368	0.2313	0.2290	0.2144	0.2315	0.2167
			0.1223	0.1232	0.1208	0.1208	0.1133	0.1214	0.1137
		III	0.2185	0.2201	0.2166	0.2136	0.2010	0.2159	0.2033
			0.1147	0.1146	0.1131	0.1129	0.1063	0.1136	0.1072
		I	0.2077	0.2259	0.2188	0.2044	0.1938	0.2073	0.1965
			0.1104	0.1187	0.1148	0.1089	0.1033	0.1101	0.1045
	40	II	0.1750	0.1754	0.1751	0.1746	0.1675	0.1744	0.1672
			0.0928	0.0919	0.0919	0.0930	0.0893	0.0927	0.0887
		III	0.1615	0.1628	0.1645	0.1604	0.1545	0.1597	0.1540
			0.0859	0.0862	0.0873	0.0853	0.0822	0.0850	0.0820
80									
		I	0.1661	0.1732	0.1686	0.1640	0.1581	0.1653	0.1591
			0.0872	0.0891	0.0875	0.0864	0.0833	0.0868	0.0837
	60	II	0.1590	0.1613	0.1588	0.1567	0.1517	0.1578	0.1527
			0.0845	0.0854	0.0841	0.0837	0.0810	0.0840	0.0813
		III	0.1463	0.1483	0.1464	0.1446	0.1404	0.1451	0.1409
			0.0763	0.0767	0.0761	0.0755	0.0733	0.0757	0.0736

Comparing the three different censoring schemes, it is clear that RMSEs, RABs and ACLs associated with the unknown parameters δ, $r(t)$, and $h(t)$ for scheme III are smaller than those based on other schemes. However, scheme I is a good choice for the parameter ζ based on the RMSEs, RABs and ACLs.

From the above outcomes, it has been observed that the performance of both the point and interval estimates is highly acceptable. The Bayesian estimates, calculated using an informative prior, yield more accurate findings in terms of point estimates and interval estimates.

Table 3: Simulated RMSEs (first row) and RABs (second row) of $r(t)$. The true value of $r(0.5) = 0.8884$.

						Lindley		MCMC	
n	m	Scheme	MLE	LSE	WLSE	Prior-I	Prior-II	Prior-I	Prior-II
		I	0.0889	0.0928	0.0909	0.0906	0.0813	0.0880	0.0794
			0.1140	0.1173	0.1156	0.1152	0.1037	0.1130	0.1022
	20	II	0.0776	0.0755	0.0752	0.0784	0.0719	0.0765	0.0705
			0.1003	0.0966	0.0967	0.1009	0.0928	0.0989	0.0914
		III	0.0750	0.0715	0.0722	0.0751	0.0693	0.0735	0.0681
			0.0971	0.0926	0.0939	0.0968	0.0896	0.0950	0.0885
40									
		I	0.0717	0.0748	0.0729	0.0719	0.0670	0.0716	0.0667
			0.0950	0.0974	0.0953	0.0952	0.0887	0.0949	0.0885
	30	II	0.0706	0.0706	0.0693	0.0708	0.0664	0.0704	0.0660
			0.0920	0.0923	0.0905	0.0923	0.0866	0.0917	0.0860
		III	0.0654	0.0652	0.0644	0.0653	0.0615	0.0650	0.0613
			0.0859	0.0856	0.0845	0.0857	0.0808	0.0854	0.0807
		I	0.0629	0.0669	0.0649	0.0630	0.0597	0.0628	0.0596
			0.0830	0.0883	0.0854	0.0828	0.0786	0.0828	0.0786
	40	II	0.0543	0.0539	0.0539	0.0548	0.0525	0.0542	0.0519
			0.0709	0.0699	0.0699	0.0716	0.0687	0.0709	0.0679
		III	0.0500	0.0501	0.0506	0.0501	0.0482	0.0495	0.0477
			0.0654	0.0652	0.0660	0.0652	0.0628	0.0647	0.0624
80									
		I	0.0504	0.0522	0.0509	0.0505	0.0487	0.0504	0.0486
			0.0657	0.0670	0.0658	0.0657	0.0634	0.0657	0.0634
	60	II	0.0479	0.0486	0.0478	0.0478	0.0463	0.0478	0.0463
			0.0634	0.0642	0.0631	0.0633	0.0613	0.0633	0.0613
		III	0.0445	0.0448	0.0443	0.0445	0.0432	0.0444	0.0431
			0.0575	0.0578	0.0573	0.0574	0.0557	0.0573	0.0557

Table 4: Simulated RMSEs (first row) and RABs (second row) of $h(t)$. The true values of $h(0.5) = 0.6572$.

						Linldey		MCMC	
n	m	Scheme	MLE	LSE	WLSE	Prior-I	Prior-II	Prior-I	Prior-II
		I	0.2430	0.2377	0.2355	0.2469	0.2197	0.2423	0.2171
			0.2307	0.2290	0.2262	0.2332	0.2112	0.2301	0.2096
	20	II	0.2145	0.2007	0.1972	0.2149	0.1958	0.2107	0.1932
			0.2093	0.2057	0.2001	0.2095	0.1934	0.2062	0.1912
		III	0.2317	0.2352	0.2262	0.2308	0.2107	0.2248	0.2064
			0.2274	0.2369	0.2290	0.2271	0.2100	0.2223	0.2065
40									
		I	0.1656	0.1754	0.1707	0.1661	0.1557	0.1656	0.1555
			0.1653	0.1754	0.1701	0.1656	0.1560	0.1653	0.1559
	30	II	0.1692	0.1693	0.1651	0.1692	0.1590	0.1685	0.1585
			0.1664	0.1718	0.1666	0.1663	0.1573	0.1659	0.1568
		III	0.1714	0.1684	0.1652	0.1708	0.1615	0.1697	0.1609
			0.1697	0.1709	0.1682	0.1691	0.1607	0.1683	0.1604
		I	0.1445	0.1557	0.1497	0.1451	0.1383	0.1451	0.1384
			0.1458	0.1584	0.1505	0.1463	0.1398	0.1462	0.1397
	40	II	0.1416	0.1464	0.1433	0.1414	0.1357	0.1410	0.1352
			0.1432	0.1509	0.1479	0.1431	0.1378	0.1428	0.1372
		III	0.1489	0.1604	0.1560	0.1483	0.1425	0.1467	0.1411
			0.1485	0.1651	0.1598	0.1481	0.1428	0.1469	0.1417
80									
		I	0.1157	0.1195	0.1157	0.1158	0.1124	0.1158	0.1124
			0.1184	0.1223	0.1184	0.1185	0.1151	0.1183	0.1149
	60	II	0.1177	0.1213	0.1179	0.1176	0.1146	0.1176	0.1146
			0.1220	0.1274	0.1237	0.1219	0.1189	0.1218	0.1187
		III	0.1158	0.1206	0.1171	0.1156	0.1128	0.1155	0.1126
			0.1172	0.1264	0.1228	0.1171	0.1144	0.1168	0.1142

Table 5: The lengths of 95% ACIs and HPD credible intervals for the parameter ζ and δ. The true value of $(\zeta, \delta) = (0.5, 1.5)$

			ζ			δ		
			ACI	**HPD**		**ACI**	**HPD**	
n	**m**	**Scheme**	**MLE**	**Prior-I**	**Prior-II**	**MLE**	**Prior-I**	**Prior-II**
40	20	I	0.3318	0.3204	0.3117	1.075	1.049	1.001
		II	0.3360	0.3249	0.3169	0.9432	0.9265	0.8897
		III	0.3546	0.3421	0.3339	0.8872	0.8719	0.8419
	30	I	0.2758	0.2669	0.2623	0.9062	0.8851	0.8543
		II	0.2770	0.2686	0.2643	0.8522	0.8328	0.8074
		III	0.2821	0.2734	0.2691	0.8153	0.7975	0.7755
80	40	I	0.2263	0.2183	0.2159	0.7747	0.7563	0.7364
		II	0.2313	0.2238	0.2216	0.6641	0.6501	0.6375
		III	0.2424	0.2343	0.2316	0.6233	0.6124	0.6019
	60	I	0.1923	0.1858	0.1843	0.6401	0.6244	0.6135
		II	0.1912	0.1848	0.1833	0.6030	0.5897	0.5809
		III	0.1926	0.1863	0.1851	0.5730	0.5615	0.5536

Table 6: The average lengths of 95% ACIs and HPD credible intervals for $r(0.5) = 0.8884$ and $h(0.5) = 0.6572$.

			$r(0.5)$			$h(0.5)$		
			ACI	**HPD**		**ACI**	**HPD**	
n	**m**	**Scheme**	**MLE**	**Prior-I**	**Prior-II**	**MLE**	**Prior-I**	**Prior-II**
40	20	I	0.1831	0.1695	0.1574	0.8302	0.7455	0.6533
		II	0.1660	0.1580	0.1484	0.7401	0.6795	0.6041
		III	0.1580	0.1529	0.1437	0.7090	0.6588	0.5774
	30	I	0.1619	0.1545	0.1448	0.7036	0.6525	0.5927
		II	0.1559	0.1504	0.1412	0.6560	0.6105	0.5516
		III	0.1443	0.1404	0.1347	0.6297	0.5890	0.5432
80	40	I	0.1366	0.1296	0.1235	0.5920	0.5580	0.5131
		II	0.1201	0.1156	0.1120	0.5135	0.4867	0.4574
		III	0.1096	0.1072	0.1041	0.4750	0.4529	0.4283
	60	I	0.1204	0.1166	0.1112	0.5032	0.4813	0.4509
		II	0.1162	0.1129	0.1094	0.4707	0.4483	0.4251
		III	0.1114	0.1088	0.1047	0.4436	0.4245	0.4014

Table 7: Simulated RMSEs (First row) and RABs (Second row) of ζ. The true values of $(\zeta, \delta) = (1, 2)$.

						Lindley		MCMC	
n	**m**	**Scheme**	**MLE**	**LSE**	**WLSE**	**Prior-I**	**Prior-II**	**Prior-I**	**Prior-II**
		I	0.1865	0.2351	0.2151	0.1792	0.1539	0.1873	0.1602
			0.1383	0.1765	0.1579	0.1342	0.1171	0.1387	0.1206
		II	0.2094	0.2312	0.2208	0.1980	0.1688	0.2060	0.1764
			0.1531	0.1729	0.1642	0.1450	0.1261	0.1506	0.1313
		III	0.2149	0.2210	0.2079	0.2032	0.1718	0.2092	0.1779
			0.1623	0.1727	0.1637	0.1561	0.1345	0.1589	0.1375
40									
		I	0.1488	0.1685	0.1541	0.1426	0.1294	0.1486	0.1346
			0.1142	0.1326	0.1215	0.1107	0.1012	0.1140	0.1041
	30	II	0.1461	0.1664	0.1540	0.1397	0.1271	0.1450	0.1319
			0.1132	0.1308	0.1212	0.1089	0.0998	0.1125	0.1031
		III	0.1594	0.1785	0.1653	0.1525	0.1385	0.1575	0.1431
			0.1224	0.1385	0.1279	0.1182	0.1083	0.1212	0.1110
		I	0.1193	0.1515	0.1355	0.1162	0.1092	0.1198	0.1125
			0.0935	0.1192	0.1050	0.0915	0.0862	0.0940	0.0884
	40	II	0.1273	0.1527	0.1381	0.1234	0.1154	0.1268	0.1182
			0.0982	0.1162	0.1048	0.0956	0.0899	0.0979	0.0917
		III	0.1339	0.1465	0.1367	0.1297	0.1208	0.1321	0.1232
			0.1032	0.1169	0.1091	0.1009	0.0943	0.1022	0.0956
80									
		I	0.1045	0.1254	0.1129	0.1023	0.0980	0.1047	0.1001
			0.0812	0.0982	0.0886	0.0802	0.0769	0.0814	0.0780
	60	II	0.1055	0.1231	0.1131	0.1030	0.0986	0.1054	0.1008
			0.0798	0.0962	0.0876	0.0787	0.0756	0.0797	0.0765
		III	0.1027	0.1187	0.1113	0.1003	0.0960	0.1023	0.0981
			0.0798	0.0946	0.0871	0.0785	0.0753	0.0796	0.0765

Table 8: Simulated RMSEs (First row) and RABs (Second row) of δ. The true values of $(\zeta, \delta) = (1, 2)$.

						Lindley		MCMC	
n	**m**	**Scheme**	**MLE**	**LSE**	**WLSE**	**Prior-I**	**Prior-II**	**Prior-I**	**Prior-II**
		I	0.3741	0.4461	0.4224	0.3550	0.2694	0.3733	0.2790
			0.1470	0.1631	0.1543	0.1409	0.1081	0.1467	0.1114
	20	II	0.3392	0.3640	0.3438	0.3230	0.2573	0.3352	0.2654
			0.1306	0.1347	0.1286	0.1257	0.1014	0.1292	0.1040
		III	0.3005	0.2774	0.2766	0.2889	0.2377	0.2962	0.2427
			0.1163	0.1084	0.1077	0.1128	0.0938	0.1147	0.0952
40									
		I	0.3187	0.3614	0.3415	0.3070	0.2492	0.3165	0.2552
			0.1249	0.1381	0.1309	0.1219	0.0998	0.1245	0.1015
	30	II	0.3051	0.3274	0.3115	0.2905	0.2382	0.3014	0.2457
			0.1179	0.1231	0.1172	0.1135	0.0943	0.1167	0.0967
		III	0.2757	0.2934	0.2749	0.2618	0.2181	0.2711	0.2249
			0.1068	0.1107	0.1047	0.1027	0.0866	0.1055	0.0887
		I	0.2445	0.2895	0.2728	0.2378	0.2056	0.2438	0.2102
			0.0978	0.1116	0.1049	0.0956	0.0829	0.0977	0.0845
	40	II	0.2164	0.2302	0.2206	0.2123	0.1900	0.2149	0.1920
			0.0849	0.0888	0.0854	0.0838	0.0752	0.0843	0.0758
		III	0.1978	0.1954	0.1935	0.1937	0.1764	0.1960	0.1781
			0.0780	0.0770	0.0765	0.0768	0.0700	0.0775	0.0705
80									
		I	0.2147	0.2305	0.2189	0.2104	0.1891	0.2138	0.1915
			0.0842	0.0910	0.0865	0.0829	0.0748	0.0839	0.0755
	60	II	0.2008	0.2174	0.2067	0.1962	0.1782	0.1994	0.1809
			0.0784	0.0846	0.0805	0.0771	0.0702	0.0780	0.0710
		III	0.1926	0.1993	0.1928	0.1888	0.1730	0.1913	0.1749
			0.0754	0.0781	0.0756	0.0745	0.0685	0.0751	0.0691

Table 9: Simulated RMSEs (first row) and RABs (second row) of $r(1)$. The true value of $r(1)$ is 0.7385.

						Lindley		MCMC	
n	**m**	**Scheme**	**MLE**	**LSE**	**WLSE**	**Prior-I**	**Prior-II**	**Prior-I**	**Prior-II**
40	20	I	0.0768	0.0843	0.0799	0.0772	0.0592	0.0768	0.0589
			0.0830	0.0906	0.0851	0.0826	0.0637	0.0830	0.0640
		II	0.0690	0.0706	0.0681	0.0688	0.0556	0.0687	0.0556
			0.0738	0.0757	0.0726	0.0732	0.0594	0.0734	0.0597
		III	0.0630	0.0596	0.0592	0.0628	0.0519	0.0626	0.0518
			0.0670	0.0635	0.0629	0.0666	0.0554	0.0665	0.0554
	30	I	0.0657	0.0717	0.0686	0.0664	0.0546	0.0660	0.0543
			0.0714	0.0784	0.0745	0.0719	0.0592	0.0717	0.0591
		II	0.0611	0.0652	0.0621	0.0608	0.0509	0.0610	0.0511
			0.0662	0.0697	0.0662	0.0656	0.0550	0.0661	0.0554
		III	0.0548	0.0575	0.0546	0.0542	0.0460	0.0546	0.0464
			0.0597	0.0622	0.0591	0.0589	0.0501	0.0595	0.0506
80	40	I	0.0513	0.0580	0.0546	0.0514	0.0446	0.0513	0.0446
			0.0561	0.0634	0.0593	0.0560	0.0486	0.0561	0.0488
		II	0.0460	0.0483	0.0466	0.0462	0.0414	0.0460	0.0412
			0.0493	0.0516	0.0495	0.0493	0.0443	0.0491	0.0442
		III	0.0419	0.0419	0.0415	0.0417	0.0380	0.0418	0.0380
			0.0452	0.0449	0.0446	0.0449	0.0410	0.0450	0.0410
	60	I	0.0453	0.0492	0.0465	0.0455	0.0410	0.0454	0.0410
			0.0486	0.0531	0.0502	0.0486	0.0439	0.0486	0.0439
		II	0.0418	0.0453	0.0432	0.0418	0.0381	0.0419	0.0382
			0.0450	0.0488	0.0464	0.0448	0.0409	0.0450	0.0411
		III	0.0401	0.0421	0.0406	0.0401	0.0369	0.0402	0.0370
			0.0434	0.0455	0.0438	0.0434	0.0399	0.0434	0.0401

Table 10: Simulated RMSEs (first row) and RABs (second row) of $h(1)$. The true values of $h(1)$ is 1.1696.

						Linldey		MCMC	
n	**m**	**Scheme**	**MLE**	**LSE**	**WLSE**	**Prior-I**	**Prior-II**	**Prior-I**	**Prior-II**
40	20	I	0.3654	0.3606	0.3602	0.3650	0.2808	0.3670	0.2824
			0.2358	0.2345	0.2319	0.2344	0.1854	0.2364	0.1866
		II	0.3588	0.3445	0.3460	0.3545	0.2843	0.3552	0.2861
			0.2259	0.2227	0.2234	0.2230	0.1831	0.2237	0.1840
		III	0.3591	0.3301	0.3237	0.3520	0.2844	0.3527	0.2866
			0.2305	0.2187	0.2156	0.2264	0.1878	0.2270	0.1890
	30	I	0.2782	0.2739	0.2691	0.2768	0.2334	0.2788	0.2355
			0.1856	0.1836	0.1813	0.1843	0.1568	0.1858	0.1582
		II	0.2530	0.2460	0.2443	0.2506	0.2156	0.2527	0.2173
			0.1688	0.1646	0.1634	0.1670	0.1447	0.1684	0.1460
		III	0.2475	0.2418	0.2395	0.2450	0.2146	0.2464	0.2162
			0.1665	0.1639	0.1629	0.1646	0.1453	0.1654	0.1463
80	40	I	0.2201	0.2254	0.2220	0.2197	0.1945	0.2205	0.1955
			0.1478	0.1521	0.1494	0.1473	0.1311	0.1479	0.1316
		II	0.2223	0.2211	0.2195	0.2207	0.1996	0.2218	0.2001
			0.1458	0.1432	0.1422	0.1448	0.1318	0.1454	0.1322
		III	0.2220	0.2260	0.2232	0.2196	0.1999	0.2202	0.2006
			0.1467	0.1518	0.1503	0.1454	0.1332	0.1458	0.1334
	60	I	0.1932	0.1971	0.1931	0.1928	0.1775	0.1939	0.1784
			0.1298	0.1333	0.1303	0.1293	0.1195	0.1300	0.1199
		II	0.1809	0.1816	0.1795	0.1801	0.1676	0.1814	0.1687
			0.1214	0.1225	0.1210	0.1208	0.1126	0.1211	0.1131
		III	0.1723	0.1743	0.1736	0.1714	0.1604	0.1727	0.1618
			0.1152	0.1174	0.1164	0.1146	0.1074	0.1153	0.1082

Table 11: The average lengths of 95% ACIs and HPD credible intervals for the parameters ζ and δ.

			ζ			δ		
			ACI	**HPD**		**ACI**	**HPD**	
n	**m**	**Scheme**	**MLE**	**Prior-I**	**Prior-II**	**MLE**	**Prior-I**	**Prior-II**
40	20	I	0.6559	0.6311	0.5925	1.3580	1.3160	1.1530
		II	0.5523	0.5324	0.5087	1.1510	1.1160	1.0110
		III	0.7100	0.6813	0.6364	1.0675	1.0425	0.9539
	30	I	0.5578	0.5339	0.3618	1.1869	1.1522	0.7905
		II	0.5506	0.5294	0.5076	1.1007	1.0673	0.9727
		III	0.5580	0.5368	0.5150	1.0496	1.0221	0.9377
80	40	I	0.4533	0.4348	0.4228	0.9674	0.9403	0.8733
		II	0.4642	0.4460	0.4332	0.8240	0.8016	0.7603
		III	0.4839	0.4665	0.4518	0.7478	0.7324	0.6996
	60	I	0.3830	0.3687	0.3616	0.8147	0.7935	0.7531
		II	0.3834	0.3696	0.3624	0.7683	0.7501	0.7152
		III	0.3867	0.3732	0.3663	0.7260	0.7098	0.6798

Table 12: The average lengths of 95% ACIs and HPD credible intervals for $r(1) = 0.7385$ and $h(1) = 1.1696$.

			$r(1)$			$h(1)$		
			ACI	**HPD**		**ACI**	**HPD**	
n	**m**	**Scheme**	**MLE**	**Prior-I**	**Prior-II**	**MLE**	**Prior-I**	**Prior-II**
40	20	I	0.2779	0.2624	0.2360	1.2130	1.1700	1.0640
		II	0.2436	0.2328	0.2123	1.1830	1.1350	1.0390
		III	0.2255	0.2176	0.2025	1.2040	1.1660	1.0690
	30	I	0.2368	0.2294	0.2095	0.9866	0.9644	0.9005
		II	0.2265	0.2208	0.2022	0.9699	0.9584	0.8782
		III	0.2162	0.2118	0.1953	0.9550	0.9291	0.8735
80	40	I	0.2060	0.1983	0.1858	0.8941	0.8684	0.8192
		II	0.1754	0.1700	0.1606	0.7910	0.7691	0.7370
		III	0.1594	0.1543	0.1487	0.8108	0.7836	0.7611
	60	I	0.1722	0.1664	0.1587	0.7094	0.6964	0.6679
		II	0.1620	0.1580	0.1509	0.6636	0.6458	0.6288
		III	0.1539	0.1499	0.1442	0.6527	0.6370	0.6211

5 Optimal Censoring Scheme

In reliability theory, the best strategy for gathering information about the lifespan of a system or censored component is to identify the optimal censoring plan. To do this, one must choose the appropriate time to monitor the system's condition, considering both the censoring time and observed failure times. In this section, the issue of the optimal censoring plan under the progressive Type-II censored sample (PT-II CS) of IPM distribution is covered. Three distinct optimality criteria have been taken into account. The trace optimality criterion (criterion A) aims to minimize the trace of the information matrix associated with the parameter estimates.

The information matrix is essentially the covariance matrix of the parameter estimates, and minimizing its trace is equivalent to maximizing the efficiency of parameter estimation. A trace-optimal censoring technique allocates censored data to yield the maximum amount of information regarding the parameters being studied. Stated differently, it minimizes the degree of uncertainty in parameter estimations. Criteria B is based on minimizing the determinant of the V-C matrix. Minimizing the determinant of the V-C matrix implies maximizing the precision of parameter estimates. Moreover, criterion C relies on maximizing the trace of the Fisher information matrix. Maximizing the trace enhances the precision of survival probability estimates derived from the available data. This criterion is tabulated in the Table 13. Numerous scholars have examined the optimal censoring plan; for instance, Elshahhat and Abu El Azm (2022); Dey and Elshahhat (2022); Sultan et al. (2019); Pradhan and Kundu (2013); Dube et al. (2016); Irfan and Sharma (2023).

Table 13: Various optimality criteria.

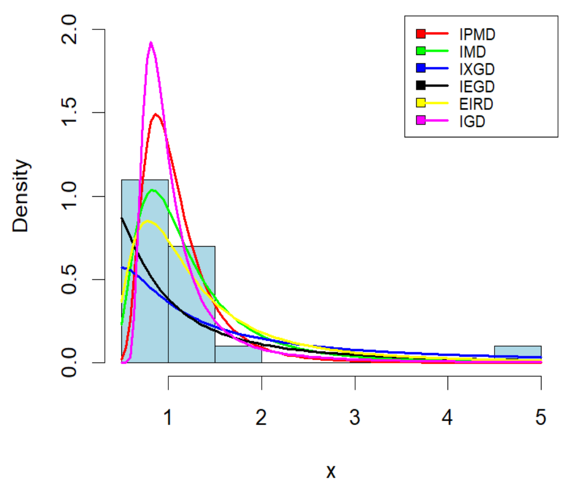
Criterion	Goal
A	minimum $trace(I^{-1}(\hat{\zeta},\hat{\delta}))$
B	minimum $det(I^{-1}(\hat{\zeta},\hat{\delta}))$
C	maximum $trace(I(\hat{\zeta},\hat{\delta}))$

6 Real Data Analysis

In this section, two real datasets are analyzed for illustrative purposes to demonstrate the practical applicability of the proposed methodology.

Dataset I

We examine 20 mechanical components' failure times as reported by Murthy et al. (2004). In order to analyze the data set, first, we check whether this data set fits our model accurately or not. We obtained various measures of goodness of fit tests such as log-likelihood value, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn information criterion (HQIC), Akaike information corrected criterion (AICC), K-S distance and p-value are presented in Table 14. The MLEs, together with the standard errors (SE) of the unknown model parameters ζ and δ, are presented in the same Table 14. It has been observed that the IPM distribution fits more accurately among all the other statistical models. For comparison purposes, we have also considered different lifetime models such as the inverse Maxwell distribution (IMD), inverse X-gamma distribution (IXGD), inverse exponentiated gamma distribution (IEGD), exponentiated inverse Rayleigh distribution (EIRD), and inverse Gompertz distribution (IGD). The histogram and fitted PDFs of these lifetime models are depicted in Figure 3.



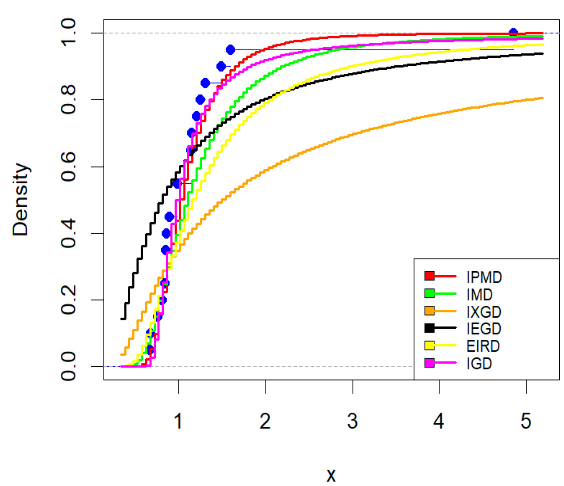


Figure 3: Histogram and fitted PDFs (left) empirical and fitted CDFs (right) of competing models under mechanical component data.

Table 14: Goodness of fit measures and MLEs with SEs (in parenthesis) for the mechanical component dataset.

Model	$\hat{\zeta}$	$\hat{\delta}$	-2 log(l)	AIC	BIC	HQIC	AICC	K-S	p-value
IPMD	1.4527(0.2610)	1.2276(0.2461)	**15.79**	**19.79**	**21.78**	**20.18**	**20.49**	**0.1221**	**0.9274**
IMD	-	1.3881(0.2532)	19.41	21.41	22.40	21.60	21.63	0.1946	0.4353
IXGD	1.8228(0.3024)	-	45.81	47.81	48.80	48.00	48.03	0.4272	0.0014
IEGD	0.6800(0.1520)	-	45.13	47.13	48.13	47.33	47.36	0.4331	0.0011
EIRD	0.9145 (NA)	1.0119(NA)	24.10	28.10	30.10	28.49	28.81	0.2668	0.1161
IGD	0.0579(0.0427)	3.6776(0.7066)	13.43	17.43	19.42	17.82	18.14	0.1335	0.8680

A significant issue with MLE is the inability to mathematically prove the existence and uniqueness of $\hat{\zeta}$ and $\hat{\delta}$. To address this, Figure 4 provides a contour plot of the log-likelihood function for the complete mechanical components dataset. The plot shows that with suitable initial values, ζ and δ converge to approximately 1.4527 and 1.2276, respectively, confirming the existence and uniqueness of the MLEs.

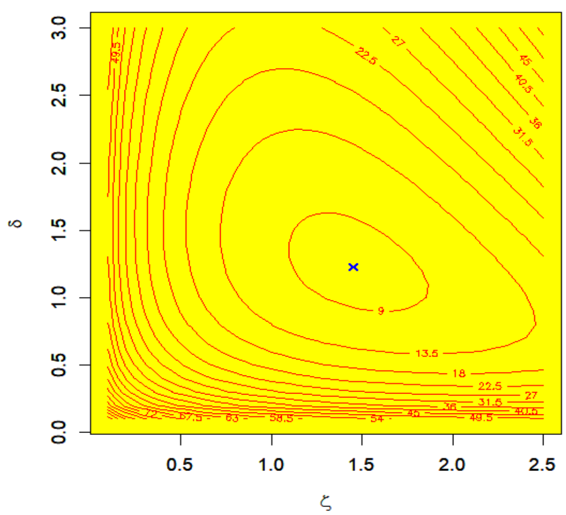


Figure 4: Contour plot of log-likelihood function for various choices of ζ and δ based on mechanical component dataset.

Three artificial datasets with $m = 10$ are generated from the complete mechanical component dataset under various censoring schemes and provided in the Table 15. For brevity, the censoring scheme $R = (3, 0, 0, 3)$ is denoted as $R = (3, 0^*2, 3)$. Bayes estimates are obtained using the Linldey and MCMC technique with non-informative priors, as no prior knowledge of the model parameters is provided. All hyperparameters are set to $a_1 = b_1 = a_2 = b_2 = 0$. A 5×10^4 sample MCMC chain is generated, discarding the first 10^4 iterations to mitigate the influence of initial values. The MLEs, LSEs, WLSEs, and Bayes estimates of the model parameters ζ and δ, as well as $r(t)$ and $h(t)$ at mission time $t = 1$, are computed using the dataset in Table 15 and presented in Table 16. Additionally, 95% ACIs and HPD credible intervals with their lengths are calculated and provided in same Table 16. The MLE, Linldey and MCMC point estimates are comparable for CS-I, while the LSE and WLSE deviate from MLE and MCMC for all censoring schemes. From Table 16, it is observed that the HPD credible intervals perform better than the ACIs in terms of the lengths of the confidence intervals.

Table 15: Three artificial PT-II CS based on mechanical component dataset.

Sample	**Scheme**	**Censored data**
1	$R = (10, 0^{*}9)$	0.67, 1.14, 1.14, 1.15, 1.21, 1.25, 1.31, 1.49, 1.60, 4.85
2	$R = (5, 0^{*}8, 5)$	0.67, 0.85, 0.86, 0.89, 0.98, 0.98, 1.14, 1.14, 1.15, 1.21
3	$R = (0^{*}9, 10)$	0.67, 0.68, 0.76, 0.81, 0.84, 0.85, 0.85, 0.86, 0.89, 0.98

Table 16: Point and Interval estimates of parameters based on mechanical component dataset.

		Point Estimate					**Interval Estimate**	
Sample	**Par.**	**MLE**	**LSE**	**WLSE**	**MCMC**	**Lindley**	**ACI**	**HPD**
I	ζ	1.1522	2.3710	2.4687	1.1013	1.1376	(0.6519,1.6525) 1.0006	(0.6833,1.6485) 0.9652
	δ	2.1533	3.6878	4.0060	2.0487	2.1576	(1.1577,3.1489) 1.991	(0.6833,1.6485) 1.949
	$r(1)$	0.7698	0.9392	0.9542	0.7489	0.7706	(0.5784,0.9612) 0.3828	(0.5761,0.9349) 0.3588
	$h(1)$	1.2392	1.0098	0.8523	1.2545	1.2206	(0.3253,2.1530) 1.8281	(0.4543,2.1868) 1.7332
II	ζ	1.5545	1.3594	1.3356	1.5044	1.5189	(0.8774,2.2316) 1.3542	(0.9015,2.2173) 1.3160
	δ	1.5925	1.6575	1.6658	1.5443	1.6074	(0.9010,2.2841) 1.3834	(0.9015,2.2173) 1.3533
	$r(1)$	0.6360	0.6544	0.6567	0.6218	0.6403	(0.4357,0.8363) 0.4006	(0.4323,0.8149) 0.3826
	$h(1)$	2.2547	1.9068	1.8654	2.2366	2.1863	(0.7727,3.7367) 2.964	(0.9038,3.6697) 2.766
III	ζ	1.7750	1.5815	1.5584	1.7475	1.7274	0.9308 2.6192 1.688	0.9406 2.5577) 1.617
	δ	1.0667	1.0841	1.1023	1.0421	1.0946	0.5212 1.6123 1.091	0.9406 2.5577) 1.055
	$r(1)$	0.4548	0.4618	0.4690	0.4449	0.4659	(0.2360,0.6736) 0.4376	(0.2388,0.6490) 0.4102
	$h(1)$	3.339	2.951	2.882	3.326	3.207	(1.057,5.621) 4.564	(1.114,5.344) 4.230

To assess the convergence of the MCMC samples, trace plots for the posterior distributions of ζ, δ, $r(t)$, and $h(t)$ under the complete mechanical component dataset are shown in Figures 5 and 6. These plots display 4×10^4 values with their sample means (solid red lines) and 95% credible interval limits (dotted red lines). The well-mixed chains indicate favourable convergence and effective exploration of the parameter space, suggesting the MCMC chain has likely converged and accurately approximates the target distribution.

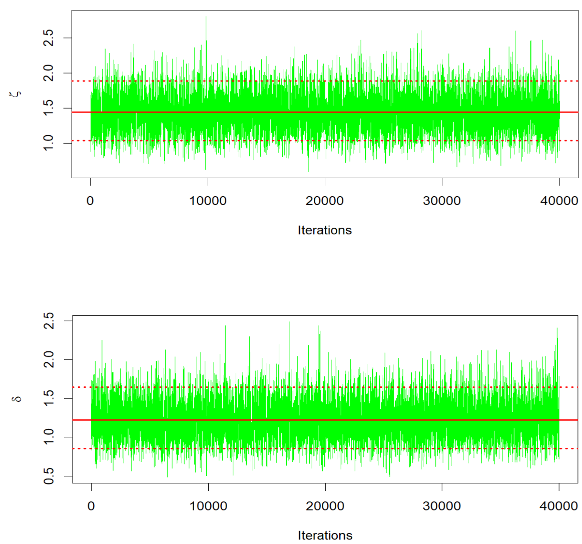


Figure 5: Trace plot of ζ (left) and δ (right) based on mechanical component dataset.

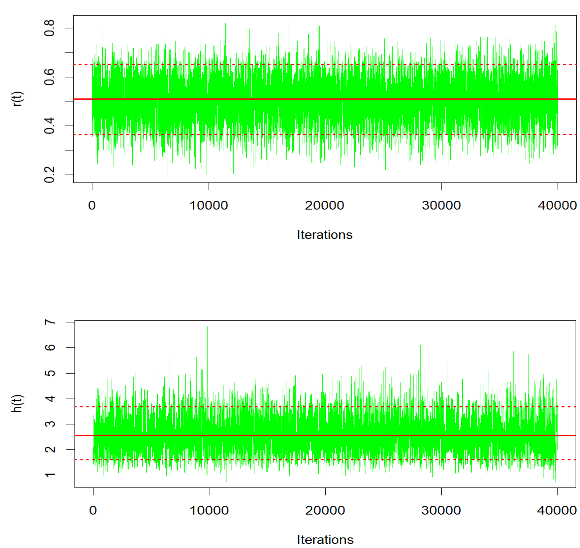


Figure 6: Trace plot of $r(t)$ (left) and $h(t)$ (right) for based on mechanical component dataset.

We now focus on selecting the optimal CS using criteria A, B, and C based on the variance-covariance matrix for the complete mechanical component dataset. The computed values, presented in Table 17, indicate that PCS $R = (5, 0^{*}8, 5)$ is optimal under criterion I due to the lowest trace of the inverse Fisher information matrix. Additionally, CS $R = (0^{*}9, 10)$ is optimal under criteria B and C, as it has the lowest value for criterion B and the highest for criterion C.

Table 17: Optimal CS using three PT-II CS based on mechanical component data.

Data	Criterion A	Criterion B	Criterion C
I	0.3548	0.02005	17.7
II	**0.3225**	0.01433	22.5
III	0.6745	**0.0027**	**248.5**

Dataset II

Dataset II consists of simulated measurements of glass fiber strengths, comprising 63 observations, as reported by Enogwe *et al.* Enogwe et al. (2021). To evaluate the suitability of the IPM distribution for modeling this dataset, the Kolmogorov-Smirnov (K-S) distance and its corresponding p-value are calculated. The results yielded a K-S distance of 0.07 with a p-value of 0.90, indicating that the IPM distribution provides a good fit to the data. The MLEs of the parameters are estimated as $\zeta = 2.156$ and $\delta = 7.122$. Additionally, the contour plot in Figure 7 confirms the existence and uniqueness of the MLEs of $\hat{\zeta}$ and $\hat{\delta}$.

To analyze Dataset II, three artificial Type-II progressive censoring scheme (T-II PCS) samples with $m = 33$ are generated and tabulated in Table 18. Point and interval estimates for the unknown parameters and reliability indices at time $t = 1.5$ are computed and presented in Table 19. In the Bayesian framework, non-informative priors ($a_1 = b_1 = a_2 = b_2 = 0$) is used since no prior information about the parameters was available. The convergence of MCMC samples is assessed using trace plots, shown in Figures 8 and 9. These plots confirm that the MCMC samples successfully converged to the target posterior distribution.

Table 18: Three artificial PT-II CS for glass fiber strength dataset.

Sample	**Scheme**	**Censored data**
1	$R = (10, 0^{*}9)$	1.014, 1.526, 1.535, 1.541, 1.568, 1.579, 1.581, 1.591, 1.593, 1.602 1.666, 1.67, 1.684, 1.691, 1.704, 1.731, 1.735, 1.747, 1.748, 1.757,1.800, 1.806, 1.867, 1.876, 1.878, 1.91, 1.916, 1.972, 2.012, 2.456, 2.592, 3.197, 4.121
2	$R = (5, 0^{*}8, 5)$	1.014, 1.306, 1.355, 1.361, 1.364, 1.379, 1.409, 1.426, 1.459, 1.46, 1.476, 1.481, 1.484, 1.501, 1.506, 1.524, 1.526, 1.535, 1.541, 1.568, 1.579, 1.581, 1.591, 1.593, 1.602, 1.666, 1.67, 1.684, 1.691, 1.704, 1.731, 1.735, 1.747
3	$R = (0^{*}32, 30)$	1.014, 1.081, 1.082, 1.185, 1.223, 1.248, 1.267, 1.271, 1.272, 1.275, 1.276, 1.278, 1.286, 1.288, 1.292, 1.304, 1.306, 1.355, 1.361, 1.364, 1.379, 1.409, 1.426, 1.459, 1.46, 1.476, 1.481, 1.484, 1.501, 1.506, 1.524, 1.526, 1.535

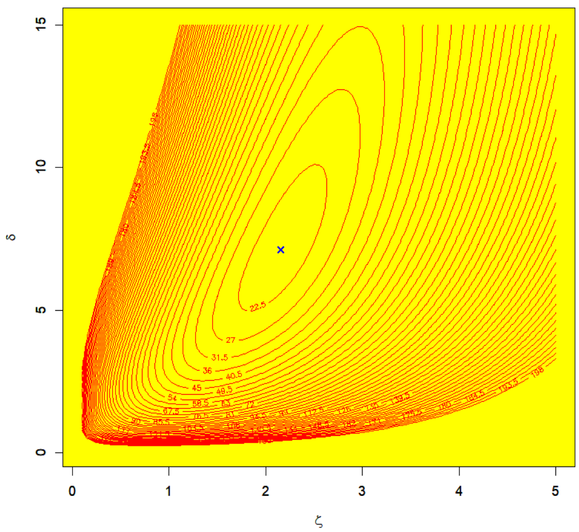


Figure 7: Contour plot of log-likelihood function for various choices of ζ and δ based on glass fiber strength dataset.

Table 19: Point and Interval estimates of parameters based on glass fiber strengths dataset.

		Point Estimate					Interval Estimate	
Sample	**Par.**	**MLE**	**LSE**	**WLSE**	**MCMC**	**Lindley**	**ACI**	**HPD**
I	ζ	1.939	3.774	3.911	1.867	1.917	(1.4910,2.3871)	(1.4863,2.3372)
							0.8958	0.8516
	δ	10.759	76.824	90.280	9.926	10.723	(5.7362, 15.7833)	(6.0943,15.6832)
							10.0460	9.5893
	$r(1.5)$	0.7848	0.9342	0.9443	0.7756	0.7904	(0.6727, 0.8968)	(0.6652,0.8774)
							0.3828	0.3588
	$h(1.5)$	1.3293	1.1349	1.0412	1.3159	1.2914	(0.7870,1.8716)	(0.8377,1.8733)
							1.0852	1.0361
II	ζ	2.274	2.923	2.957	2.191	2.256	(1.752, 2.796)	(1.7492, 2.765)
							1.043	1.016
	δ	11.154	20.234	20.917	10.367	11.153	(6.271, 16.038)	(6.618, 16.161)
							9.767	9.544
	$r(1.5)$	0.6829	0.7139	0.7166	0.6804	0.6896	(0.5752,0.7906)	(0.5682, 0.7827)
							0.2154	0.2146
	$h(1.5)$	2.011	2.418	2.430	1.947	1.968	(1.305, 2.717)	(1.249, 2.657)
							1.412	1.408
III	ζ	2.118	2.018	2.071	2.042	2.094	(1.583,2.653)	(1.586, 2.639)
							1.071	1.052
	δ	7.053	6.656	6.955	6.697	7.024	(4.573, 9.533)	(4.705, 9.509)
							4.961	4.804
	$r(1.5)$	0.5305	0.5411	0.5413	0.5348	0.5375	(0.4159, 0.6450)	(0.4245, 0.6430)
							0.2291	0.2185
	$h(1.5)$	1.3293	1.1349	1.0412	1.3159	1.2914	(0.7870,1.8716)	(0.8377,1.8733)
							1.773	1.699

Furthermore, the optimal progressive censoring schemes is determined based on the three generated samples presented in Table 20. From this table, it is observed that Sample III is optimal under criteria A and B, while sample I is optimal under criterion C.

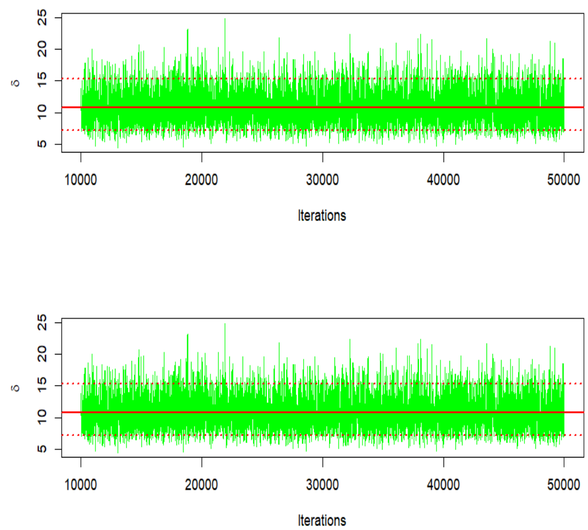


Figure 8: Trace plot of MCMC chains for ζ and δ based on glass fiber strengths dataset.

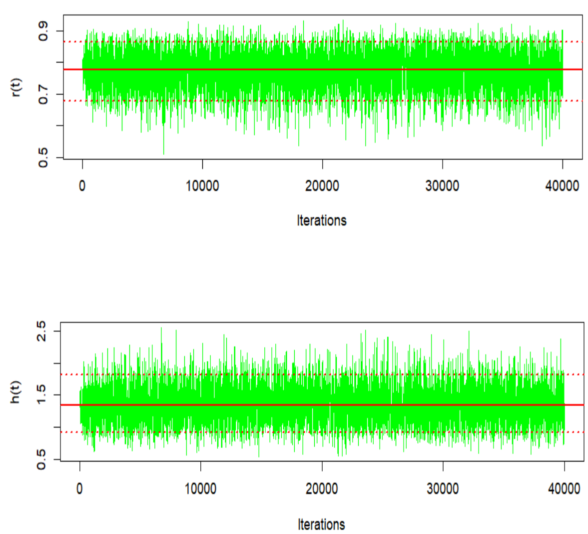


Figure 9: Trace plot of MCMC chains for $r(t)$ (left) and $h(t)$ (right) based on glass fiber strengths dataset.

7 Conclusions

In this article, the reliability and parameter estimation for the inverse power Maxwell distribution under progressive Type-II censored data have been developed. Based on both frequentist and Bayesian approaches, point and interval estimation procedures are proposed. In the classical approach, the MLE, LSE, and WLSE methods are utilized. Since the MLEs, LSEs, and WLSEs of the unknown parameters cannot be derived explicitly, Newton's iterative method has been implemented for this purpose. Similarly, due to the complex form of the likelihood function, Bayesian estimates have been obtained using Lindley's approximation and the MCMC method. The convergence of MCMC samples has been tested using various diagnostic plots. A Monte Carlo simulation study has been conducted to evaluate the performance of the proposed estimation methods. It has been found that the Bayesian MCMC method provides better results compared to the classical methods. Furthermore, HPD credible intervals outperform asymptotic confidence intervals (ACIs) in terms of shorter average lengths. To demonstrate the practical applicability of the proposed methodologies, two real dataset have been analyzed. The findings and methodologies presented in this study are expected to be valuable for practitioners in mechanical and reliability engineering.

It would be worthwhile to extend the proposed methods discussed in this study to investigate their applicability under competing risk analysis for the inverse power Maxwell distribution. Future research could also explore their performance under complex schemes, such as adaptive progressive hybrid and improved adaptive Type-II progressive censoring schemes.

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Data Availability: The real data used in this study were obtained from Murthy *et al.* Murthy et al. (2004). Also, the simulated data were generated by the authors using R software.

Author Contribution: Authors Mohd Irfan and Anup Kumar Sharma contributed equally to the conception and design of the study. First author conducted mathematical formulation, data description and numerical simulation, while second authors contributed to the interpretation of theoretical and empirical results. Both authors were actively involved in drafting and revising the manuscript, providing critical intellectual content, and approving the final version for publication.

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