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Saddlepoint Method in Integer-valued Bilinear Models

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Abstract. Saddlepoint techniques have proven effective across various applications due to their markable accuracy in approximating densities. In this work, through considering the integer-valued bilinear model, the saddlepoint maximum likelihood method is applied to parameter estimation. Simulation studies show that this method provides an efficient approach to parameter estimation. The analysis of practical cases highlight the usefulness and adequacy of the proposed model in applications.

Keywords. Integer-valued Bilinear Process, Quasi-Likelihood, Saddlepoint Approximation, Thinning Operator.

MSC: 62M10, 62M20.

1 Introduction

In numerous nonlinear models, directly providing the probability mass function for the likelihood function poses a challenge. To address this issue, the use of saddlepoint approximation has been suggested. The introduction of saddlepoint techniques to the statistical domain by Daniels (1954) has been extended by Jensen (1995) and Butler (2007). The effectiveness of saddlepoint techniques lies in their ability to accurately approximate complex densities and tail probabilities, making them successful in various applications. An alternative approach proposed by Pedeli et al. (2015) applied saddlepoint approximation to the log-likelihood function and utilized saddlepoint maximum likelihood (SPML) method for parameter estimation in the INAR model, highlighting the versatility of this technique. Furthermore, Xu et al. (2023) employed the saddle-

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point technique in a modified multiplicative thinning-based INARCH model to derive the probability mass function. In this work, the saddlepoint method is applied to the integer-valued bilinear model, providing an efficient approach to parameter estimation.

Count time series modeling has seen many developments in recent years and the thinning operator is an effective approach for handling them. Conventional bilinear models can also be adapted to the integer case using the thinning operator, resulting in the class of integer-valued bilinear (INBL) models, as introduced by Doukhan et al. (2006). Drost et al. (2008) reconsidered the nonnegative integer-valued bilinear processes introduced by Doukhan et al. (2006). Latour and Truquet (2009) introduced an integer valued bilinear model based on the signed thinning operator to handle integer cases with negative values. In this paper, similar to many others, the thinning operator is utilized. The signed thinning operator, denoted by $\alpha\circ$, is defined as follows

$$\alpha \circ X = \begin{cases} \operatorname{sign}(X)\sum_{j=1}^{|X|} Y_j, & \text{if } X \neq 0 \\ 0, & \text{otherwise} \end{cases}, \alpha \in (0,1),$$

where X is an integer-valued random variable, $\left\{Y_j\right\}_{j=1}^{X}$ is a sequence of independent and identically distributed (IID) integer-valued random variables called the counting series, with mean α independent of X.

The structure of the paper is as follows. In Section 2, an integer-valued bilinear model is presented and its features are discussed. In Section 3, we discuss the statistical method called saddlepoint maximum likelihood and its asymptotic behaviors, which is used to estimate the parameters of the model. In Section 4, the analysis of practical case are presented to highlight the usefulness of the proposed model in application.

2 The Model and its Properties

Consider the second order count process $\{X_t\}$, $t = 1, 2, \ldots$ introduced by Latour and Truquet (2009), as follows

$$X_t = \sum_{j=1}^{p} \alpha_j \circ X_{t-j} + \varepsilon_t \sum_{j=1}^{p} \beta_j \circ X_{t-j} + \eta_t, \tag{2.1}$$

where the operator "$\circ$" represents the signed binomial thinning, which acts on the counting series $\{Y_j\}$ and $\{\tilde{Y}_j\}$, each following a Bernouli distribution with parameters α_i and β_i, respectively. For a fixed t, the IID innovation ε_t and η_t are assumed to be independent of $\{X_s\}_{s<t}$ with non-negative support, finite means μ_ε and μ_η, and variances σ_ε^2 and σ_η^2. We refer to this model as an INBL with the signed binomial thinning operator of order p (INBLS(p)).

The process is second order stationary if $\sum_{j=1}^{p}\left(E|Y_j| + (E|\varepsilon|^2)^{\frac{1}{2}} E|\tilde{Y}_j|\right) < 1$, as shown by Latour and Truquet (2009).

As the conditional moments are crucial for time series forecasting, we will address them in the following theorem.

Theorem 2.1. *For the stationary process (2.1), we have*

$$\mu_t := E\Big[X_t|\mathscr{F}_{t-1}\Big] = \sum_{j=1}^{p} \alpha_j X_{t-j} + \mu_\varepsilon \sum_{j=1}^{p} \beta_j X_{t-j} + \mu_\eta, \tag{2.2}$$

and

$$\sigma_t^2 := Var\Big[X_t|\mathscr{F}_{t-1}\Big] = \sum_{j=1}^{p} \Big(\alpha_j(1-\alpha_j) + (\mu_\varepsilon^2 + \sigma_\varepsilon^2)\beta_j(1-\beta_j)\Big)|X_{t-j}| + \sigma_\varepsilon^2\Big(\sum_{j=1}^{p} \beta_j X_{t-j}\Big)^2 + \sigma_\eta^2,$$

where $\mathscr{F}_{t-1}$ is the σ-field generated by $X_{t-1}, X_{t-2}, \ldots$.

Proof. $\mathrm{E}\Big[X_t|\mathscr{F}_{t-1}\Big]$ can obtain easily. For the variance we have

$$\begin{aligned}
\sigma_t^2 &:= \mathrm{Var}\Big[X_t|\mathscr{F}_{t-1}\Big] \\
&= \mathrm{Var}\Big[\sum_{j=1}^{p} \alpha_j \circ X_{t-j}|\mathscr{F}_{t-1}\Big] + \mathrm{Var}\Big[\varepsilon_t \sum_{j=1}^{p} \beta_j \circ X_{t-j}|\mathscr{F}_{t-1}\Big] + \mathrm{Var}\Big[\eta_t|\mathscr{F}_{t-1}\Big] \\
&+ 2\mathrm{Cov}\Big[\sum_{j=1}^{p} \alpha_j \circ X_{t-j}, \varepsilon_t \sum_{j=1}^{p} \beta_j \circ X_{t-j}|\mathscr{F}_{t-1}\Big] + 2\mathrm{Cov}\Big[\sum_{j=1}^{p} \alpha_j \circ X_{t-j}, \eta_t|\mathscr{F}_{t-1}\Big] \\
&+ 2\mathrm{Cov}\Big[\varepsilon_t \sum_{j=1}^{p} \beta_j \circ X_{t-j}, \eta_t|\mathscr{F}_{t-1}\Big] \\
&= \Big(\sum_{j=1}^{p} \alpha_j X_{t-j}\Big)^2 + \sum_{j=1}^{p} Var[Y_j]|X_{t-j}| - \Big(\sum_{j=1}^{p} \alpha_j X_{t-j}\Big)^2 \\
&+ \mathrm{E}[\varepsilon_t^2]\Big(\sum_{j=1}^{p} \beta_j X_{t-j}\Big)^2 + \mathrm{E}[\varepsilon_t^2]\sum_{j=1}^{p} \mathrm{Var}[\tilde{Y}_j]|X_{t-j}| - (\mathrm{E}[\varepsilon_t])^2\Big(\sum_{j=1}^{p} \beta_j X_{t-j}\Big)^2 + \mathrm{Var}[\eta_t] \\
&= \sum_{j=1}^{p} \mathrm{Var}[Y_j]|X_{t-j}| + \mathrm{Var}[\varepsilon_t]\Big(\sum_{j=1}^{p} \beta_j X_{t-j}\Big)^2 + \mathrm{E}[\varepsilon_t^2]\sum_{j=1}^{p} \mathrm{Var}[\tilde{Y}_j]|X_{t-j}| + \mathrm{Var}[\eta_t] \\
&= \sum_{j=1}^{p} \Big(\mathrm{Var}[Y_j] + \mathrm{E}[\varepsilon_t^2]Var[\tilde{Y}_j]\Big)|X_{t-j}| + \mathrm{Var}[\varepsilon_t]\Big(\sum_{j=1}^{p} \beta_j X_{t-j}\Big)^2 + \mathrm{Var}[\eta_t],
\end{aligned}$$

where $\mathrm{Var}[Y_j] = \alpha_j(1-\alpha_j)$ and $\mathrm{Var}[\tilde{Y}_j] = \beta_j(1-\beta_j)$. □

Furthermore, the INBLS(p) model is considered under three different marginal distributions of innovations: Poisson (P-INBLS), geometric (G-INBLS), and Poisson-Lindley (PL-INBLS). The conditional mean and variance for each model are provided in Table 1.

Table 1: Conditional mean and variance of the INBLS process under different marginal distributions of innovations.

Model	Conditional Mean and Variance
P-INBLS(p) $\varepsilon_t \sim P(\mu_\varepsilon)$ and $\eta_t \sim P(\mu_\eta)$	$\mu_t = \sum_{j=1}^{p} \alpha_j X_{t-j} + \mu_\varepsilon \sum_{j=1}^{p} \beta_j X_{t-j} + \mu_\eta$ $\sigma_t^2 = \sum_{j=1}^{p} \Big(\alpha_j(1-\alpha_j) + (\mu_\varepsilon^2 + \mu_\varepsilon)\beta_j(1-\beta_j)\Big)\lvert X_{t-j}\rvert + \mu_\varepsilon\Big(\sum_{j=1}^{p} \beta_j X_{t-j}\Big)^2 + \mu_\eta$
G-INBLS(p) $\varepsilon_t \sim Ge(p_\varepsilon)$ and $\eta_t \sim Ge(p_\eta)$	$\mu_t = \sum_{j=1}^{p} \alpha_j X_{t-j} + \frac{1-p_\varepsilon}{p_\varepsilon} \sum_{j=1}^{p} \beta_j X_{t-j} + \frac{1-p_\eta}{p_\eta}$ $\sigma_t^2 = \sum_{j=1}^{p} \Big(\alpha_j(1-\alpha_j) + ((\frac{1-p_\varepsilon}{p_\varepsilon})^2 + \frac{1-p_\varepsilon}{p_\varepsilon^2})\beta_j(1-\beta_j)\Big)\lvert X_{t-j}\rvert + \frac{1-p_\varepsilon}{p_\varepsilon^2}\Big(\sum_{j=1}^{p} \beta_j X_{t-j}\Big)^2 + \frac{1-p_\eta}{p_\eta^2}$
PL-INBLS(p) $\varepsilon_t \sim PL(\theta_\varepsilon)$ and $\eta_t \sim PL(\theta_\eta)$	$\mu_t = \sum_{j=1}^{p} \alpha_j X_{t-j} + \Big(\frac{\theta_\varepsilon+2}{\theta_\varepsilon(\theta_\varepsilon+1)}\Big)\sum_{j=1}^{p} \beta_j X_{t-j} + \Big(\frac{\theta_\eta+2}{\theta_\eta(\theta_\eta+1)}\Big)$ $\sigma_t^2 = \sum_{j=1}^{p} \Big(\alpha_j(1-\alpha_j) + \Big[\frac{2(\theta_\varepsilon+2)}{\theta_\varepsilon(\theta_\varepsilon+1)} + \frac{2(\theta_\varepsilon+3)}{\theta_\varepsilon^2(\theta_\varepsilon+1)}\Big]\beta_j(1-\beta_j)\Big)\lvert X_{t-j}\rvert +$ $+\Big[\Big(\frac{2(\theta_\varepsilon+2)}{\theta_\varepsilon(\theta_\varepsilon+1)} + \frac{2(\theta_\varepsilon+3)}{\theta_\varepsilon^2(\theta_\varepsilon+1)}\Big) - \Big(\frac{\theta_\varepsilon+2}{\theta_\varepsilon(\theta_\varepsilon+1)}\Big)^2\Big]\Big(\sum_{j=1}^{p} \beta_j X_{t-j}\Big)^2$ $+\Big[\Big(\frac{2(\theta_\eta+2)}{\theta_\eta(\theta_\eta+1)} + \frac{2(\theta_\eta+3)}{\theta_\eta^2(\theta_\eta+1)}\Big) - \Big(\frac{\theta_\eta+2}{\theta_\eta(\theta_\eta+1)}\Big)^2\Big]$

Theorem 2.2. *For the INBLS(1) process (2.1), the first and second moments are as follows*

$$E[X_t] = \frac{\mu_\eta}{1-(\alpha+\mu_\varepsilon\beta)},$$

and

$$E[X_t^2] = \frac{\alpha(1-\alpha)E[\lvert X_t\rvert] + E[\varepsilon_t^2]\beta(1-\beta)E[\lvert X_t\rvert] + 2\alpha\mu_\eta E[X_t] + 2\beta\mu_\eta\mu_\varepsilon E[X_t] + E[\eta_t^2]}{1-(\alpha^2 + 2\alpha\beta\mu_\varepsilon + E[\varepsilon_t^2]\beta^2)}.$$

Proof. $\mathrm{E}[X_t]$ can obtain easily. For $\mathrm{E}[X_t^2]$, we have

$$\begin{aligned} \mathrm{E}[X_t^2] &= \mathrm{E}\Big[(\alpha \circ X_{t-1} + \varepsilon_t(\beta \circ X_{t-1}) + \eta_t)^2\Big] \\ &= E[B_{t-1}^2] + 2\mathrm{E}[B_{t-1}]\mathrm{E}[\eta_t] + \mathrm{E}[\eta_t^2], \end{aligned} \tag{2.3}$$

where $B_{t-1} = \alpha \circ X_{t-1} + \varepsilon_t(\beta \circ X_{t-1})$. Under the properties of the sign operator,

$$\mathrm{E}[B_{t-1}] = \mathrm{E}\Big[\alpha \circ X_{t-1} + \varepsilon_t(\beta \circ X_{t-1})\Big] = \alpha\mathrm{E}[X_{t-1}] + \beta\mu_\varepsilon\mathrm{E}[X_{t-1}], \tag{2.4}$$

and

$$\begin{aligned} \mathrm{E}[B_{t-1}^2] &= \mathrm{E}\Big[\alpha \circ X_{t-1} + \varepsilon_t(\beta \circ X_{t-1})\Big]^2 \\ &= \mathrm{E}[\alpha \circ X_{t-1}]^2 + 2\mathrm{E}[(\alpha \circ X_{t-1})(\varepsilon_t(\beta \circ X_{t-1}))] + \mathrm{E}[\varepsilon_t(\beta \circ X_{t-1})]^2 \\ &= \alpha^2\mathrm{E}[X_{t-1}^2] + \mathrm{Var}[Y_j]\mathrm{E}[\lvert X_{t-1}\rvert] + 2\alpha\beta\mu_\varepsilon\mathrm{E}[X_{t-1}^2] + \mathrm{E}[\varepsilon_t^2]\beta^2\mathrm{E}[X_{t-1}^2] \\ &+ \mathrm{E}[\varepsilon_t^2]\mathrm{Var}[\tilde{Y}_j]\mathrm{E}[\lvert X_{t-1}\rvert]. \end{aligned} \tag{2.5}$$

Substituting Eqns. (2.4) and (2.5) in Eqn. (2.3), $\mathrm{E}[X_t^2]$ is concluded. □

3 Saddlepoint Maximum Likelihood Method

Now, we apply the saddlepoint maximum likelihood method. Let the unknown parameter vector as $\theta = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p, \mu_\varepsilon, \mu_\eta)^T$. The maximum likelihood estimator of θ can be obtained by maximizing the conditional log-likelihood function

$$l(\theta) = \sum_{t=1}^{n} \log f_{X_t|X_{t-1}=x_{t-1},\ldots,X_{t-p}=x_{t-p}}(x_t). \tag{3.1}$$

However, implementing the above procedure is difficult because providing the likelihood function is a challenge due to the thinning operations involved.

Now we investigate the SPML procedure. Consider the conditional cumulant generating function of X_t as $K_t(u) = \log\left[\mathrm{E}(e^{uX_t}|X_{t-1} = x_{t-1}, ..., X_{t-p} = x_{t-p})\right]$. The saddlepoint approximation gives a precise estimation of the conditional mass function of X_t at x_t as

$$\tilde{f}_{X_t|X_{t-1}=x_{t-1},\ldots,X_{t-q}=x_{t-q}}(x_t) = (2\pi K_t''(\tilde{u}_t))^{-\frac{1}{2}} \exp\left\{K_t(\tilde{u}_t) - \tilde{u}_t x_t\right\}, \tag{3.2}$$

where K_t' and K_t'' represent the first and second derivatives of K_t with respect to u and the value of u that satisfies the saddlepoint equation $K'(u) = x_t$ is denoted by $\tilde{u}_t$. It should be noted that finding a solution to the saddlepoint equation $K_t'(u) = x_t$ analytically can be challenging. As mentioned in Pedeli et al. (2015), we can apply the Newton-Raphson method to solve this equation. The first-order Taylor expansion of $K_t'(u)$ at $u = 0$ is given by

$$K_t'(u) \simeq K_t'(0) + uK_t''(0) \simeq \mu_t(\theta) + u\sigma_t^2(\theta), \tag{3.3}$$

where $\mu_t(\theta)$ and $\sigma_t^2(\theta)$ represent the conditional mean and conditional variance of X_t. Utilizing Eqn. (3.3), we obtain

$$\tilde{u}_t \simeq \frac{x_t - \mu_t(\theta)}{\sigma_t^2(\theta)}. \tag{3.4}$$

Next, we can derive the second-order Taylor expansion of $K_t'(u)$ at $u = 0$ as follows

$$K_t(u) \simeq uK_t'(0) + \frac{u^2}{2}K_t''(0) \simeq u\mu_t(\theta) + \frac{u^2}{2}\sigma_t^2(\theta). \tag{3.5}$$

By focusing on the exponent in the saddlepoint approximation (3.2), Eqn. (3.3) yields

$$K_t(u) - ux_t \simeq u(\mu_t(\theta) - x_t) + \frac{u^2}{2}\sigma_t^2(\theta).$$

Then, using Eqn. (3.4), we obtain

$$K_t(\tilde{u}_t) - \tilde{u}_t x_t \simeq \frac{-(x_t - \mu_t(\theta))^2}{2\sigma_t^2(\theta)}. \tag{3.6}$$

Thus, based on Eqns. (3.5) and (3.6), the conditional probability function can be approximated as

$$\begin{aligned}\tilde{f}_{X_t|X_{t-1}=x_{t-1},\dots,X_{t-q}=x_{t-q}}(x_t) &= (2\pi K_t''(\tilde{u}_t))^{-\frac{1}{2}} \exp\left\{K_t(\tilde{u}_t) - \tilde{u}_t x_t\right\} \\ &= (2\pi\sigma_t^2(\theta))^{-\frac{1}{2}} \exp\left\{\frac{-(x_t-\mu_t(\theta))^2}{2\sigma_t^2(\theta)}\right\}. \end{aligned} \tag{3.7}$$

The quasi-likelihood function $\tilde{L}_n(\theta) := \sum_{t=1}^{n} \log \tilde{f}_{X_t|X_{t-1}=x_{t-1},\dots,X_{t-q}=x_{t-q}}(x_t)$ can be utilized for the estimation of θ. The maximum value of θ that optimizes this expression is referred to as the saddlepoint maximum likelihood estimator. The maximization is accomplished using Newton–Raphson algorithm.

Theorem 3.1. *The SPML estimate $\hat{\theta}$ is asymptotically normal, i.e.*

$$\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, \sum\nolimits^{-1}),$$

where

$$\sum = -E\Big[\frac{\partial^2 l_t(\theta)}{\partial\theta\partial\theta^T}\Big],$$

and the second derivatives of the likelihood function is as follows

$$\begin{aligned}\frac{\partial^2 l_t(\theta)}{\partial\theta\partial\theta^T} &= \Big(\frac{(x_t-\mu_t(\theta))}{\sigma_t^2(\theta)} - \frac{1}{\sigma_t^2(\theta)}\Big)\frac{\partial^2\mu_t(\theta)}{\partial\theta\partial\theta^T} - 2\frac{(x_t-\mu_t(\theta))}{\sigma_t^4(\theta)}\frac{\partial\mu_t(\theta)}{\partial\theta}\frac{\partial\sigma_t^2(\theta)}{\partial\theta^T} \\ &+ \Big(\frac{(x_t-\mu_t(\theta))^2-\sigma_t^2(\theta))}{2\sigma_t^4(\theta)} - \frac{(x_t-\mu_t(\theta))^2}{\sigma_t^6(\theta)} + \frac{1}{2\sigma_t^4(\theta)}\Big)\frac{\partial^2\sigma_t^2(\theta)}{\partial\theta\partial\theta^T}.\end{aligned}$$

Proof. The proof is very similar to the proof given to Xu et al. (2023). □

3.1 Simulation Study

To evaluate the performance of the SPML estimates, we generated sample paths for the P-INBLS, G-INBLS, and PL-INBLS processes with varying sample sizes. Replications were conducted to verify the robustness of the estimates. Algorithm 1 outlines the procedure used to estimate the model parameters.

Our findings are summarized in Table 2, where the results demonstrate that the estimators perform satisfactorily in terms of the mean absolute deviation from the estimated (MADE) values. The sample size is $n = 100, 500, 1000$ with the number of replications $s = 1000$. We used the following parameter combinations as the true values for generating the random samples:

- For the P-INBLS(1) model: $(\alpha, \beta, \mu_\varepsilon, \mu_\eta)^T = (0.85, 0.35, 1, 1)^T$,
- For the G-INBLS(1) model: $(\alpha, \beta, p_\varepsilon, p_\eta)^T = (0.65, 0.25, 0.6, 0.4)^T$,
- For the PL-INBLS(1) model: $(\alpha, \beta, \theta_\varepsilon, \theta_\eta)^T = (0.5, 0.15, 0.7, 0.8)^T$.

Algorithm 1

Initialize:
Set sample sizes $n = 100, 500, 1000$.
Set the number of simulations $s = 1000$.
Define the processes: **P-INBLS(1)**, **G-INBLS(1)** and **PL-INBLS(1)**.

for each process in {**P-INBLS(1)**, **G-INBLS(1)**, **PL-INBLS(1)**} **do**
  for each sample size n **do**
    for simulation run $k = 1$ to s **do**
      Generate a sample of size n from the specified process.
      Estimate the parameters using the `optim` function in R via SPML.
    end for
    Compute the mean of the estimated parameters over s simulations.
    Calculate the Mean Absolute Deviation Error (MADE) of the estimates.
  end for
end for

The results of these simulations, presented in Table 2, indicate that as the sample sizes increase, the MADEs decrease, suggesting that the estimates converge toward the true parameter values. These findings underscore the effectiveness of the SPML method in providing accurate parameter estimates in the context of the INBLS models studied.

Table 2: Mean and MADE of estimates for P-INBLS, G-INBLS and PL-INBLS models with SPML.

Model			α	β	μ_ε	μ_η
P-INBLS	$n = 100$	Mean	0.8567255	0.3407647	1.000743	0.996782
		MADE	0.001092514	0.001230265	0.003447479	0.003284437
	$n = 500$	Mean	0.851304	0.3495557	0.9898882	1.001781
		MADE	0.0002617064	0.000251983	0.002871727	0.002298623
	$n = 1000$	Mean	0.8527711	0.35137	0.9810123	1.000379
		MADE	0.0001473274	0.0001512291	0.002624402	0.00223726
			α	β	p_ε	p_η
G-INBLS	$n = 100$	Mean	0.6602072	0.4930343	0.5924451	0.3945554
		MADE	0.001963711	0.02254635	0.003515865	0.003187796
	$n = 500$	Mean	0.6556229	0.5496167	0.6016027	0.403974
		MADE	0.001086858	0.01167803	0.003191457	0.002576685
	$n = 1000$	Mean	0.6538663	0.5526363	0.5988895	0.4024485
		MADE	0.000726796	0.01247545	0.003364151	0.002159636
			α	β	θ_ε	θ_η
PL-INBLS	$n = 100$	Mean	0.5537954	0.1204166	0.6931677	0.7867049
		MADE	0.004172623	0.002055575	0.003735458	0.003319334
	$n = 500$	Mean	0.5721151	0.1175136	0.6968855	0.8043694
		MADE	0.005608268	0.001477007	0.003188854	0.002924123
	$n = 1000$	Mean	0.5724368	0.1189308	0.6998202	0.8186343
		MADE	0.005591804	0.00126325	0.003055911	0.002790475

4 Real Example

In this section, we focus on the monthly incidence of police reports related to cannabis trafficking in Liverpool, Australia. This time series data, spanning from January 1995 to March 2024, is part of the New South Wales police reports dataset. The dataset is named "Recorded crime by offence" and has a geographic breakdown by "LGA" in the "Datasets" section, which may be accessed at http://www.bocsar.nsw.gov.au. The sample paths, autocorrelation functions (ACFs) and partial autocorrelation functions (PACFs) of the data set are displayed in Figure 1. The data has a sample mean of 1.5499 and a variance of 4.9168. Notably, the data set contains some instances of exceptionally high observations, specifically in January 2001 and October 2012 where the number of cannabis trafficking reports spiked. Consequently, our model appears well-suited to capture the auto-correlation structure and accommodate these outliers within the dataset. The p-value 0.01 for the Augmented Dickey Fuller test of stationarity confirms that the data sets are stationary. Also, the Keenan test for non-linearity, gets p-value 0.0004 which confirm the non-linearity of the process. Thus, linear models are unable to adequately represent the data.

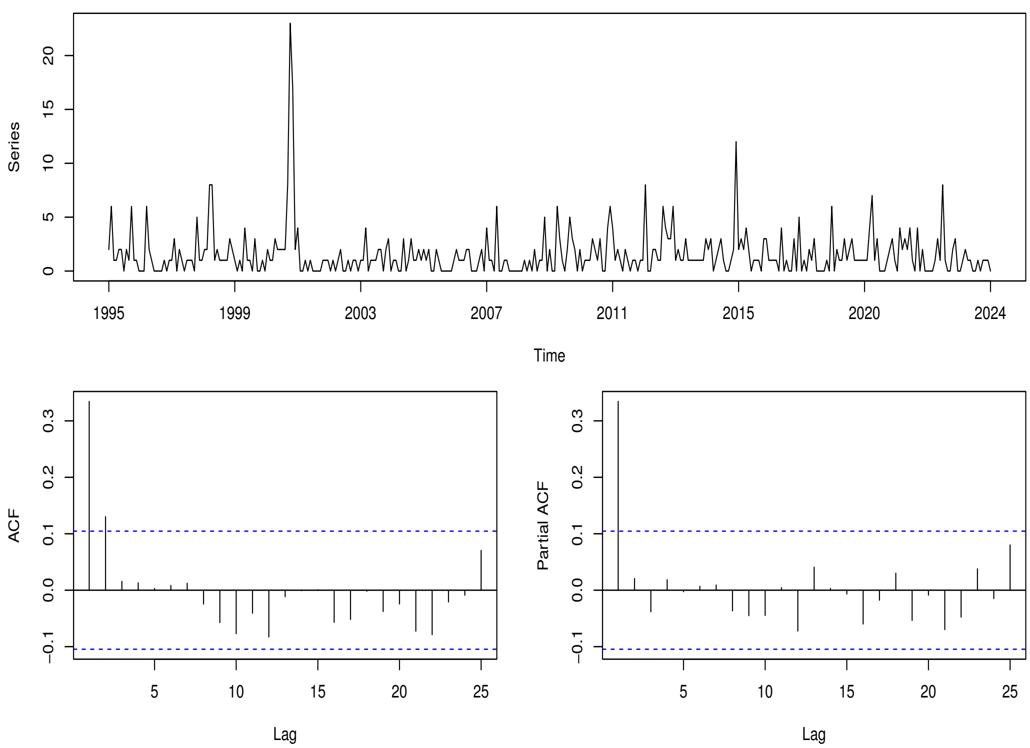

Figure 1: The sample path, ACF and PACF of the cannabis trafficking in Liverpool.

We conducted an analysis of the data using several models: the P-INBLS(1), G-INBLS(1), PL-INBLS(1), Poisson INBL(1) (abbreviated as P-INBL(1)), geometric

INBL(1) (G-INBL(1)), and Poisson-Lindley INBL(1) (PL-INBL(1)), PMthINARCH(1), and GMthINARCH(1). The INBL model was referenced from Doukhan et al. (2006), while the PMthINARCH(1) and GMthINARCH(1) models were referenced from Xu et al. (2023). Parameters for these models were estimated using the SPML method. Table 3 presents the SPML estimates, with approximate standard errors (in parentheses, derived from the Hessian matrix of the log-likelihood using R software), alongside the corresponding Akaike Information Criterion (AIC) values. The results in Table 3 indicate that the AIC values for the G-INBLS(1) models are lower than those for the other models.

To evaluate the suitability and accuracy of the G-INBLS(1) models, the real data series and their predicted values are plotted and displayed in Figure 2. The close correspondence between the predicted values and the actual data series indicates that the model is effective in providing reliable forecasts.

Table 3: Estimated parameters with approximate standard errors and AIC of the cannabis trafficking in Liverpool.

Model	SPML	AIC
P-INBLS(1)	$\hat{\alpha}$ = 0.676 (0.2147) $\hat{\beta}$ = 0.314 (0.0736) $\hat{\mu}_\varepsilon$ = 2.403 (0.3604) $\hat{\mu}_\eta$ = 1.464 (0.0872)	1425.422
G-INBLS(1)	$\hat{\alpha}$ = 0.088 (0.2988) $\hat{\beta}$ = 2.136 (5.2080) $\hat{p}_\varepsilon$ = 0.940 (0.2567) $\hat{p}_\eta$ = 0.470 (0.0140)	1407.478
PL-INBLS(1)	$\hat{\alpha}$ = 0.237 (0.0927) $\hat{\beta}$ = 2.181 (2.2955) $\hat{\theta}_\varepsilon$ = 36.695 (70.6414) $\hat{\theta}_\eta$ = 1.495 (0.0863)	1414.261
P-INBL(1)	$\hat{\alpha}$ = 0.057 (0.0462) $\hat{\beta}$ = 0.049 (0.0082) $\hat{\mu}_\varepsilon$ = 1.600 (0.0769)	1529.03
G-INBL(1)	$\hat{\alpha}$ = 0.058 (0.0661) $\hat{\beta}$ = 0.046 (0.0103) $\hat{p}_\varepsilon$ = 0.441 (0.0128)	1443.508
PL-INBL(1)	$\hat{\alpha}$ = 0.112 (0.0696) $\hat{\beta}$ = 0.041 (0.0104) $\hat{\theta}_\varepsilon$ = 1.325 (0.0709)	1444.848
PMthINARCH(1)	$\hat{\alpha}$ = 0.256 (0.0390) $\hat{\omega}$ = 0.540 (0.0392)	1412.955
GMthINARCH(1)	$\hat{\alpha}$ = 0.196 (0.0323) $\hat{\omega}$ = 0.395 (0.0316)	1438.692

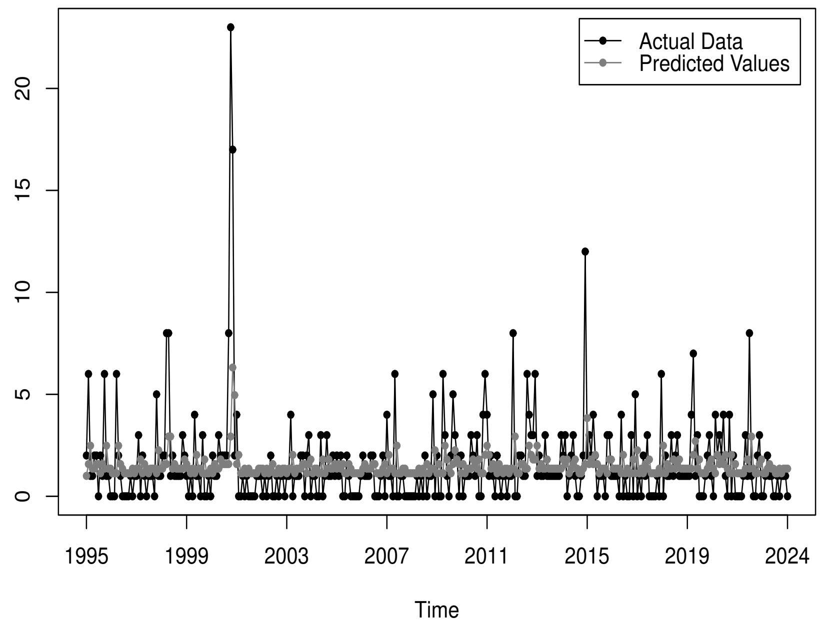

Figure 2: Actual data and its predicted values of the cannabis trafficking in Liverpool.

4.1 The Residual Analysis

We evaluate the suitability of the suggested model and its fit to the data set. Diagnostic methods for examining the model's dynamic aspects often utilize standardized Pearson residuals, denoted as

$$r_t = \frac{X_t - \mathrm{E}\left[X_t|\mathscr{F}_{t-1}\right]}{\sqrt{\mathrm{Var}\left[X_t|\mathscr{F}_{t-1}\right]}},$$

where the population quantities are replaced by their estimated counterparts in the G-INBLS model. Figure 3 depicts the cumulative periodogram and the sample autocorrelation function of the Pearson residuals. The residuals show no indication of correlation, and the Ljung-Box test's p-value of 0.5197 confirms such conclusions. Cumulative periodogram plots in Figure 3 make it abundantly evident that the residuals are randomly distributed and lack a discernible trend.

5 Conclusion

In this paper, we applied the saddlepoint maximum likelihood method for parameter estimation in the integer-valued bilinear model. Our findings demonstrate that the

SPML method provides highly accurate estimates, as confirmed by both theoretical analysis and simulation studies. Notably, the SPML approach offers a robust alternative to conventional methods, especially in cases where direct likelihood calculations are challenging. The application of the model to real-world data, specifically the monthly cannabis trafficking reports in Liverpool, highlights its practical utility. Furthermore, the residual analysis confirmed the adequacy of the model, demonstrating its suitability for forecasting purposes.

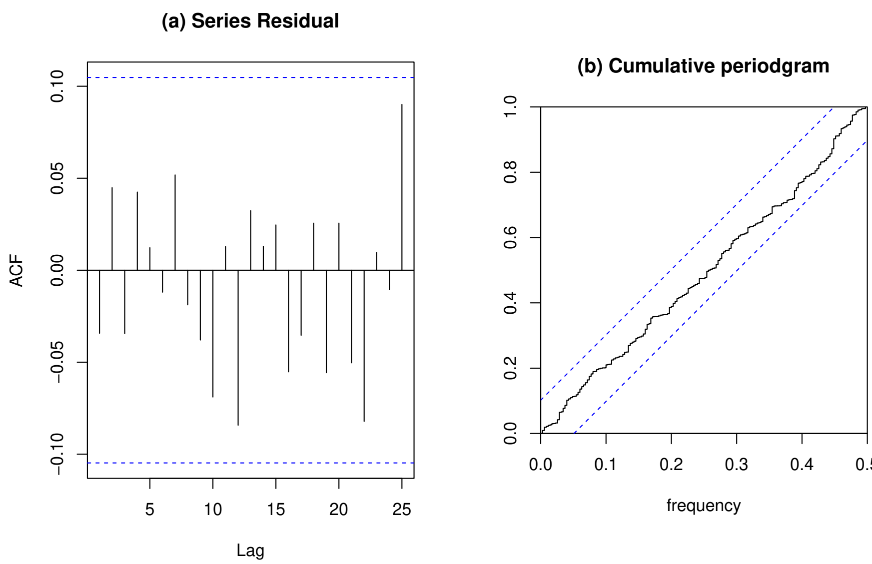

Figure 3: ACF and Cumulative periodogram plots of the Pearson residuals of the cannabis trafficking in Liverpool.

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