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Some Results of Extropy-Based Measure of Divergence for Record Statistics

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Abstract. It is interesting to learn that the complementary dual of the Shannon entropy measure exists and has some common properties. This new measure of uncertainty has been introduced by Lad et al. (2015) and is known as extropy. Although there are some mathematical analogies between the two measures, extropy typically has different uses and interpretations than entropy. Taking into account the importance of extropy measure, and its various generalizations, in the present communication, we consider and study Kullback-Leibler based "divergence-extropy" measure between the distribution of n^{th} upper k-record and m^{th} upper k-record values. Characterization problems for the proposed "divergence-extropy" measure have been studied. Further, some specific lifetime distributions used in lifetime testing, physical sciences, survival analysis and reliability engineering have been studied using the proposed "divergence-extropy" measure. At the end, we study the proposed "divergence-extropy" measure between the distribution of k-record value and order statistics.

Keywords. K-L Information Measure, Extropy Measure, Divergence-Extropy Measure, Order Statistics, Record Value.

MSC: 62N05, 90B25.

1 Introduction

Lad et al. (2015) proposed and studied the complementary dual of Shannon's (1948)

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entropy measure called the extropy measure. This proposed measure is a new approach to measure uncertainty sharing some common characterization results and properties with Shannon's measure. Both of these measures are fundamentally interrelated and have different approaches to measure uncertainty. Divergence entropy and extropy are fundamental concepts in information theory and statistics that offer powerful tools for quantifying the difference between probability distributions. Their primary motivation lies in the need to measure the uncertainty, disorder, and information content of data. The applications of divergence measure can be found in the analysis of contingency tables see Gokhale and Kullback (1976), in approximation of probability distributions refer to Chow and Liu (1968) and Lin and Wong (1988), Kazakos and Cotsidas (1980), in signal processing refer to Kadota and Shepp (1967), and Kailath (1967), and in pattern recognition see Ben (1978), and Chen (1973).

Generalized extropy like Renyi extropy, Tsallis extropy, and their continuous versions have been studied by many researchers, refer to Liu and Xiao (2021), Xue and Deng (2021), and Mohamed et al. (2023) respectively.

The information measures for record values have been investigated by several authors, including Baratpour et al. (2007), Ahmadi and Fashandi (2009), Zarezadeh and Asadi (2010), Arghami et al. (2011), Baratpour and Khammar (2016), Goel et al. (2018a), Goel et al. (2018b), and Kumar and Dangi (2023).

Guoxin Qiu (2017) studied the concept of extropy of order statistics and record value. Also, the residual extropy of order statistics has been studied by Qiu and Jia (2018). The residual extropy of k-record values has been studied by Jose and Sathar (2019), and Sathar and Jose (2020). Testing symmetry based on the extropy of record values has been studied by Xiong et al. (2021). Further, weighted extropies and past extropy of order statistics and k-record values studied by Bansal and Gupta (2022). Maximum extropy negation of a probability distribution is studied by Liu and Xiao (2024), and Hashempour and Mohammadi (2024), study a new measure of inaccuracy based on extropy between distributions of the nth upper (lower) record value and parent random variable and discuss some properties of it, and stochastic inequalities involving past extropy of order statistics and past extropy of record values have been studied by Shrahili et al. (2024).

With the divergent viewpoints, both measures (entropy and extropy) give insightful information that might enhance our comprehension of the data Lad et al. (2015). Some characterizations of continuous symmetric distributions based on the extropy of record values have been studied by Gupta and Chaudhary (2023). More properties and characterization results for complementary dual of *K-L* divergence measure were studied by Lad et al. (2018).

An interesting question is whether extropy-based measure of divergence for record statistics can be interpreted as the amount of information lost when approximating one distribution with another. Study for the *"divergence-extropy"* measures between the distribution of n^{th} upper k-record value and m^{th} upper k-record found to be an open problem throughout the literature.

In this paper, the *"divergence-extropy"* measure between the distribution of n^{th} upper k-record value and m^{th} upper k-record value and its characterization result are discussed in Section 2, "divergence-extropy" measure based on k-record values for specific distribution is studied in Section 3. In section 4, the "divergence-extropy" measure between k-record value and order statistics is studied. Finally, Section 5 concludes this paper.

1.1 Preliminaries

The extropy Lad et al. (2015) measure for a discrete random variable with corresponding probability mass function (pmf) P is given as

$$J(P) = -\sum_{i=1}^{n}(1-p_i)\log(1-p_i). \tag{1.1}$$

Suppose that the vector S represents an alternative pmf in comparison to P. Then complementary dual for Kullback-Leible (1967) directed distance measure known as relative extropy given as

$$J(P||S) = -\sum_{i=1}^{n}(1-p_i)\log\frac{(1-p_i)}{(1-s_i)}, \tag{1.2}$$

refer to Lad et al. (2015). More properties and characterization results for complementary dual of K-L divergence measure (1.2) were studied by Lad et al. (2018). In case of continuous random variable X with probability density function (pdf) $f(x)$, the extropy measure as follows

$$J(f) = -\frac{1}{2}\int_0^\infty f^2(x)dx. \tag{1.3}$$

Hashempour and Mohammadi (2023) proposed the "extropy-inaccuracy" measure between the distribution of continuous random variables X and Y defined as

$$J(f,g) = -\frac{1}{2}\int_0^\infty f(x)g(x)dx. \tag{1.4}$$

Further divergence information measure based on extropy between pdfs $f(x)$ and $g(x)$ is defined as

$$J(f||g) = \frac{1}{2}\int_0^\infty \{f(x)-g(x)\}f(x)dx. \tag{1.5}$$

Record values are order statistics drawn from a sample of identically distributed and independent random variables $X_1, X_2, ..., X_n$, the size of which depends on the values and the observational variable's order of appearance. An observational variable X_k will be referred to as an upper record value, if the value of X_k is greater than all the prior observational variables. Thus, the pdf of n^{th} upper record values are

$$f_n^u(x) = \frac{\{-\log(\bar{F}(x))\}^{n-1}}{\Gamma(n)} f(x), \ -\infty < x < +\infty. \tag{1.6}$$

Dziubdziela and Kopociński (1976) introduced the notion of k-record values. The pdf of the n^{th} upper k-record value is given as

$$f_{n,k}^u(x) = \frac{k^n\{-\log(\bar{F}(x))\}^{n-1}\{\bar{F}(x)\}^{k-1}}{\Gamma(n)} f(x), \ -\infty < x < +\infty, \tag{1.7}$$

In analogous to (1.3), the extropy measure of the n^{th} upper k-record has been studied by Xiong et al. (2021), which is defined as

$$J(f_{n,k}^u) = -\frac{1}{2}\int_0^\infty f_{n,k}^{u\,2}(x)dx. \tag{1.8}$$

Recently, Mohammadi and Hashempour (2023) suggested and investigated a new measure of inaccuracy called ″extropy-inaccuracy" measure between the distribution of n^{th} upper record value and parent distribution given as

$$J(f_n^u, f) = -\frac{1}{2}\int_0^\infty f_n^u(x)f(x)dx = J(f_n^u\|f) + J(f), \tag{1.9}$$

where $J(f_n^u\|f) = \frac{1}{2}\int_0^\infty f_n^u(x)\{f(x) - f_n^u(x)\}dx$ is "divergence-extropy" measure between the distribution of n^{th} upper record value and parent distribution.
Next section we introduce the concept of "divergence-extropy" measure between k-record statistics.

2 "Divergence-extropy" Measure Between k-Record Values

Let X and Y be continuous random variables with their corresponding n^{th} upper k-record value's pdf $f_{n,k}^u(x)$ and m^{th} upper k-record value's pdf $f_{m,k}^u(x)$. If $f_{n,k}^u(x)$ is the actual distribution and $f_{m,k}^u(x)$ is the experimental distribution, then the *″divergence-extropy″* measure $DJ(f_{n,k}^u\|f_{m,k}^u)$ between the distribution of n^{th} upper k-record value and m^{th} upper k-record value is defined as

$$\begin{aligned} DJ(f_{n,k}^u\|f_{m,k}^u) =& \frac{1}{2}\int_0^\infty f_{n,k}^u(x)\{f_{n,k}^u(x) - f_{m,k}^u(x)\}dx \\ =& \frac{1}{2}\int_0^\infty \{f_{n,k}^u(x)\}^2dx - \frac{1}{2}\int_0^\infty f_{n,k}^u(x)f_{m,k}^u(x)dx. \end{aligned} \tag{2.1}$$

Remark 1. If pdf of m^{th} upper k-record value is replaced by the pdf of parent random variable, then (2.1) reduce to

$$DJ(f_{n,k}^u\|f) = \frac{1}{2}\int_0^\infty \{f_{n,k}^u(x)\}^2dx - \frac{1}{2}\int_0^\infty f_{n,k}^u(x)f(x)dx, \tag{2.2}$$

a result obtained by Gupta and Chaudhary (2023) and Xiong et al. (2021), where $-\frac{1}{2}\int_0^\infty f^u_{n,k}(x)f(x)dx$ is known as the inaccuracy-extropy measure.

Theorem 2.1. *The "divergence-extropy" measure (2.1) can be expressed as*

$$\begin{aligned} DJ(f^u_{n,k}\|f^u_{m,k}) = &-\frac{1}{2}\frac{\Gamma(m+n-1)}{\Gamma(m)\Gamma(n)}\frac{k^{n+m}}{(2k-1)^{n+m-1}}E[f\{F^{-1}(1-e^{T})\}] \\ &+\frac{1}{2}\frac{k^{2n}}{(2k-1)^{2n-1}}\frac{\Gamma(2n-1)}{\{\Gamma(n)\}^2}E[f\{F^{-1}(1-e^{-U})\}], \end{aligned} \tag{2.3}$$

where $T \sim \Gamma(n+m-1, 2k-1)$, *and* $U \sim \Gamma(2n-1, 2k-1)$ *and E is the expectation.*

Proof. Using (1.7), "divergence-extropy" measure (2.1) can be express as

$$\begin{aligned} DJ(f^u_{n,k}\|f^u_{m,k}) = &\frac{1}{2}\int_0^\infty \frac{k^{2n}}{(\Gamma(n))^2}\{-\log(\bar{F}(x))\}^{2(n-1)}\{\bar{F}(x)\}^{2(k-1)}f^2(x)dx \\ &-\frac{1}{2}\int_0^\infty \frac{k^{n+m}}{\Gamma(n)\Gamma(m)}\{-\log(\bar{F}(x))\}^{m+m-2}\{\bar{F}(x)\}^{2(k-1)}f^2(x)\,dx. \end{aligned}$$

Taking $-\log\{\bar{F}(x)\} = t,\ \bar{F}(x) = e^{-t}$ and $x = f\{F^{-1}(1-e^{-t})\}$, we obtain

$$\begin{aligned} DJ(f^u_{n,k}\|f^u_{m,k}) = &\frac{1}{2}\int_0^\infty \frac{k^{2n}}{(\Gamma(n))^2}t^{2(n-1)}e^{-(2k-1)t}f\{F^{-1}(1-e^{-t})\}\,dt \\ &-\frac{1}{2}\int_0^\infty \frac{k^{n+m}}{\Gamma(n)\Gamma(m)}t^{n+m-2}e^{-(2k-1)t}f\{F^{-1}(1-e^{-t})\}\,dt. \end{aligned} \tag{2.4}$$

The equation (2.4) can be written as

$$\begin{aligned} DJ(f^u_{n,k}\|f^u_{m,k}) = &\frac{1}{2}\frac{k^{2n}}{(2k-1)^{2n-1}}\frac{\Gamma(2n-1)}{\{\Gamma(n)\}^2}E[f\{F^{-1}(1-e^{-U})\}] \\ &-\frac{1}{2}\frac{\Gamma(m+n-1)}{\Gamma(m)\Gamma(n)}\frac{k^{n+m}}{(2k-1)^{n+m-1}}E[f\{F^{-1}(1-e^{T})\}]. \end{aligned}$$

where U, V follow the gamma distribution with pdfs $\frac{(2k-1)^{2n-1}}{\Gamma(2n-1)}t^{2n-2}e^{-(2k-1)t}$ and $\frac{(2k-1)^{n+m-1}}{\Gamma(m+n-1)}t^{n+m-2}e^{-(2k-1)t}$ respectively. So the result follows. □

In the next subsection, we show that the proposed "divergence-extropy" measure (2.1) characterizes the distribution function of parent random variable uniquely. To prove this we use the following lemma, refer to Goffman and Pedrick (1965, pp. 192-193).

2.1 Characterization Results

Lemma 2.1. *Golffman and Pedrick (1965) A complete orthogonal system for the space* $L_2(0,\infty)$ *is given the sequence of Laguerre functions*

$$\phi_n(x) = \frac{1}{n!}e^{-\frac{x}{2}}L_n(x),\ \forall\ n \geq 0,$$

where $L_n(x)$ is the Laguerre polynomial defined as, the sum of coefficients of e^{-x} in the n^{th} derivative of $e^{-x}\, x^n$ that is

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}).$$

The compactness of the Laguerre functions in $L_2(0,\infty)$ means that if $f \in L_2(0,\infty)$ and $\int_0^\infty f(x)e^{-\frac{x}{2}} L_n(x)dx = 0$, then f is zero almost everywhere.

Theorem 2.2. Let X_1 and X_2 be two absolutely continuous random variables with pdfs f_1, f_2 and cdfs F_1, F_2, respectively. If "divergence-extropy" measures between distribution of n^{th} and m^{th} upper k-record values satisfy $DJ(f_{n,k}^{u(i)} \| f_{m,k}^{u(i)}) < \infty,\ i = 1, 2$. Then F_1, F_2 belong to the same family of distribution (but with different location) if and only if

$$DJ(f_{n,k}^{u(1)} \| f_{m,k}^{u(1)}) = DJ(f_{n,k}^{u(2)} \| f_{m,k}^{u(2)}),$$

where $f_{n,k}^{u(i)}$ and $f_{m,k}^{u(i)}$ are the pdfs of n^{th} and m^{th} upper k-record values corresponding to f_i for $i = 1, 2$.

Proof. The necessary is evident; we will demonstrate the sufficient. To see this, if $DJ(f_{n,k}^{u(1)} \| f_{m,k}^{u(1)}) = DJ(f_{n,k}^{u(2)} \| f_{m,k}^{u(2)})$, or $DJ(f_{n,k}^{u(1)} \| f_{m,k}^{u(1)}) - DJ(f_{n,k}^{u(2)} \| f_{m,k}^{u(2)}) = 0$. Using (1.7) and (2.1) in above, we get

$$\Rightarrow \frac{1}{2}\int_0^\infty \frac{k^n}{\Gamma(n)}\{-\log \bar{F}_1(x)\}^{n-1}\{\bar{F}_1(x)\}^{2(k-1)} f_1^2(x) \left\{\frac{k^n}{\Gamma(n)}\{-\log \bar{F}_1(x)\}^{n-1} - \frac{k^m}{\Gamma(m)}\{-\log \bar{F}_1(x)\}^{m-1}\right\} dx$$
$$-\frac{1}{2}\int_0^\infty \frac{k^n}{\Gamma(n)}\{-\log \bar{F}_2(x)\}^{n-1}\{\bar{F}_2(x)\}^{2(k-1)} f_2^2(x) \left\{\frac{k^n}{\Gamma(n)}\{-\log \bar{F}_2(x)\}^{n-1} - \frac{k^m}{\Gamma(m)}\{-\log \bar{F}_2(x)\}^{m-1}\right\} dx = 0.$$

Put $u = -\log \bar{F}(x)$, this gives $e^{-u} = \bar{F}(x)$ and $f(x)dx = e^{-u}du$. Substituting these values in above expression, we obtain

$$\Rightarrow \frac{1}{2}\int_0^\infty u^{n-1} e^{-2(k-1)u} e^{-u} f_1(x) \left\{\frac{k^n}{\Gamma(n)} u^{n-1} - \frac{k^m}{\Gamma(m)} u^{m-1}\right\} du$$
$$-\frac{1}{2}\int_0^\infty u^{n-1} e^{-2(k-1)u} e^{-u} f_2(x) \left\{\frac{k^n}{\Gamma(n)} u^{n-1} - \frac{k^m}{\Gamma(m)} u^{m-1}\right\} du = 0;$$
$$\Rightarrow \frac{1}{2}\int_0^\infty u^{n-1} e^{(-2k+1)u} \left\{\frac{k^n}{\Gamma(n)} u^{n-1} - \frac{k^m}{\Gamma(m)} u^{m-1}\right\} \{f_1(x) - f_2(x)\} du = 0;$$
$$\Rightarrow \frac{1}{2}\int_0^\infty u^{n-1} e^{(-2k+1)u} \left\{\frac{k^n}{\Gamma(n)} u^{n-1} - \frac{k^m}{\Gamma(m)} u^{m-1}\right\} [f_1\{F_1^{-1}(1-e^{-u})\} - f_2\{F_2^{-1}(1-e^{-u})\}] du = 0;$$
$$\Rightarrow \frac{1}{2}\int_0^\infty [f_1\{F_1^{-1}(1-e^{-u})\} - f_2\{F_2^{-1}(1-e^{-u})\}] L_n(u) e^{-\frac{u}{2}} du = 0.$$

Using lemma 2.1, we have $f_1\{F_1^{-1}(1-e^{-u})\} = f_2\{F_2^{-1}(1-e^{-u})\}$. By putting $(1 - e^{-u}) = v$, we get

$$\frac{d}{dv} F^{-1}(v) = \frac{1}{f(F^{-1}(v))}.$$

It follows that

$$(F_1^{-1})'(v) = (F_2^{-1})'(v)\,, \forall\, v \in (0,1),$$
$$F_1^{-1}(v) = F_2^{-1}(v) + constant.$$

Hence the result is complete. □

To demonstrate the impact of monotone transformations on the proposed "divergence-extropy" measure (2.1), we provide the following result.

Theorem 2.3. *Consider two continuous random variables X and Y with corresponding cdf $F(x)$, $G(x)$ pdfs $f(x)$, $g(x)$ respectively. Let $Y = \phi(x)$, where $\phi(.)$ is a real-valued, increasing, and convex (concave) function. Also $X_{n,k}$ and $Y_{m,k}$ be the n^{th} upper k-upper record value and m^{th} upper k-upper record value associated with X and Y respectively. Then $DJ(g^u_{n,k}\|g^u_{m,k}) = \frac{1}{\phi'(x)} DJ(f^u_{n,k}\|f^u_{m,k})$.*

Proof. From (2.1), we have

$$\begin{aligned} DJ(g^u_{n,k}\|g^u_{m,k}) =& \frac{1}{2}\int_0^\infty g^u_{n,k}(y)(g^u_{n,k}(y) - g^u_{m,k}(y))dy \\ =& \frac{1}{2}\int_0^\infty \{g^u_{n,k}(y)\}^2 dy - \frac{1}{2}\int_0^\infty g^u_{n,k}(y) g^u_{m,k}(y) dy. \end{aligned}$$

Using (1.7), we obtain

$$\begin{aligned} DJ(g^u_{n,k}\|g^u_{m,k}) =& \frac{1}{2}\int_0^\infty \frac{k^{2n}}{\Gamma(n)^2}\{-\log \bar{G}(y)\}^{2(n-1)}\{\bar{G}(y)\}^{2(k-1)} g^2(y)\, dy \\ &- \frac{1}{2}\int_0^\infty \frac{k^{n+m}}{\Gamma(m)\,\Gamma(n)}\{-\log \bar{G}(y)\}^{m+n-2}\{\bar{G}(y)\}^{2(k-1)} g^2(y)\, dy. \end{aligned} \tag{2.5}$$

We have the probability density function of $Y = \phi(X)$ is $g(y) = \frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))}$. Then the "divergence-extropy" (2.5) measure becomes

$$\begin{aligned} DJ(g^u_{n,k}\|g^u_{m,k}) =& \frac{1}{2}\int_0^\infty \frac{k^{2n}}{\Gamma(n)^2}\{-\log \bar{F}(\phi^{-1}(y))\}^{2(n-1)}\{\bar{F}(\phi^{-1}(y))\}^{2(k-1)} \left\{\frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))}\right\}^2 dy \\ &- \frac{1}{2}\int_0^\infty \frac{k^{n+m}}{\Gamma(m)\,\Gamma(n)}\{-\log \bar{F}(\phi^{-1}(y))\}^{m+n-2}\{\bar{F}(\phi^{-1}(y))\}^{2(k-1)} \left\{\frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))}\right\}^2 dy. \end{aligned}$$

By taking $x = \phi^{-1}(y)$, we obtain

$$\begin{aligned} DJ(g^u_{n,k}\|g^u_{m,k}) =& \frac{1}{2}\int_0^\infty \frac{k^{2n}}{\Gamma(n)^2}\{-\log \bar{F}(x)\}^{2(n-1)}\{\bar{F}(x)\}^{2(k-1)} \frac{f^2(x)}{\phi'(x)}\, dx \\ &- \frac{1}{2}\int_0^\infty \frac{k^{n+m}}{\Gamma(m)\,\Gamma(n)}\{-\log \bar{F}(x)\}^{m+n-2}\{\bar{F}(x)\}^{2(k-1)} \frac{f^2(x)}{\phi'(x)}\, dx. \end{aligned} \tag{2.6}$$

The equation (2.6) can be written as

$$DJ(g^u_{n,k}||g^u_{m,k}) = \frac{1}{2}\int_0^\infty \frac{1}{\phi'(x)}\{f^u_{n,k}(x)\}^2\,dx - \frac{1}{2}\int_0^\infty \frac{1}{\phi'(x)} f^u_{n,k}(x) f^u_{m,k}(x)\,dx;$$
$$= \frac{1}{2}\int_0^\infty \frac{1}{\phi'(x)} f^u_{n,k}(x)\{f^u_{n,k}(x) - f^u_{m,k}(x)\}dx.$$

$$DJ(g^u_{n,k}||g^u_{m,k}) = \frac{1}{\phi'(x)} DJ(f^u_{n,k}||f^u_{m,k}).$$

This proves the result. □

Remark 2. For any continuous random variable X, define $Y = aX + b$, where $a > 0$, b are constants. Then

$$DJ(g^u_{n,k}||g^u_{m,k}) = \frac{1}{a} DJ(f^u_{n,k}||f^u_{m,k}).$$

The "divergence-extropy" measure defined in (2.1) is invariant under scale but not under location transformation.

Considering the importance of some specific probability distributions in real-life events like Exponential, Uniform and Weibull distribution etc., we study the proposed "divergence-extropy" measure (2.3) for these distributions in next section.

3 "Divergence-extropy" Measure for Some Probability Distributions

3.1 $DJ(f^u_{n,k}||f^u_{m,k})$ for Exponential Distribution

Mathematical properties make the exponential distribution valuable for modeling a wide range of phenomena involving time or distance between events that is the exponential distribution is memoryless. So the study of the "divergence-extropy" measure (2.1) for the exponential distribution is important. Consider a random variable X following exponential distribution with pdf and survival function are given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \lambda > 0, \\ 0 & x \le 0, \end{cases}$$

$$\bar{F}(x) = e^{-\lambda x}.$$

Using the expression (2.3), the "divergence-extropy" measure (2.1) for exponential distribution is given as

$$DJ(f^u_{n,k}||f^u_{m,k}) = \frac{\lambda k}{2^n \Gamma(n)}\left(\frac{\Gamma(2n-1)}{2^n\,\Gamma(n)} - \frac{\Gamma(m+n-1)}{2^m\,\Gamma(m)}\right). \tag{3.1}$$

In particular, $k = 1$ and $m = 1$ in (3.1); we obtain the "divergence-extropy" measure between the distribution of n^{th} upper record value and parent random variable

$$DJ(f_n^u \| f) = \frac{\lambda}{2^{2n}\{\Gamma(n)\}^2}\Gamma(2n-1) - \frac{\lambda}{2^n}.$$

Similarly, the "divergence-extropy" measure between the distributions of n^{th} upper k-record value and parent random variable is obtained as follows

$$DJ(f_{n,k}^u \| f) = \frac{\lambda k}{2^{2n}\{\Gamma(n)\}^2}\Gamma(2n-1) - \frac{1}{2}\frac{k^n \lambda}{(k+1)^n}.$$

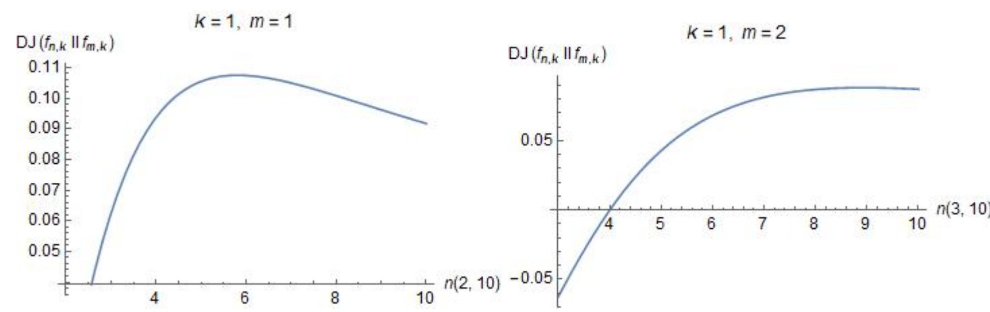

Figure 1: Eponential distributed "divergence-extropy" measure (2.1) with mean 0.5

3.2 $DJ(f_{n,k}^u \| f_{m,k}^u)$ for Uniform Distribution

The uniform distribution is a simple probability distribution where all outcomes are equally likely. It's often used to model situations where we have no prior knowledge about the distribution of a random variable. So the study of the proposed "divergence-extropy" measure (2.1) for uniform distribution is important.

Let us consider X be uniform distribution random variable over (a, b) with pdf and cdf $f(x) = \frac{1}{b-a}$, and $F(x) = \frac{x-a}{b-a}$, $\forall a \leq x \leq b$ respectively. Using transformation $-\log \bar{F}(x) = t$, for uniform distribution we get $E[f\{F^{-1}(1-e^{-t})\}] = \frac{1}{b-a}$. Thus "divergence-extropy" measure (2.3) for uniform distribution is

$$DJ(f_{n,k}^u \| f_{m,k}^u) = \frac{k^n}{2\Gamma(n)(b-a)}\left(\frac{k^n\Gamma(2n-1)}{\Gamma(n)(2k-1)^{2n-1}} - \frac{k^m\Gamma(m+n-1)}{\Gamma(m)(2k-1)^{m+n-1}}\right). \quad (3.2)$$

For $m = 1,\ k = 1$, (3.2) reduces to

$$J(f_n^u \| f) = \frac{\Gamma(2n-1)}{2(b-a)\{\Gamma(n)\}^2} - \frac{1}{2(b-a)}, \quad (3.3)$$

the equation (3.3) provides the "divergence-extropy" measure between the distribution of n^{th} upper record value and parent distribution.

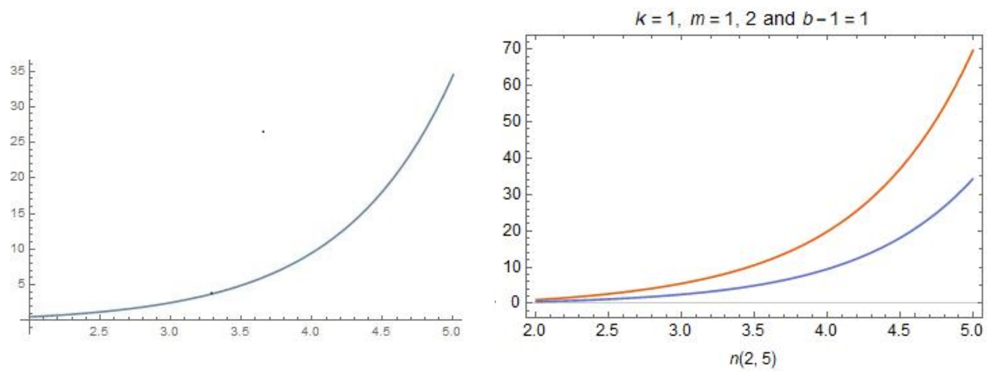

Figure 2: Uniformally distributed "divergence-extropy" measure (2.1) for $k = 1$, $m = 1$, $b - a = 1$ at left side and for $k = 1$, $m = 1, 2$, $b - a = 1$ are at right side of the figure.

3.3 $DJ(f^u_{n,k} || f^u_{m,k})$ for Weibull Distribution

The Weibull distribution is a flexible probability distribution that can be used to model a wide range of phenomena, including failure times. So the study of the "divergence-extropy" measure (2.1) for the Weibull distribution is important.

A non-negative random variable X is Weibull distributed, if its pdf is

$$f(x) = \lambda\, b\, x^{b-1} e^{-\lambda x^b},\ \lambda,\ b > 0,\ x > 0,$$

where $\lambda,\ b > 0$ are the shape parameters respectively. The survival function is

$$\bar{F}(x) = e^{-\lambda x^b}\ or\ -\log \bar{F}(x) = \lambda x^b.$$

By using the transformation $-\log \bar{F}(x) = t =$, this gives $x = F^{-1}(1 - e^{-t}) = (\frac{t}{\lambda})^{\frac{1}{b}}$, and for computing $DJ(f^u_{n,k} || f^u_{m,k})$, we have

$$f\{F^{-1}(1 - e^{-t})\} = f\left\{\left(\frac{t}{\lambda}\right)^{\frac{1}{b}}\right\} = b\lambda^{\frac{1}{b}} t^{1-\frac{1}{b}} e^{-t}.$$

Thus the "divergence-extropy" measure (2.3) for Weibull distribution obtain as

$$DJ(f_{n,k} || f_{m,k}) = \frac{b(\lambda)^{\frac{1}{b}} k^{\frac{1}{b}}}{2\Gamma(n)} \left\{ \frac{\Gamma(2n - \frac{1}{b})}{2^{2n-\frac{1}{b}}\, \Gamma(n)} - \frac{\Gamma(m + n - 1)}{2^{m+n-\frac{1}{b}}\, \Gamma(m)} \right\}, \tag{3.4}$$

Remark 3. For $b = 1$, (3.4) reduces to (3.1), a result of "divergence-extropy" measure for the exponential distribution.

4 "Divergence-extropy" Measure Between k-Record Value and Order Statistic

Record values and order statistics have close connections to concepts in probability theory and statistics. Both originate from researching data sequences and extreme values. Record values are special examples of order statistics. The k-upper record value and the k^{th} greatest order statistic in the data sequence are equivalent. Despite their disparate applications, these concepts are related, which depends our understanding of exceptional events and data characteristics.

Let $f_{i:n}(x)$ and $f_{1:n}(x)$ represents the pdfs of i^{th} order statistic and first-order statistics given as

$$f_{i:n}(x) = \frac{1}{\beta(i, n-i+1)}\{\bar{F}(x)\}^{n-1}\{F(x)\}^{i-1}f(x), \tag{4.1}$$

and

$$f_{1:n}(x) = n\{\bar{F}(x)\}^{n-1}f(x), \tag{4.2}$$

respectively. Analogous to (2.1), we propose the "divergence-extropy" measure between the distributions of m^{th} upper k-record value and the i^{th} order statistics, which is defined as follow

$$\begin{aligned} DJ(f^u_{m,k}\|f_{i:n}) =& \frac{1}{2}\int_0^\infty f^u_{m,k}(x)\{f^u_{m,k}(x) - f_{i:n}(x)\}dx \\ =& \frac{1}{2}\int_0^\infty \{f^u_{m,k}(x)\}^2dx - \frac{1}{2}\int_0^\infty f^u_{m,k}(x)f_{i:n}(x)dx. \end{aligned} \tag{4.3}$$

Similarly, the"divergence-extropy" measure between the distributions of m^{th} upper k-record value and first order statistics is given as

$$DJ(f^u_{m,k}\|f_{1:n}) = \frac{1}{2}\int_0^\infty \{f^u_{m,k}(x)\}^2dx - \frac{1}{2}\int_0^\infty f^u_{m,k}(x)f_{1:n}(x)dx. \tag{4.4}$$

Theorem 4.1. *The "divergence-extropy" measure $DJ(f^u_{m,k}\|f_{1:n})$ can be expressed as*

$$\begin{aligned} DJ(f_{m,k}\|f_{1:n}) =& \frac{k^{2m}}{2(2k-1)^{2m-1}}\frac{\Gamma(2m-1)}{\{\Gamma(m)\}^2}E[f\{F^{-1}(1-e^{-U})\}] \\ &- \frac{nk^m}{2(n+k-1)^m}E[f\{F^{-1}(1-e^{-V})\}], \end{aligned} \tag{4.5}$$

where $U \sim \Gamma(2m-1, 2k-1)$, $V \sim \Gamma(m, n+k-1)$ and E is the expectation.

Proof. Using (1.7) and (4.2), the "divergence-extropy" measure (4.4) can be expressed as

$$\begin{aligned} DJ(f^u_{m,k}\|f_{1:n}) =& \frac{1}{2}\frac{k^{2m}}{\{\Gamma(m)\}^2}\int_0^\infty \{-\log\bar{F}(x)\}^{2(m-1)}\{\bar{F}(x)\}^{2(k-1)}f^2(x)dx \\ &- \frac{1}{2}\frac{nk^m}{\Gamma(m)}\int_0^\infty \{-\log\bar{F}(x)\}^{(m-1)}\{\bar{F}(x)\}^{n+k-2}f^2(x)\,dx. \end{aligned}$$

Substituting $-\log\{\bar{F}(x)\} = t,\ \bar{F}(x) = e^{-t}$ and $x = f\{F^{-1}(1-e^{-t})\}$ in the above expression, we get

$$\begin{aligned} DJ(f^u_{m,k}\|f_{1:n}) =& \frac{1}{2}\frac{k^{2m}}{(2k-1)^{2m-1}}\int_0^\infty \frac{1}{\{\Gamma(m)\}^2}t^{2(m-1)}e^{-t}f\{F^{-1}(1-e^{-t})\}\,dt \\ &- \frac{1}{2}\frac{nk^m}{(k+n-1)^m}\int_0^\infty \frac{1}{\Gamma(m)}t^{(m-1)}e^{-t}f\{F^{-1}(1-e^{-t})\}\,dt. \end{aligned}$$

This can be written as follows

$$DJ(f^u_{m,k}\|f_{1:n}) = \frac{1}{2}\frac{k^{2m}}{(2k-1)^{2m-1}}\frac{\Gamma(2m-1)}{\{\Gamma(m)\}^2}E[f\{F^{-1}(1-e^{-U})\}]-\frac{1}{2}\frac{nk^m}{(k+n-1)^m}E[f\{F^{-1}(1-e^{-V})\}],$$

where U follow the gamma distribution with pdf $\frac{(2k-1)^{2m-1}}{\Gamma(2m-1)}t^{2m-2}e^{-(2k-1)t}$ and V follow the gamma distribution with pdf $\frac{(k+n-1)^m}{\Gamma(m)}t^{m-1}e^{-(k+n-1)t}$ respectively. Hence proved. □

To demonstrate the impact of monotone transformations on the "divergence-extropy" measure (4.4), we provide the following theorem.

Theorem 4.2. *Consider two continuous random variables X and Y with corresponding cdf $F(x)$, $G(x)$ and pdfs $f(x)$, $g(x)$ respectively. If $\phi(.)$ is a real-valued, increasing, and convex (concave) function defined as $Y = \phi(x)$, and $X_{m,k}$ be the m^{th} upper k-record value associated with the random variable X and $Y_{1:n}$ be the first order statistics value associated with the random variable Y. Then $DJ(g^u_{m,k}\|g_{1:n}) = \frac{1}{\phi'(x)}DJ(f^u_{m,k}\|f_{1:n})$.*

Proof. From (4.4), we have

$$DJ(g^u_{m,k}\|g_{1:n}) = \frac{1}{2}\int_0^\infty \{g^u_{m,k}(y)\}^2 dy - \frac{1}{2}\int_0^\infty g^u_{m,k}(y)g_{1:n}(y)dy.$$

Using (1.7) and (4.2), we obtain

$$\begin{aligned}DJ(g^u_{m,k}\|g_{1:n}) =&\frac{1}{2}\frac{k^{2m}}{\{\Gamma(m)\}^2}\int_0^\infty \{-\log \bar{G}(y)\}^{2(m-1)}\{\bar{G}(y)\}^{2(k-1)}g^2(y)dy\\ &-\frac{1}{2}\frac{nk^m}{\Gamma(m)}\int_0^\infty \{-\log \bar{G}(y)\}^{(m-1)}\{\bar{G}(y)\}^{n+k-2}g^2(y)\,dy.\end{aligned} \tag{4.6}$$

We have the probability density function of $Y = \phi(X)$ is $g(y) = \frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))}$. Then the "divergence-extropy" measure (4.6) becomes

$$\begin{aligned}DJ(g^u_{m,k}\|g_{1:n}) =&\frac{1}{2}\frac{k^{2m}}{\{\Gamma(m)\}^2}\int_0^\infty \{-\log \bar{F}(\phi^{-1}(y))\}^{2(n-1)}\{\bar{F}(\phi^{-1}(y))\}^{2(k-1)}\left\{\frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))}\right\}^2 dy\\ &-\frac{1}{2}\frac{nk^m}{\Gamma(m)}\int_0^\infty \{-\log \bar{F}(\phi^{-1}(y))\}^{(m-1)}\{\bar{F}(\phi^{-1}(y))\}^{n+k-2}\left\{\frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))}\right\}^2 dy.\end{aligned}$$

By taking $x = \phi^{-1}(y)$, we obtain

$$\begin{aligned}DJ(g^u_{m,k}\|g_{1:n}) =&\frac{1}{2}\frac{k^{2m}}{\{\Gamma(m)\}^2}\int_0^\infty \{-\log \bar{F}(x)\}^{2(m-1)}\{\bar{F}(x)\}^{2(k-1)}\frac{f^2(x)}{\phi'(x)}dx\\ &-\frac{1}{2}\frac{nk^m}{\Gamma(m)}\int_0^\infty \{-\log \bar{F}(x)\}^{(m-1)}\{\bar{F}(x)\}^{n+k}\frac{f^2(x)}{\phi'(x)}\,dx,\\ =&\frac{1}{2}\int_0^\infty \frac{f^u_{m,k}(x)}{\phi'(x)}\{f^u_{m,k}(x)-f_{1:n}(x)\}\,dx.\end{aligned} \tag{4.7}$$

The equation (4.7) can be written as

$$DJ(g^u_{m,k} \| g_{1:n}) = \frac{1}{\phi'(x)} DJ(f^u_{m,k} \| f_{1:n}).$$

This proves the result.

Remark 4. For any continuous random variable X, define $Y = cX + d$, where $c > 0$, d are constants. Then

$$DJ(g^u_{m,k} \| g_{1:n}) = \frac{1}{c} DJ(f^u_{m,k} \| f_{1:n}).$$

The "divergence-extropy" measure (4.4) is invariant under scale but not under location transformation. □

Examples 4.1. Consider a random variable X following exponential distribution with parameter $\lambda > 0$, then pdf and survival function are given by $f(x) = \lambda e^{-\lambda x}$ and $\bar{F}(x) = e^{-\lambda x}$ respectively. Then "divergence-extropy" measure (4.3) for exponential distribution is defined as

$$\begin{aligned} DJ(f^u_{m,k} \| f_{i:n}) =& \frac{1}{2} \frac{k^{2m} (\lambda)^{2m}}{\{\Gamma(m)\}^2} \int_0^\infty e^{-2k\lambda x} x^{2(m-1)} dx \\ &- \frac{k^m}{\Gamma(m)} \frac{\lambda^{m+1}}{\beta(i, n-i+1)} \int_0^\infty x^{m-1} e^{-\lambda(k+n-i+1)x} (1 - e^{-\lambda x})^{i-1} dx. \end{aligned} \tag{4.8}$$

For $i = 1$, above expression reduces to

$$\begin{aligned} DJ(f^u_{m,k} \| f_{1:n}) =& \frac{1}{2} \frac{k^{2m} (\lambda)^{2m}}{\{\Gamma(m)\}^2} \int_0^\infty e^{-2k\lambda x} x^{2(m-1)} dx - \frac{k^m}{\Gamma(m)} \frac{\lambda^{m+1}}{2} \int_0^\infty x^{m-1} e^{-\lambda(k+n)x} dx \\ =& \frac{\Gamma(2m-1) k\lambda}{2^{2m} \{\Gamma(m)\}^2} - \frac{\lambda k^m}{2n(k+n)^m}. \end{aligned}$$

5 Conclusion

This work presents the "divergence-extropy" measure between the distribution of n^{th} upper k-record value and m^{th} upper k-record value. Some theorems and some lifetime distributions have been studied through the proposed "divergence-extropy" measure. Further, the "divergence-extropy" measure for the k-record value and order statistics has been studied. The results studied in this manuscript can be useful for further exploring the concept of various generalized "divergence-extropy" measures based on order and record statistics.

Conflict of Interest

The corresponding author declares that there is no conflict of interest on behalf of all authors.

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