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Analysis of the Three-parameter Weibull Distributed Lifetime Data based on Progressive Type-II Right Censoring: a New Approach to Determine the Random Removals

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Abstract. An important challenge in using progressive Type-II right censoring is to determine a removal scheme. It can be predetermined or randomly chosen per discrete distributions. This paper considers the random removal problem and proposes two scenarios for determining the removal vector without introducing any parameter to a model when progressively Type-II censored samples are available from the three-parameter Weibull distribution. The proposed scenarios are based on the normalized spacings with random and fixed coefficients according to progressively Type-II censored order statistics from an exponential distribution. The joint probability mass functions of removal vectors are provided as well as expected experimental time under the proposed two methods. Moreover, the maximum likelihood estimators (MLEs) and corrected maximum likelihood estimators (corrected MLEs) of parameters are obtained. The new approaches are compared with the patterns of removal derived from the discrete uniform and binomial distributions using a Monte Carlo simulation study. This comparison is based on their estimated biases, estimated mean squared errors and expected total time on the experiment. Finally, a real data example is given to show the practical applications of the paper.

Keywords. Corrected Maximum Likelihood Estimator, Expected Test Time, Monte Carlo Simulation, Progressive Censoring, Random Removals, Weibull Distribution.

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1 Introduction

In a life-testing experiment, a censoring scheme occurs when the failure times of all units put on a life test are not observed. This can occur due to time and cost constraints or other unavoidable circumstances. There are several types of censored data. The two most common censoring schemes are Type-I and Type-II censoring. However, they do not allow for the removal of live experimental units at points other than the final terminal point of the experiment. A more general type of censoring scheme, known as the progressive Type-II right censoring, has been examined. It can be explained as follows: Suppose that a total of n independent units are placed on a life test with continuous, identically distributed failure times represented by $X_1, \cdots, X_n$. Where m units are completely observed and $n-m$ units are withdrawn from the experiment at various time points. Notice that m is predetermined and fixed. After observing the first failure $X_{1:m;n}$, r_1 units are randomly selected and removed from the remaining $n-1$ units. Then the second failure will be the smallest lifetime among the $n-r_1-1$ units. After the second failure $X_{2:m:n}$, r_2 items are randomly selected and removed from the $n-r_1-2$ remaining units. This process continues until observing the m-th failure $X_{m:m:n}$. Then all r_m units, where $r_m = n - \sum_{i=1}^{m-1}(r_i - 1)$, will be removed from the experiment. One concern with this type of censoring is how to determine the removals vector $r = (r_1, r_2, \cdots, r_m)$.

In general, two approaches are used to determine the removal scheme in the literature: it can be either fixed and determined in advance or randomly chosen according to some discrete probability distributions. Readers can find more details on the methods and applications of the first topic in the books written by Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014). The research on random removals dates back to the late 1990s, when Yuen and Tse (1996) introduced the progressive Type-II censoring plan with random removals. They indicated that, for example, number of patients dropping out from a clinical test at each stage is random and cannot be predetermined. For certain life test studies, an experimenter might decide to stop testing some units, even if they haven't failed, due to safety concerns. In these instances, removal pattern at each failure is random. They discussed the issue of estimation when lifetimes follow a two-parameter Weibull distribution and random removals follow a discrete uniform distribution. In this scenario, statistical inference can be conducted without introducing any additional parameters to the model. In the particular research study, two types of censoring were compared: Type-II censoring and/or progressive Type-II censoring with random removals (PCR). The researchers numerically compared expected test times or derived MLEs of the parameters and their asymptotic variances. Later, Tse et al. (2000) discussed the inference problem for a situation where each experimental unit being dropped out from the life test is independent of the others but with the same removal probability, p. Thus, the number of test units removed at each failure time follows a binomial distribution. Inferential issues, for progressively censored samples and binomial removals, have been addressed in numerous papers considering different lifetime distributions. For further reading, we refer to e.g., Tse and Xiang (2003), Tse and Yang (2003), Wu and Chang (2003), Yang and Tse (2005), Dey and Dey (2014), Gunasekera (2018), Budhiraja and Biswabrata (2019), Qin and Gui (2020), Azizi and MirMostafaee (2021), Ghahramani et al. (2020) and Sharafi (2022),. The Bayesian inference approach has been used for this model as well as classic inference by Singh et al. (2013, 2016); Kaushik et al. (2017) and Chacko and Mohan (2019). For the Weibull lifetime distribution, Ding et al. (2010) and Ding and Tse (2013) investigated the optimal designs of accelerated life tests under progressive Type-I and Type-II interval censoring schemes with binomial random removals, respectively.

In reliability literature, analyzing data from known discrete distributions for random removals can pose challenges and lead to unreliable inferences. For instance, the discrete uniform distribution gives an equal chance for the number of removals, while binomial removals have a fixed removal probability p at each stage. The choice of small or large values of p can significantly impact statistical inference. Furthermore, assuming p to be random and following a specific probability distribution, such as the beta distribution (refer to Singh et al. (2013)), can introduce additional complications by imposing more parameters on the model or verifying the true probability distribution of p. Additionally, we have found that the lifetime distribution does not have any effect on determining the distribution for random removals in the current probability mass functions. To address these issues, we suggest a new method for determining removals based on the occurred failure times. This method does not require adding any additional parameters to the model. It assumes that the lifetimes of the n units placed on the life test follow a three-parameter Weibull distribution. We have used the relationship between the Weibull and exponential distributions, and propose the removal scheme obtained by progressively Type-II censored exponential lifetimes.

The proposed method is based on two approaches: the normalized spacings with random and fixed coefficients according to progressively Type-II censored order statistics from the exponential distribution. The rest of this article is organized as follows: In Section 2, we provide a strategy for determining the number of removals at each stage, denoted as R_j, based on a spacing random variable. We then obtain the joint probability mass functions of the random removals under two different approaches, and provide a brief description of the removal models under two distributions: the discrete uniform and binomial. In Section 3, we review the MLEs and corrected MLEs (CMLEs) of the model parameters. Then, in Section 4, we discuss the expected experiment time of the proposed patterns. Section 5, a Monte Carlo simulation study is presented to evaluate the performance of the proposed schemes and compare them with the other schemes described in this paper. The results show that the new approaches generally outperform the existing ones and significantly reduce the expected total time on test. In Section 6, we analyze a real dataset representing the endurance of deep groove ball bearings to illustrate the proposed procedures discussed in the previous sections. Finally, we provide concluding remarks and also highlight some open problems while suggesting possible future work, for the benefit of readers interested in this area of research.

2 The Model

The Weibull distribution is widely used for modeling lifetime data, especially in applications related to the durability of manufactured items such as ball bearings, automobile components, and electrical insulation. The three-parameter Weibull distribution is highly flexible in fitting data and can be empirically fitted to different types of datasets. It is commonly used in life-testing, reliability analysis, meteorology, hydrology, and economics when it is reasonable to assume a minimum threshold below which no event can occur. Thus, this study considers that the failure times follow the three-parameter Weibull distribution with a probability density function

$$f(x;\alpha,\theta,\beta) = \frac{\beta}{\theta}\Big(\frac{x-\alpha}{\theta}\Big)^{\beta-1}\exp\Big[-\Big(\frac{x-\alpha}{\theta}\Big)^{\beta}\Big], \tag{2.1}$$

and corresponding cumulative distribution function

$$F(x;\alpha,\theta,\beta) = 1-\exp\Big[-\Big(\frac{x-\alpha}{\theta}\Big)^{\beta}\Big], \tag{2.2}$$

for $x \geq \alpha$, $\beta > 0$, $\theta > 0$, $\alpha \in R$, wherein α is the location parameter (failure-free life), θ is the scale parameter (characteristic life). Also, β is the shape parameter (slope) because the value of β equals the slope of the regressed line in a probability plot. The slope of the Weibull distribution is important as it determines which member of the family of Weibull failure distributions best fits or describes the data. The case where $\alpha = 0$ and $\theta = 1$ is called the standard Weibull distribution and when $\alpha = 0$ is called the two-parameter Weibull distribution.

2.1 Removal Models

Let $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$ denote progressively Type-II right censored sample from the three-parameter Weibull distribution with random removals $\mathcal{R} \equiv (R_1, R_2, \cdots, R_m)$. It should be noted that we have used the simplified notation $X_{i:m:n}, i = 1, \cdots, m$ even though these failure times depend on the particular choice of $\mathcal{R}$. Further, let $Y_{i:m:n} = (\frac{X_{i:m:n}-\alpha}{\theta})^{\beta}$, $i = 1, \cdots, m$. It can be seen that $Y_{1:m:n} < \cdots < Y_{m:m:n}$ are a progressively Type-II right censored sample from an exponential distribution with mean 1. Let us consider the following transformation:

$$W_i^d = \begin{cases} nY_{1:m:n}, & i = 1, \\ \varphi_{d.i}(Y_{i:m:n} - Y_{i-1:m:n}), & i = 2, ..., m, \end{cases}$$

where

$$\varphi_{d.i} = \begin{cases} n - \sum_{j=1}^{i-1} R_j - (i-1), & d = 1, \\ n - \sum_{j=1}^{i-1} r_j - (i-1), & d = 2. \end{cases}$$

This shows that coefficients are random and fixed for $d = 1$ and $d = 2$, respectively, in the progressive censoring scheme. Now, we define the ratio, $V_i^d = W_i^d / W_1^d$, $d = 1, 2$.

Thus, the method of the i-th random removal is proposed by

$$R_i^d = \lfloor \sqrt{V_i^d} \rfloor, \quad i = 1, \cdots, m,$$

in which $\lfloor . \rfloor$ is the floor function and $d = 1, 2$. With this strategy, R_1^d is degenerated at point 1, and also the removal number R_m^d is deterministic by definition as $R_m^d = n - m - 1 - \sum_{j=2}^{m-1} R_j^d$. In the following section, we derive the joint probability mass functions $(R_2^d, \cdots, R_{m-1}^d)$ using two methods. The first method involves normalized spacing with random coefficients ($d = 1$), known as the model with random coefficients (MRC). The second method uses normalized spacing with fixed coefficients ($d = 2$), denoted as the model with fixed coefficients (MFC). For simplicity in notation, we'll refer to two types of random vectors R_{m-1}^{MRC} and R_{m-1}^{MFC}.

Lemma 2.1. *For $i = 2, \cdots, k$, $k = 2, \cdots, m-1$ and $d = 1, 2$, the joint density function of $V_2^d, \cdots, V_m^d$ is given by*

$$f_{V_2^d, \cdots, V_k^d}(v_2, \ldots, v_k) = \frac{\Gamma(k)}{(1 + \sum_{j=2}^k v_j)^k}, \quad v_i > 0.$$

Proof. Cramer and Iliopoulos (2010) and Thomas and Wilson (1972) proved that $W_1^d, \cdots, W_m^d$ are independent and identically distributed (*IID*) as an exponential distribution with mean 1 for $d = 1$ and $d = 2$, respectively. Therefore, we find that $V_i^d (= \frac{W_i^d}{W_1^d})$ has the density function $f_{V_i^d}(v_i) = \frac{1}{(1+v_i)^2}$, which is the F-distribution with parameters 2 and 2. Furthermore, by selecting $Z \equiv W_1^d$ and using the Jacobian transformation method, we have

$$f_{Z, V_2^d, \cdots, V_k^d}(z, v_2, \cdots, v_k) = z^{k-1} e^{-z(\sum_{j=2}^k v_j + 1)}.$$

Consequently, the proof is completed by integrating with respect to (w.r.t.) z. □

Now, by using the above lemma, we obtain the following representation for the joint probability mass function $\mathbf{R}_k^{MRC} \equiv (R_2^{d=1}, \cdots, R_k^{d=1})$, $k = 2, \cdots, m-1$, under the MRC approach.

Theorem 2.1. *The joint probability mass function $\mathbf{R}_k^{MRC}$, for $k = 2, \cdots, m-1$, is given by*

$$Pr(\mathbf{R}_k^{MRC}) = Pr(R_2^{d=1} = r_2, R_3^{d=1} = r_3, \cdots, R_k^{d=1} = r_k) = \tau_k^{-1} \sum_{i=1}^{2^{k-1}} \frac{(-1)^{\sum_{j=1}^{k-1} a_{i,j}}}{1 + \sum_{j=1}^{k-1} (r_{j+1} + a_{i,j})^2}, \quad (2.3)$$

where $a_{i,j}$ denotes the jth element of the ith row in the matrix A_{k-1} for $i = 1, \cdots, 2^{k-1}$ and $j = 1, \cdots, k-1$. Here $A_{k-1} = [a_{i,j}]$ is the matrix that contains all possible sequences of 0's and 1's of length $k-1$. Therefore, the matrix dimension is $2^{k-1} \times (k-1)$. The constant τ_k^{-1} serves *as the normalizing constant that ensures the joint density function integrates into one. This constant is given by*

$$\tau_k = \left\{Pr(R_2^{d=1} \le \ell_0, R_3^{d=1} \le \ell_0 - r_2, \cdots, R_k^{d=1} \le \ell_0 - \sum_{j=2}^{k-1} r_j)\right\}$$
$$= \left\{\sum_{r_{d_0}(d_0=2,\cdots,k)=0}^{\ell_0+1-\sum_{j=1}^{d-1} r_j} \sum_{i=1}^{2^{k-1}} \frac{(-1)^{\sum_{j=1}^{k-1} a_{i,j}}}{1+\sum_{j=1}^{k-1}(r_{j+1}+a_{i,j})^2}\right\},$$

where $\ell_0 \equiv n-m-1$ *and* $r_1 = 1$.

Proof. For the sake of simplicity and without loss of generality, we check for case $k = 2$ as

$$Pr(R_2^{d=1} = r_2) = Pr([\sqrt{V_2}^{d=1}] = r_2) = \tau_2^{-1} Pr(r_2^2 \le V_2^{d=1} < (r_2+1)^2) = \tau_2^{-1} \int_{r_2^2}^{(r_2+1)^2} \frac{1}{(1+v_2)^2} dv_2$$
$$= \tau_2^{-1}[\frac{1}{(1+r_2^2)} - \frac{1}{1+(1+r_2)^2}] = \tau_2^{-1} \sum_{i=1}^{2} \frac{(-1)^{\sum_{j=1}^{1} a_{i,j}}}{1+\sum_{j=1}^{1}(r_{j+1}+a_{i,j})^2}, \tag{2.4}$$

where here, there are 2 possible sequences of 0's and 1's of length 1. Therefore, we define a matrix as follows: $A_1 = \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The joint probability mass function can be also obtained for $k = 3$ as

$$\begin{aligned}
Pr(R_2^{d=1} = r_2, R_3^{d=1} = r_3) &= \tau_3^{-1} P([\sqrt{V}_2^{d=1}] = r_2, [\sqrt{V}_3^{d=1}] = r_3) \\
&= \tau_3^{-1} P(r_2^2 \le V_2^{d=1} < (r_2+1)^2, r_3^2 \le V_3^{d=1} < (r_3+1)^2)) \\
&= \tau_3^{-1} \int_{r_3^2}^{(r_3+1)^2} \int_{r_2^2}^{(r_2+1)^2} f(v_2, v_3) dv_2 dv_3 \\
&= \tau_3^{-1} \int_{r_3^2}^{(r_3+1)^2} \left(\frac{1}{(1+r_2^2+v_3)^2} - \frac{1}{(1+(1+r_2)^2+v_3)^2}\right) dv_3 \\
&= \tau_3^{-1}\Big[\frac{1}{1+r_2^2+r_3^2} - \frac{1}{1+r_2^2+(r_3+1)^2} - \frac{1}{1+(r_2+1)^2+r_3^2} \\
&\qquad + \frac{1}{1+(r_2+1)^2+(r_3+1)^2}\Big] \\
&= \tau_3^{-1} \sum_{i=1}^{4} \frac{(-1)^{\sum_{j=1}^{2} a_{i,j}}}{1+\sum_{j=1}^{2}(r_{j+1}+a_{i,j})^2},
\end{aligned}$$

in which, $A_2 = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \\ a_{4,1} & a_{4,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$ and

$$\tau_3 = \left\{Pr(R_2^{d=1} \le \ell_0, R_3^{d=1} \le \ell_0 - r_2)\right\} = \left\{ \sum_{r_{d_0}(d_0=2,3)=0}^{\ell_0+1-\sum_{j=1}^{d_0-1} r_j} \sum_{i=1}^{2^2} \frac{(-1)^{\sum_{j=1}^{2} a_{i,j}}}{1+\sum_{j=1}^{2}(r_{j+1}+a_{i,j})^2} \right\}.$$

Then, by using a similar argument for $k = 4, \cdots, m-1$, we find

$$\begin{aligned} Pr\left(R_2^{d=1} = r_2, R_3^{d=1} = r_3, \cdots, R_k^{d=1} = r_k\right) &= \tau_k^{-1} P\left([\sqrt{V}_2^{d=1}] = r_2, [\sqrt{V}_3^{d=1}] = r_3, \cdots, [\sqrt{V}_k^{d=1}] = r_k\right) \\ &= \tau_k^{-1} \int_{r_k^2}^{(r_{k+1})^2} \sum_{i=1}^{2^{k-2}} (-1)^{\sum_{j=1}^{k-2} a_{i,j}} \left[1 + \sum_{j=1}^{k-2} (r_{j+1}+a_{i,j})^2 + v_k\right]^{-2} dv_k \\ &= \tau_k^{-1} \sum_{i=1}^{2^{k-1}} \frac{(-1)^{\sum_{j=1}^{k-1} a_{i,j}}}{1+\sum_{j=1}^{k-1}(r_{j+1}+a_{i,j})^2}. \end{aligned}$$

There are 2^{k-1} possible sequences of length $k-1$. In other words, $A_{k-1} = [a_{i,j}]_{2^{k-1}\times(k-1)}$ is the matrix of all possible sequences of 0's and 1's of length $k-1$. □

Theorem 2.2. *The joint probability mass function* $\mathbf{R}_k^{MFC}$*, for* $k = 2, \cdots, m-1$*, is given by*

$$Pr(\mathbf{R}_k^{MFC}) = Pr\left(R_2^{d=2} = r_2, R_3^{d=2} = r_3, \cdots, R_k^{d=2} = r_k\right) = \prod_{i=2}^{k} h_i \left(\frac{1}{1+r_i^2} - \frac{1}{1+(1+r_i)^2}\right), \tag{2.5}$$

where h_i *is the normalizing constant defined as*

$$h_i = \left(1 + \frac{1}{(n-m-\sum_{l=1}^{i-1} r_l)^2}\right).$$

Proof. Since $R_1^d = 1$, for $d = 1, 2$, and $Pr\left(R_2^{d=2} = r_2\right)$ is equal to (2.4), it is thus seen that for $i = 3, \cdots, k$ and $k = 3, \cdots, m-1$,

$$\begin{aligned} Pr\left(R_i^{d=2} = r_i \mid R_2^{d=2} = r_2, \cdots, R_{i-1}^{d=2} = r_{i-1}\right) &= Pr([\sqrt{V}_i^{d=2}] = r_i \mid R_2^{d=2} = r_2, \cdots, R_{i-1}^{d=2} = r_{i-1}) \\ &= Pr(r_i^2 \le V_i^{d=2} < (r_i+1)^2 \mid R_2^{d=2} = r_2, \cdots, R_{i-1}^{d=2} = r_{i-1}) \\ &= \delta_i^{-1} \int_{r_i^2}^{(r_i+1)^2} \frac{1}{(1+v_i)^2} dv_i = \delta_i^{-1} \left\{\frac{1}{1+r_i^2} - \frac{1}{1+(1+r_i)^2}\right\}, \end{aligned} \tag{2.6}$$

where normalizing constant δ_i in (2.6), for $i = 3, \cdots, k$ and $k = 3, \cdots, m-1$ is defined by

$$\delta_i = \sum_{r_j=0}^{n-m-1-\sum_{\ell=2}^{i-1} r_\ell} \left(\frac{1}{1+r_j^2} - \frac{1}{1+(1+r_j)^2} \right)$$
$$= \frac{(n-m-\sum_{\ell=2}^{i-1} r_\ell)^2}{1+(n-m-\sum_{\ell=2}^{i-1} r_\ell)^2}.$$

On the other hand, the joint probability mass function can be obtained as

$$\begin{aligned} Pr(R_2^{d=2} = r_2, R_3^{d=2} = r_3, \cdots, R_k^{d=2} = r_k) &= Pr(R_2^{d=2} = r_2) Pr(R_3^{d=2} = r_3 \mid R_2^{d=2} = r_2) \\ &\times Pr(R_4^{d=2} = r_4 \mid R_2^{d=2} = r_2, R_3^{d=2} = r_3) \times \cdots \\ &\times Pr(R_k^{d=2} = r_k \mid R_2^{d=2} = r_2, R_3^{d=2} = r_3, \cdots, R_{k-1}^{d=2} = r_{k-1}). \end{aligned} \quad (2.7)$$

The proof of the theorem is completed by replacing (2.6) in (2.7) and setting $h_i = \delta_i^{-1}$. □

As mentioned in the introduction of progressive censoring with random removals, two commonly used random models are the binomial and discrete uniform distributions. In the following subsections, we will briefly review them.

2.2 Removal Model with Discrete Uniform Distribution

Suppose that the number of units removed at each failure time follows a discrete uniform distribution such that

$$Pr(R_1 = r_1) = \frac{1}{n-m+1},$$

and

$$Pr(R_i = r_i \mid R_1 = r_1, R_2 = r_2, \cdots, R_{i-1} = r_{i-1}) = \frac{1}{n-m-\sum_{j=1}^{i-1} r_j + 1}, \quad (2.8)$$

where, $0 \leq r_1 \leq n-m$ and $0 \leq r_i \leq n-m-\sum_{j=1}^{i-1} r_j$, $i = 2, \cdots, m-1$.

2.3 Removal Model with Binomial Distribution

Assume that each test unit being dropped out from the life test is independent of the others but with the same removal probability p. Therefore, the number of removed units at each failure time follows the binomial distribution as

$$Pr(R_1 = r_1) = \binom{n-m}{r_1} p^{r_1} (1-p)^{n-m-r_1},$$

and

$$Pr(R_i = r_i \mid R_1 = r_1, R_2 = r_2, \cdots, R_{i-1} = r_{i-1}) = \binom{n_i - m}{r_i} p^{r_i} (1-p)^{n_{i+1}-m}, \quad (2.9)$$

where, $0 \leq r_i \leq n_i - m$ and $n_i = n - \sum_{j=1}^{i-1} r_j$ for $i = 1, \ldots, m-1$. Furthermore, the joint probability mass function of $\mathbf{R} \equiv (R_1, R_2, \cdots, R_{m-1})$ is as follows:

$$Pr(\mathbf{R} = \mathbf{r}) = \frac{(n-m)!}{(n-m-\sum_{j=1}^{m-1} r_j)! \prod_{j=1}^{m-1} r_j!} p^{\sum_{j=1}^{m-1} r_j} (1-p)^{(m-1)(n-m) - \sum_{j=1}^{m-1}(m-j) r_j}.$$

3 Maximum Likelihood Estimation

Let $(X_{1:m:n}, R_1)$, $(X_{2:m:n}, R_2)$, $\cdots$, $(X_{m:m:n}, R_m)$ be the progressive Type-II right censored sample from the three-parameter Weibull lifetimes, where $\mathbf{R}$ is chosen according to four given discrete distributions. Given that $\mathbf{x}_m = (x_{1:m:n}, \cdots, x_{m:m:n})$ and $\mathbf{r} = (r_1, \cdots, r_m)$, the likelihood function can be defined by

$$L(\alpha, \theta, \beta; \mathbf{x}_m, \mathbf{r}) = L_1(\alpha, \theta, \beta; \mathbf{x}_m \mid \mathbf{R} = \mathbf{r}) Pr(\mathbf{R} = \mathbf{r}).$$

Since $Pr(\mathbf{R} = \mathbf{r})$ does not include the parameters α, θ, and β, therefore the maximum likelihood estimates can be derived from maximizing equation $L_1(\alpha, \theta, \beta; \mathbf{x}_m \mid \mathbf{R})$. The likelihood function for progressive Type-II censoring can be defined according to Cohen (1963) as:

$$L_1(\alpha, \theta, \beta; \mathbf{x}_m \mid \mathbf{R} = \mathbf{r}) = c \prod_{i=1}^{m} f(x_{i:m:n}; \alpha, \theta, \beta)(1 - F(x_{i:m:n}; \alpha, \theta, \beta))^{r_i}, \tag{3.1}$$

where $c_i = n - \sum_{j=1}^{i-1}(r_j + 1)$ for $i = 1, \ldots, m$, and $c = \prod_{i=1}^{m} c_i$.

Substituting (2.1) and (2.2) into (3.1), the likelihood function takes the following form

$$L_1(\alpha, \theta, \beta; \mathbf{x}_m \mid \mathbf{R} = \mathbf{r}) = c \prod_{i=1}^{m} \frac{\beta}{\theta} \left(\frac{x_{i:m:n} - \alpha}{\theta} \right)^{\beta - 1} exp \left[-(r_i + 1) \left(\frac{x_{i:m:n} - \alpha}{\theta} \right)^{\beta} \right].$$

The log-likelihood function may then be written as

$$\ell = \log c + m \log \frac{\beta}{\theta} + (\beta - 1) \sum_{i=1}^{m} \log \left(\frac{x_{i:m:n} - \alpha}{\theta} \right) - \sum_{i=1}^{m} (r_i + 1) \left(\frac{x_{i:m:n} - \alpha}{\theta} \right)^{\beta}.$$

The normal equations are obtained by equating the first-order partial derivatives of the log-likelihood function w.r.t. the parameters to zero, i.e.,

$$\frac{\partial \ell}{\partial \alpha} = -(\beta - 1) \sum_{i=1}^{m} \frac{1}{x_{i:m:n} - \alpha} + \frac{\beta}{\theta} \sum_{i=1}^{m} (r_i + 1) \left(\frac{x_{i:m:n} - \alpha}{\theta} \right)^{\beta - 1} = 0,$$

$$\frac{\partial \ell}{\partial \theta} = -m \frac{\beta}{\theta} + \sum_{i=1}^{m} (r_i + 1) \frac{\beta}{\theta} \left(\frac{x_{i:m:n} - \alpha}{\theta} \right)^{\beta} = 0,$$

$$\frac{\partial \ell}{\partial \beta} = \frac{m}{\beta} + \sum_{i=1}^{m} \log(\frac{x_{i:m:n} - \alpha}{\theta}) - \sum_{i=1}^{m} (r_i + 1) \left(\frac{x_{i:m:n} - \alpha}{\theta} \right)^{\beta} \log \left(\frac{x_{i:m:n} - \alpha}{\theta} \right) = 0.$$

We can obtain numerical solutions for the three likelihood equations while adhering to the constraint $\alpha < x_{1:m:n}$. Ng et al. (2012) extensively discussed different methods for estimating parameters of the three-parameter Weibull distribution. They also proposed an efficient and reliable approach for obtaining initial estimates based on censored estimation, as suggested by Harter and Moore (1966), along with a one-step bias correction for the location parameter. We employ their algorithm for computing the estimates of all three parameters. The analytic Hessian of the log-likelihood function is provided by

$$I^o_{11} = \frac{\partial^2 \ell}{\partial \alpha^2} = -(\beta-1)\sum_{i=1}^{m}\frac{1}{(x_{i:m:n}-\alpha)^2} - \frac{\beta(\beta-1)}{\theta^2}\sum_{i=1}^{m}(r_i+1)(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta-2},$$

$$I^o_{22} = \frac{\partial^2 \ell}{\partial \theta^2} = \frac{m\beta}{\theta^2} - \frac{\beta(\beta+1)}{\theta^2}\sum_{i=1}^{m}(r_i+1)(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta},$$

$$I^o_{33} = \frac{\partial^2 \ell}{\partial \beta^2} = \frac{-m}{\beta^2} - \sum_{i=1}^{m}(r_i+1)(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta}(\log(\frac{x_{i:m:n}-\alpha}{\theta}))^2,$$

$$I^o_{12} = I^o_{21} = \frac{\partial^2 \ell}{\partial \alpha \partial \theta} = \frac{-\beta^2}{\theta^2}\sum_{i=1}^{m}(r_i+1)(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta-1},$$

$$I^o_{23} = I^o_{32} = \frac{\partial^2 \ell}{\partial \theta \partial \beta} = \frac{-m}{\theta} + \sum_{i=1}^{m}\frac{(r_i+1)}{\theta}(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta}[1+\beta\log(\frac{x_{i:m:n}-\alpha}{\theta})],$$

$$I^o_{13} = I^o_{31} = \frac{\partial^2 \ell}{\partial \alpha \partial \beta} = \sum_{i=1}^{m}\frac{-1}{x_{i:m:n}-\alpha} + \sum_{i=1}^{m}\frac{(r_i+1)}{\theta}(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta-1}[1+\beta\log(\frac{x_{i:m:n}-\alpha}{\theta})].$$

3.1 Corrected Maximum Likelihood Estimation

In some cases, the maximum likelihood method may not work because the classical regularity conditions described in Cramer (1946) and Wald (1949) are not met, for more details see e.g Le Cam (1990). These situations are often called non-regular problems and involve different levels of violations of regularity conditions. Here, model (2.1) has unbounded likelihood, and its support depends on the parameter. Therefore, we will only discuss results related to this type of non-regularity. Cheng and Iles (1987) pointed out that the likelihood tends to infinity when α approaches to the smallest observation, $x_{1:m;n}$, and the principle of maximum likelihood then requires the choice $\hat{\alpha} = X_{1:m:n}$; whilst this leads to inconsistent MLEs of the other two parameters. To overcome this problem, they proposed to maximize a corrected likelihood function by replacing $f(x_{1:m:n})$ with $\int_{x_{1:m:n}}^{x_{1:m:n}+\epsilon} f(x)dx$. By choosing $\epsilon = x_{2:m:n}$ for progressively right-censored data, the corrected likelihood can be written as

$$L_{1.c}(\alpha,\theta,\beta;\mathbf{x}_m \mid \mathbf{R}=\mathbf{r}) = c\Big[F(x_{2:m:n};\alpha,\theta,\beta) - F(x_{1:m:n};\alpha,\theta,\beta)\Big] \times \Big(1-F(x_{1:m:n};\alpha,\theta,\beta)\Big)^{r_1}\prod_{i=2}^{m} f(x_{i:m:n};\alpha,\theta,\beta)\Big(1-F(x_{i:m:n};\alpha,\theta,\beta)\Big)^{r_i}.$$

Thus, the corrected likelihood function for progressively Type-II right censored data with random removals can be obtained as follows:

$$L_c(\alpha,\theta,\beta;\mathbf{x}_m,\mathbf{r}) = L_{1.c}(\alpha,\theta,\beta;\mathbf{x}_m \mid \mathbf{R}=\mathbf{r})Pr(\mathbf{R}=\mathbf{r}).$$

For simplicity in notation, we use the simplified notations $F(x_{i:m:n}) \equiv F(x_{i:m:n};\alpha,\theta,\beta)$, and $f(x_{i:m:n}) \equiv f(x_{i:m:n};\alpha,\theta,\beta)$ for $i=2,\cdots,m$, and $\ell_{1.c} \equiv \log\left(L_{1.c}(\alpha,\theta,\beta;\mathbf{x}_m \mid \mathbf{R})\right)$. Note that $Pr(\mathbf{R}=\mathbf{r})$ does not involve the parameters of the model, α,θ and β. Therefore, the CMLEs of parameters can be found by maximizing $L_{1.c}(\alpha,\theta,\beta;\mathbf{x}_m \mid \mathbf{R}=\mathbf{r})$ directly subject to the constraint $\alpha < x_{1:m:n}$. In order to simplify the notation, let $\ell_{1.c} = \log(c) + \log(c_5) - r_1c_2 + \ell_{-1}$, where

$$c_2 = \Big(\frac{x_{1:m:n}-\alpha}{\theta}\Big)^{\beta}, \quad c_5 = \exp\Big(-\Big(\frac{x_{1:m:n}-\alpha}{\theta}\Big)^{\beta}\Big) - \exp\Big(-\Big(\frac{x_{2:m:n}-\alpha}{\theta}\Big)^{\beta}\Big),$$

and

$$\ell_{-1} = \log\left(\prod_{i=2}^{m} f(x_{i:m:n})\Big(1-F(x_{i:m:n})\Big)^{r_i}\right).$$

The gradients and Hessians of the log-corrected likelihood function, $\ell_{1.c}$, w.r.t. the three parameters are given by

$$\frac{\partial \ell_{1.c}}{\partial\alpha} = \frac{\partial c_5}{\partial\alpha}\cdot\frac{1}{c_5} - r_1\frac{\partial c_2}{\partial\alpha} - (\beta-1)\sum_{i=2}^{m}\frac{1}{x_{i:m:n}-\alpha} + \frac{\beta}{\theta}\sum_{i=2}^{m}(r_i+1)(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta-1},$$

$$\frac{\partial \ell_{1.c}}{\partial\theta} = \frac{\partial c_5}{\partial\theta}\cdot\frac{1}{c_5} - r_1\frac{\partial c_2}{\partial\theta} - (m-1)\frac{\beta}{\theta} + \sum_{i=2}^{m}(r_i+1)\frac{\beta}{\theta}(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta},$$

$$\frac{\partial \ell_{l.c}}{\partial\beta} = \frac{\partial c_5}{\partial\beta}\cdot\frac{1}{c_5} - r_1\frac{\partial c_2}{\partial\beta} + \frac{m-1}{\beta} + \sum_{i=2}^{m}\log(\frac{x_{i:m:n}-\alpha}{\theta}) - \sum_{i=2}^{m}(r_i+1)(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta}\log(\frac{x_{i:m:n}-\alpha}{\theta}),$$

and

$$I^{o}_{c.11} = \frac{\partial^2 \ell_{l.c}}{\partial\alpha^2} = \frac{\partial^2 c_5}{\partial\alpha^2}\cdot\frac{1}{c_5} - (\frac{\partial c_5}{\partial\alpha})^2\cdot\frac{1}{c_5^2} - r_1\frac{\partial^2 c_2}{\partial\alpha^2} - (\beta-1)\sum_{i=2}^{m}\frac{1}{(x_{i:m:n}-\alpha)^2}$$
$$- \frac{\beta(\beta-1)}{\theta^2}\sum_{i=2}^{m}(r_i+1)(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta-2},$$

$$I^{o}_{c.22} = \frac{\partial^2 \ell_{1.c}}{\partial\theta^2} = \frac{\partial^2 c_5}{\partial\theta^2}\cdot\frac{1}{c_5} - (\frac{\partial c_5}{\partial\theta})^2\cdot\frac{1}{c_5^2} - r_1\frac{\partial^2 c_2}{\partial\theta^2} + \frac{(m-1)\beta}{\theta^2} - \frac{\beta(\beta+1)}{\theta^2}\sum_{i=2}^{m}(r_i+1)(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta},$$

$$I^{o}_{c.33} = \frac{\partial^2 \ell_{l.c}}{\partial\beta^2} = \frac{\partial^2 c_5}{\partial\beta^2}\cdot\frac{1}{c_5} - (\frac{\partial c_5}{\partial\beta})^2\cdot\frac{1}{c_5^2} - r_1\frac{\partial^2 c_2}{\partial\beta^2} + \frac{-m+1}{\beta^2} - \sum_{i=2}^{m}(r_i+1)(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta}(\log(\frac{x_{i:m:n}-\alpha}{\theta}))^2,$$

$$I^o_{c.12} = I^o_{c.21} = \frac{\partial^2 \ell_{l.c}}{\partial\alpha\partial\theta} = \frac{\partial^2 c_5}{\partial\alpha\partial\theta}\cdot\frac{1}{c_5} - \frac{\partial c_5}{\partial\alpha}\frac{\partial c_5}{\partial\theta}\cdot\frac{1}{c_5^2} - r_1\frac{\partial^2 c_2}{\partial\alpha\partial\theta} - \frac{\beta^2}{\theta^2}\sum_{i=2}^{m}(r_i+1)(\frac{x_i-\alpha}{\theta})^{\beta-1},$$

$$I^o_{c.13} = I^o_{c.31} = \frac{\partial^2 \ell_{l.c}}{\partial\alpha\partial\beta} = \frac{\partial^2 c_5}{\partial\alpha\partial\beta}\cdot\frac{1}{c_5} - \frac{\partial c_5}{\partial\alpha}\frac{\partial c_5}{\partial\beta}\cdot\frac{1}{c_5^2} - r_1\frac{\partial^2 c_2}{\partial\alpha\partial\beta} + \sum_{i=2}^{m}\frac{-1}{x_{i:m:n}-\alpha}$$
$$+\sum_{i=2}^{m}\frac{(r_i+1)}{\theta}(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta-1}[1+\beta\log(\frac{x_{i:m:n}-\alpha}{\theta})],$$

$$I^o_{c.23} = I^o_{c.23} = \frac{\partial^2 \ell_{l.c}}{\partial\theta\partial\beta} = \frac{\partial^2 c_5}{\partial\theta\partial\beta}\cdot\frac{1}{c_5} - \frac{\partial c_5}{\partial\theta}\frac{\partial c_5}{\partial\beta}\cdot\frac{1}{c_5^2} - r_1\frac{\partial^2 c_2}{\partial\theta\partial\beta} + \frac{-m+1}{\theta}$$
$$+\sum_{i=2}^{m}\frac{(r_i+1)}{\theta}(\frac{x_{i:m:n}-\alpha}{\theta})^{\beta}[1+\beta\log(\frac{x_{i:m:n}-\alpha}{\theta})],$$

where all the first-order and second- order partial derivatives are given in Appendix A.

4 Expected Test Time

In practical applications, the costs associated with conducting a life test depend heavily on the duration of the experiment. Therefore, it is crucial for an experimenter to choose a suitable sampling plan based on the expected duration of the experiment. The superiority criterion for different plans that have the same conditions, such as the total number of units in the experiment (denoted as n) and the number of observed units (denoted as m), is the reduction of the experiment's duration. This, in turn, helps to lower research costs and save time. The expected value of $X_{m:m:n}$ compares effectiveness of schemes. In this case, the random variable $X_{m:m:n}$ depends on the lifetime distribution, sample size, removal numbers, and the distribution of random removals. Below, we will describe this issue in detail.

Cramer (2014) developed an extreme value analysis for progressively Type-II censored order statistics, utilizing the results of generalized order statistics obtained by Cramer (2003). The mth progressively Type-II censored order statistic given $\{\mathbf{R} = \mathbf{r}\}$, denoted as $X^{\mathbf{r}}_{m:m:n}$, can be expressed as a quantile transformation of a product of *IID* transformed uniform random variables $U_1, U_2, \cdots, U_m$, i.e.,

$$X^{\mathbf{r}}_{m:m:n} \overset{d}{=} F^{\leftarrow}\left(1 - \prod_{j=1}^{m} U_j^{1/\lambda_j}\right),$$

with $\lambda_j = \sum_{i=j}^{m}(r_i+1)$, $1 \le j \le m$. Therefore, the cumulative distribution function $F_{X^{\mathbf{r}}_{m:m:n}}$ has the representation

$$F_{X^{\mathbf{r}}_{m:m:n}}(x_m) = P\left(F^{\leftarrow}\left(1-\prod_{j=1}^{m} U_j^{1/\lambda_j}\right) \le x_m\right)$$
$$= P\left(\sum_{i=1}^{m} T_i < -\log \overline{F}(x_m)\right), \tag{4.1}$$

where $T_1, \cdots, T_m$ are independent random variables with $T_j = -\frac{1}{\lambda_j}\log U_j \sim Exp(\lambda_j)$, $1 \le j \le m$, i.e., $F_{T_j}(t) = 1 - e^{-\lambda_j t}$ and $\bar{F}(.) = 1 - F(.)$. For more details, see Cramer (2014) and Balakrishnan and Cramer (2014). Since $\lambda_1 > \lambda_2 > \cdots > \lambda_m$, we can use Lemma 2.1 in Kordecki (1997), and the convolution distribution function of $T_i s$ in (4.1) may then be written as

$$F_S(s) = \sum_{i=1}^{m}\left(1-e^{-\lambda_i s}\right)\prod_{j=1,j\neq i}^{m}\frac{\lambda_j}{\lambda_j-\lambda_i}, \tag{4.2}$$

where $S \equiv \sum_{i=1}^{m} T_i$. Consequently the density function of $X^{\mathbf{r}}_{m:m:n}$ can be obtained by replacing s by $-\log \bar{F}(x_m)$ in (4.2) and the derivative w.r.t. x_m from the relation (4.1) as

$$f_{X^{\mathbf{r}}_{m:m:n}}(x_m) = \sum_{i=1}^{m}\lambda_i\, f(x_m)\,(\bar{F}(x_m))^{\lambda_i-1}\prod_{j=1,j\neq i}^{m}\frac{\lambda_j}{\lambda_j-\lambda_i}.$$

Therefore, the conditional expectation $E\left(X^{r}_{m:m:n}\right)$, which is the expected time required to complete an experiment under the Type-II progressive censoring with fixed removals, can be expressed as

$$E\left(X^{r}_{m:m:n}\right) = \sum_{i=1}^{m}\prod_{j=1,j\neq i}^{m}\frac{\lambda_i\lambda_j}{\lambda_j-\lambda_i}\int_0^{\infty} x_m\, f(x_m)\left(\overline{F}(x_m)\right)^{\lambda_i-1}dx_m, \tag{4.3}$$

where $f(.)$ and $\overline{F}(.) = 1 - F(.)$ are defined in (2.1) and (2.2). Then, the expected duration of an experiment under progressive Type-II censoring with random removals can be computed by taking expectations on both sides of (4.3) w.r.t. $\mathbf{R}$. Namely, we have

$$E(X_{m:m:n}) = E_{\mathbf{R}}\left(E\left(X^{r}_{m:m:n}\right)\right) = \sum_{r_1=0}^{h(r_1)}\sum_{r_2=0}^{h(r_2)}\cdots\sum_{r_{m-1}=0}^{h(r_{m-1}-1)} E\left(X^{r}_{m:m:n}\right)Pr\left(\mathbf{R}=r\right),$$

where $h(r_1) = n - m$, $h(r_i) = n - m - \sum_{j=1}^{i-1} r_j$, $i = 2, \cdots, m$, and here $E\left(X^{r}_{m:m:n}\right)$ in (4.3), and $Pr\left(\mathbf{R}=r\right)$ are given by four scenarios in (2.3), (2.5), (2.8) and (2.9). Note that always $r_1 = 1$ in the proposed approaches.

It is obvious that analytically comparing these four expected test times is very difficult and cumbersome. An alternative is to calculate them numerically. In the next section, we calculate the values of $E[X_{m:m:n}]$ numerically for various n, m, p and β with the two proposed removal patterns as well as the binomial and discrete uniform patterns based on progressively Type-II right censored samples from the three-parameter Weibull lifetime distribution.

5 Simulation Study

A Monte Carlo simulation study is performed to compare the performance of the proposed removal models, the discrete uniform removals, binomial random removals, and the different fixed removal schemes which are listed in Table 1 for notational convenience. For each set of simulated data, we have computed the estimators discussed in Section 2. The program codes of simulation results are written in R Version 3.5.2 (R Core Team, 2018) and the maxLik function that is in the maxLik package is used for the constrained optimization (see Henningsen and Toomet, 2011). We have also provided the analytic gradient vector and Hessian matrices described in Section 3 and subsection 3.1 for faster convergence.

In order to find the initial starting points for α_0, θ_0, and β_0, we used Ng et al. (2012)'s method. This method involves using censored estimators with a one-step bias correction. We conducted simulations for sample sizes of $n = 10, 20, 40$, with different choices of the effective sample size m, along with different parameter values of $\alpha = 300, \theta = 100$, and $\beta = 0.5, 1, 2$. We also applied progressive censoring schemes by using four removal patterns with removal probabilities of $p = 0.4$ and 0.7 in the binomial case.

In addition, there are different schemes for censoring with fixed removals which are listed in Table 1 and show the early, middle and last stages. Tables 2 through 7 present the estimated biases (EBs) and estimated root-mean-square errors (ERMSEs) for estimators using both maximum likelihood and corrected maximum likelihood methods. Additionally, Table 8 shows the expected total time, denoted as $E(X_{m:m:n})$. All the averages were calculated based on 1000 replications.

Based on the simulation results, it is generally observed that as the effective sample size $\frac{m}{n}$ increases, the absolute values of EB and ERMSE for the CMLE estimators decrease in most cases and in the other cases, values are close to each other. When n and m are fixed, a random removal plan is used, and $\beta \leq 1$, the MRC and MFC estimators are shown to be superior in terms of EB and ERMSE for most estimation scenarios. However, as n increases, these values are very close under the different removal plans. When estimating α and θ, the CMLE provides the smallest ERMSE values, while for $\beta \leq 1$ and a small effective sample size m, the MLE provides the smallest ERMSEs. In large effective sample sizes, the uniform and binomial ($p = 0.7$) approaches exhibit the lowest ERMSEs. On the other hand, when $\beta \leq 1$, the MRC and MFC approaches are the most effective in estimating all three parameters in terms of ERMSE. It should be noted that with a sample size of $n = 40$ and small $n - m$, due to the numerous occurrences of zero values, the performances of different random removal methods are close to each other.

The main aim of this paper is to compare different methods of generating random removals and analyze how they affect the termination point of an experiment. In Table 8, we have observed that the expected total time value $E(X_{m:m:n})$ increases as the fixed sample size n and the effective sample size m increase, which is not surprising. When β remains fixed and $m > \lfloor \frac{n}{2} \rfloor$, the value increases with an increase in the sample size n. It's important to note that this increase is more significant for β values that are less than 1. In the random plans and for $m \geq \lfloor \frac{n}{2} \rfloor$., the smallest value of $E(X_{m:m:n})$ occurs at $\beta = 2$. Furthermore, we have discovered that values of $E(X_{m:m:n})$ under MRC and MFC are lower than those from uniform and binomial models in Table 1, except at the last stages. In the last stage case, $n - m$ remaining survival units are removed at the final termination point of the experiment. However, the concept of progressive censoring involves removing live experimental units at points other than the final termination point of the experiment, and this is motivated by the definition and existential philosophy. By leaving this stage, the MRC approach has the lowest expected test times compared to other approaches. This preference is maintained even for a small effective sample size m. Note that, this paper focuses on random removals, and only the experiment forces us to consider the type of scheme. The proposed methods depend on lifetime observations and the removal scheme, which can occur early, in the middle, or later in every situation.

6 A Real Data Example

This section presents the practical application of different distributions of random removals to a real-life data set. The data set is the result of a test conducted on the endurance of deep-groove ball bearings. It was originally discussed by Lieblein and Zelen in 1956, who attributed it to tests on the endurance of deep-groove ball bearings. The data set consists of 22 observations, each representing the number of million revolutions before failure. These different observations are ordered by failure times and provided in Table 9. The data were analyzed by Thoman et al. (1969); Lee et al. (2011) and others using two-parameter Weibull distribution. EL-Adll (2011) provided a better fitting by using the three-parameter Weibull distribution and estimated the parameters as $\hat{\alpha} = 15.98674$, $\hat{\theta} = 61.59874$ and $\hat{\beta} = 1.44$.

Here, we introduce a seven-stage pattern to derive progressive Type-II right censored samples from ball bearings data with $m = 10$. The steps are as follows:

(i) Generate 10000 sets of removal vectors under the different distributions and with $(n, m) = (22, 10)$.

(ii) Obtain progressively Type-II right censored samples from the three-parameter Weibull lifetime distribution with parameters $\hat{\alpha} = 15.98674$, $\hat{\theta} = 61.59874$ and $\hat{\beta} = 1.44$.

(iii) Choose the progressively censored vectors whose first components fall from 17.875 to 17.8844.

(iv) Calculate the average vector by taking the average of the first elements and second elements, and so on.

(v) Compute the coefficients of variation for both the average vector and the vectors obtained from the third stage.

(vi) Choose the vector whose coefficient of variation is closest to the corresponding value of the average vector as the desirable sample, and also consider its removal scheme.

(vii) Replace the progressive sample with the nearest values to the ball bearings data.

The results are presented in Table 9. We noticed that the performance of different removal scenarios varies. The MRC and MFC approaches yield a smaller maximum failure time compared to the discrete uniform and binomial distributions. This indicates that the proposed models can help reduce the total time of the experiment. It is worth noting that although different samples are provided each time, the optimal distribution of random removals is the one that yields the lowest maximum observation. The proposed distributions are always optimal and perform well. Nevertheless, the MRC outperforms the MFC to a certain extent, thus we recommend using the MRC.

7 Concluding Remarks

In this paper, we have presented two approaches, MRC and MFC, for choosing removal strategies of progressively Type-II right-censored samples under the three-parameter Weibull lifetimes. These methods utilize progressive censoring schemes that are built upon exponential lifetimes. Both the simulation results and the real data analysis demonstrate that binomial removals with $p = 0.4$ have a better performance than uniform removals in terms of $E(X_{m:m:n})$ and for $p = 0.7$ this result is reversed. However, the expected test time depends greatly on the value of the removal probability p, especially for small sample sizes. The parameter p is an important factor in the expected test time and inferential results. When p is too small, there is a low chance of removing a surviving unit in each stage of the experiment. This leads to most units being dropped out during the last stage and then making the progressive censoring scheme similar to the Type-II censoring plan. Therefore, the philosophy behind implementing the progressive Type-II censoring plan is being challenged. For example, if $p = 0.05$ and $n = 10, m = 4$, $Pr(R_1 + R_2 + R_3 = 1) = 0.3964$ but for $p = 0.4$, $Pr(R_1 + R_2 + R_3 = 1) = 0.0022$. On the other hand, if p is too large and close to 1, $n - m$ of the test units would be dropped out at the early stage of the life test. This results in the observed lifetimes being much closer to the tail of the failure time distribution. Thus, the expected test time of progressive Type-II censoring with binomial removals is close to that of the complete sample. It makes sense that a fair and practical choice for p is a value that is neither too close to zero nor too close to 1. The proposed plans do not impose any additional parameters to the model and have the shortest expected total time compared to other approaches. Thus, they have the potential to decrease testing duration and expenses, positioning them as preferred methods for application in fields such as engineering, medicine, and other sectors that necessitate life testing. It is worth noting that the MRC approach performs slightly better than the MFC approach and can be applied to the literature related to random removals with Weibull lifetimes. using information on binomial parameter p can find better approaches for statistical inference in the future. For example, we can use an informative prior that relies on failure times or a family of unit-distribution. Furthermore, in upcoming research on random censoring schemes, we can utilize adaptive censoring, particularly the adaptive Type-II progressive hybrid censoring.

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Table 1: Progressive censoring schemes with fixed removals used in the Monte Carlo simulation study

(n, m)	$(R_1, , ..., R_k)$	scheme
(10,4)	(6, 3*0)	[1]
	(0, 6, 2*0)	[2]
	(0, 2*3, 0)	[3]
	(3*0, 6)	[4]
(10,7)	(3, 6*0)	[5]
	(0, 3, 5*0)	[6]
	(3*0,3,3*0)	[7]
	(6*0,3)	[8]
(20,7)	(13, 6*0)	[9]
	(0,13, 5*0)	[10]
	(2*0,4,5,4, 2*0)	[11]
	(6*0,13)	[12]
(20,10)	(10, 9*0)	[13]
	(0, 10, 8*0)	[14]
	(4*0, 5,5,4*0)	[15]
	(9*0, 10)	[16]
(20,15)	(5, 14*0)	[17]
	(0, 5, 13*0)	[18]
	(7*0, 5, 7*0)	[19]
	(14*0,5)	[20]
(40,20)	(20, 19*0)	[21]
	(0,20, 18*0)	[22]
	(9*0,10,10,9*0)	[23]
	(19*0,20)	[24]
(40, 35)	(5, 34*0)	[25]
	(0, 5, 33*0)	[26]
	(17*0, 5, 17*0)	[27]
	(34*0,5)	[28]

Table 2: Simulated results under different approaches of generating progressive censoring schemes with $\alpha = 300$, $\theta = 100$ and $\beta = 0.5$

				α		θ		β	
n	m	Removal plan	method	EB	ERMSE	EB	ERMSE	EB	ERMSE
10	4	MRC	MLE	−0.743	12.902	−24.345	71.955	0.848	4.405
			CMLE	−5.371	14.015	−28.190	77.099	2.577	6.623
10	4	MFC	MLE	−1.693	15.910	−20.779	75.691	1.019	4.697
			CMLE	−5.371	14.015	−26.005	81.984	2.500	6.960
10	4	uniform	MLE	−0.417	14.250	−7.401	77.830	0.878	4.592
			CMLE	−3.248	12.801	−19.230	74.335	1.935	6.128
10	4	binomial	MLE	−0.687	14.872	−8.362	74.679	0.591	3.697
		with $p = 0.4$	CMLE	−5.101	19.457	−17.867	73.513	2.530	7.827
10	4	binomial	MLE	−1.073	20.74	8.815	90.417	0.281	1.475
		with $p = 0.7$	CMLE	−4.302	21.853	−5.672	72.369	2.147	6.755
10	4	[1]	MLE	−3.919	31.432	18.606	94.189	0.341	1.769
			CMLE	−10.798	36.291	10.640	85.170	3.547	9.655
10	4	[2]	MLE	0.952	8.033	0.369	77.670	0.322	1.901
			CMLE	0.354	8.957	−5.756	76.637	0.873	4.559
10	4	[3]	MLE	−0.031	11.488	−12.362	74.370	0.563	2.986
			CMLE	−0.143	10.935	−14.0647	78.814	1.646	5.969
10	4	[4]	MLE	−1.031	16.219	−28.843	70.596	0.969	4.627
			CMLE	−1.713	11.013	−42.130	75.003	3.774	8.469
10	7	MRC	MLE	0.858	14.303	12.346	75.321	0.099	0.546
			CMLE	−3.204	9.277	15.944	75.017	0.057	1.641
10	7	MFC	MLE	1.311	12.092	18.606	74.683	0.064	0.332
			CMLE	−3.662	11.745	16.897	79.0988	0.023	1.375
10	7	uniform	MLE	1.404	6.759	18.745	73.693	0.063	0.351
			CMLE	−0.552	8.079	16.218	74.581	−0.051	2.034
10	7	binomial	MLE	1.327	9.171	16.299	74.140	0.076	0.519
		with $p = 0.4$	CMLE	−1.480	9.331	15.182	73.316	0.011	2.205
10	7	binomial	MLE	1.0360	9.257	22.634	80.601	0.080	0.458
		with $p = 0.7$	CMLE	−2.068	13.355	24.017	86.133	−0.047	1.920
10	7	[5]	MLE	1.368	8.950	28.169	82.699	0.059	0.304
			CMLE	−0.502	11.859	21.026	78.398	0.033	2.569
10	7	[6]	MLE	1.609	8.120	24.830	83.691	0.067	0.278
			CMLE	1.791	5.650	21.172	78.794	−0.219	1.590
10	7	[7]	MLE	0.884	13.458	20.239	78.724	0.091	0.552
			CMLE	1.426	9.019	18.410	79.887	−0.131	2.126
10	7	[8]	MLE	1.096	8.761	9.725	70.594	0.107	0.573
			CMLE	1.050	10.453	6.620	69.129	0.194	3.711
20	7	MRC	MLE	0.280	2.357	−20.658	61.890	0.220	1.016
			CMLE	−0.338	2.610	−22.727	63.948	−0.213	0.514
20	7	MFC	MLE	0.387	2.295	−20.737	63.969	0.187	0.791
			CMLE	−0.306	2.528	−17.525	64.104	−0.134	1.163
20	7	uniform	MLE	0.456	1.520	−0.218	66.377	0.097	0.298
			CMLE	0.283	2.055	1.950	68.220	−0.219	1.177
20	7	binomial	MLE	0.451	2.164	3.870	70.118	0.089	0.226
		with $p = 0.4$	CMLE	0.460	1.457	2.833	66.359	−0.300	0.529
20	7	binomial	MLE	0.091	7.225	9.120	69.179	0.101	0.530
		with $p = 0.7$	CMLE	0.500	1.143	11.980	76.264	−0.303	0.501
20	7	[9]	MLE	0.532	1.361	10.039	67.885	0.070	0.151
			CMLE	0.294	4.378	15.753	75.570	−0.234	1.188
20	7	[10]	MLE	0.215	4.630	19.758	79.640	0.072	0.216
			CMLE	0.472	1.122	9.570	69.752	−0.292	0.527
20	7	[11]	MLE	0.405	1,936	−8.581	62.249	0.115	0.398
			CMLE	0.494	1.540	−6.920	67.691	−0.218	1.218
20	7	[12]	MLE	−0.024	4.978	−25.291	65.607	0.278	1.143
			CMLE	0.464	1.739	−27.673	64.100	−0.106	1.115
20	10	MRC	MLE	0.318	0.995	−1.712	61.150	0.108	0.340
			CMLE	−0.001	1.646	−4.997	60.691	−0.237	0.267
20	10	MFC	MLE	0.400	1.696	−0.136	58.897	0.079	0.224
			CMLE	0.047	1.948	−3.150	58.279	−0.273	0.301
20	10	uniform	MLE	0.444	0.991	10.097	61.343	0.057	0.138
			CMLE	0.403	1.330	15.210	66.260	−0.302	0.313
20	10	binomial	MLE	0.508	1.253	10.144	62.314	0.055	0.134
		with $p = 0.4$	CMLE	0.510	1.365	11.235	63.454	−0.306	0.316
20	10	binomial	MLE	0.524	1.192	22.537	73.719	0.062	0.130
		with $p = 0.7$	CMLE	0.500	1.172	16.485	69.138	−0.288	0.304
20	10	[13]	MLE	0.324	5.160	18.722	71.271	0.064	0.168
			CMLE	0.420	1.855	17.542	70.226	−0.294	0.628
20	10	[14]	MLE	0.500	1.292	14.442	67.513	0.053	0.128
			CMLE	0.511	1.408	16.817	69.620	−0.303	0.311
20	10	[15]	MLE	0.492	2.081	8.572	61.425	0.582	0.511
			CMLE	0.444	1.242	9.211	62.320	−0.293	0.560
20	10	[16]	MLE	0.461	1.139	−4.768	61.564	0.093	0.200
			CMLE	0.497	1.187	−7.502	61.022	−0.316	0.323

Table 3: Simulated results under different approaches of generating progressive censoring schemes with $\alpha = 300$, $\theta = 100$ and $\beta = 0.5$

				α		θ		β	
n	m	Removal plan	method	EB	ERMSE	EB	ERMSE	EB	ERMSE
20	15	MRC	MLE	0.383	0.948	13.220	58.584	0.053	0.124
			CMLE	0.092	1.469	13.051	60.538	−0.252	0.274
20	15	MFC	MLE	0.523	1.676	12.912	55.050	0.042	0.116
			CMLE	0.189	1.838	13.535	58.014	−0.281	0.300
20	15	uniform	MLE	0.520	1.248	18.205	60.853	0.040	0.108
			CMLE	0.354	1.563	19.310	62.871	−0.291	0.304
20	15	binomial	MLE	0.470	1.178	16.596	60.582	0.043	0.110
		with $p = 0.4$	CMLE	0.363	1.321	19.072	61.017	−0.299	0.310
20	15	binomial	MLE	0.464	1.136	13.992	56.463	0.043	0.112
		with $p = 0.7$	CMLE	0.553	1.390	20.852	65.597	−0.308	0.317
20	15	[17]	MLE	0.495	1.151	17.624	56.458	0.048	0.117
			CMLE	0.458	1.219	23.301	66.917	−0.306	0.315
20	15	[18]	MLE	0.475	1.209	19.238	61.445	0.040	0.107
			CMLE	0.519	1.346	16.038	59.519	−0.306	0.314
20	15	[19]	MLE	0.494	1.432	10.894	53.631	0.037	0.112
			CMLE	0.518	1,268	18.802	60.363	−0.031	0.313
20	15	[20]	MLE	0.517	1.296	12.848	58.397	0.052	0.137
			CMLE	0.452	1.082	11.253	56.302	−0.310	0.317
40	20	MRC	MLE	0.132	0.356	8.593	46.995	0.047	0.102
			CMLE	0.125	0.295	8.452	46.737	−0.266	0.285
40	20	MFC	MLE	0.138	0.340	8.140	48.464	0.042	0.096
			CMLE	0.133	0.313	7.620	47.300	−0.263	0.282
40	20	uniform	MLE	0.116	0.344	12.353	50.003	0.042	0.093
			CMLE	0.131	0.302	12.520	48.987	−0.259	0.280
40	20	binomial	MLE	0.107	0.240	12.049	50.547	0.040	0.094
		with $p = 0.4$	CMLE	0.132	0.306	11.018	51.049	−0.272	0.290
40	20	binomial	MLE	0.129	0.331	12.153	50.170	0.047	0.101
		with $p = 0.7$	CMLE	0.123	0.308	12.072	50.731	−0.267	0.285
40	20	[21]	MLE	0.124	0.289	11.921	51.002	0.044	0.098
			CMLE	0.125	0.306	11.981	51.413	−0.259	0.278
40	20	[22]	MLE	0.122	0.231	10.945	51.073	0.045	0.096
			CMLE	0.134	0.364	12.476	50.283	−0.266	0.286
40	20	[23]	MLE	0.133	0.313	5.296	46.448	0.044	0.097
			CMLE	0.134	0.320	6.773	46.696	−0.263	0.282
40	20	[24]	MLE	0.129	0.313	−3.195	44.388	0.073	0.130
			CMLE	0.125	0.305	−3.911	45.166	−0.266	0.284
40	35	MRC	MLE	0.133	0.351	11.223	37.617	0.031	0.080
			CMLE	0.115	0.244	11.520	40.639	−0.246	0.267
40	35	MFC	MLE	0.123	0.293	10.605	40.208	0.033	0.078
			CMLE	0.103	0.260	9.754	38.447	−0.238	0.260
40	35	uniform	MLE	0.121	0.288	13.309	40.492	0.032	0.078
			CMLE	0.125	0.326	11.122	40.570	−0.233	0.255
40	35	binomial	MLE	0.120	0.331	8.320	38.602	0.028	0.076
		with $p = 0.4$	CMLE	0.122	0.296	10.560	39.775	−0.241	0.262
40	35	binomial	MLE	0.125	0.300	12.093	40.389	0.029	0.074
		with $p = 0.7$	CMLE	0.115	0.270	8.490	38.420	−0.250	0.271
40	35	[25]	MLE	0.118	0.287	11.865	40.963	0.030	0.074
			CMLE	0.116	0.280	10.107	39.804	−0.253	0.274
40	35	[26]	MLE	0.125	0.337	9.805	39.115	0.029	0.080
			CMLE	0.113	0.279	10.274	40.040	−0.245	0.266
40	35	[27]	MLE	0.130	0.301	13.404	39.998	0.026	0.075
			CMLE	0.120	0.307	10.237	38.300	0.028	0.075
40	35	[28]	MLE	0.125	0.289	8.970	37.217	0.032	0.082
			CMLE	0.133	0.341	9.163	38.987	0.300	0.081

Table 4: Simulated results under different approaches of generating progressive censoring schemes with $\alpha = 300$, $\theta = 100$ and $\beta = 1$

n	m	Removal plan	method	α EB	α ERMSE	θ EB	θ ERMSE	β EB	β ERMSE
10	4	MRC	MLE	0.614	27.098	−11.130	51.849	0.754	3.267
			CMLE	−4.883	18.087	−26.040	57.238	5.056	9.655
10	4	MFC	MLE	0.390	28.042	−9.559	52.765	0.760	4.002
			CMLE	−5.773	17.589	−26.577	55.352	4.750	10.123
10	4	uniform	MLE	0.350	27.960	−9.206	51.322	0.802	3.803
			CMLE	−3.374	18.680	−22.145	54.059	4.391	9.322
10	4	binomial	MLE	2.109	27.151	−5.625	52.221	0.440	2.657
		with $p = 0.4$	CMLE	−5.806	23.814	−23.984	54.144	5.769	11.105
10	4	binomial	MLE	−0.255	33.143	1.745	62.914	0.372	2.611
		with $p = 0.7$	CMLE	−8.428	30.095	−19.400	50.687	6.770	12.323
10	4	[1]	MLE	−3.182	42.420	9.878	74.035	0.275	2.028
			CMLE	2.011	20.214	−24.423	54.065	4.352	10.214
10	4	[2]	MLE	2.076	23.541	−11.198	51.946	0.451	3.077
			CMLE	2.339	19.023	−23.960	52.966	4.277	9.843
10	4	[3]	MLE	1.619	27.538	−6.476	52.347	0.619	3.118
			CMLE	0.918	20.467	−27.011	55.4003	5.572	11.331
10	4	[4]	MLE	0.210	27.503	−12.467	52.569	0.970	4.100
			CMLE	−0.943	20.215	−33.496	60.194	7.672	12.653
10	7	MRC	MLE	4.135	23.023	3.606	46.370	0.013	1.233
			CMLE	−4.553	17.265	−6.486	39.431	1.022	4.687
10	7	MFC	MLE	4.222	25.049	0.974	43.999	0.080	1.388
			CMLE	−4.614	17.366	−6.037	39.537	0.947	4.522
10	7	uniform	MLE	5.899	19.313	−0.037	44.290	−0.044	1.013
			CMLE	1.172	18.470	−8.277	42.035	1.497	7.024
10	7	binomial	MLE	5.040	21.800	1.256	42.503	0.010	1.022
		with $p = 0.4$	CMLE	−2.561	19.488	−6.319	39.976	1.645	6.451
10	7	binomial	MLE	3.617	25.282	3.888	49.444	0.006	1.239
		with $p = 0.7$	CMLE	−4.771	21.661	−4.109	42.081	1.812	6.809
10	7	[5]	MLE	4.470	22.501	4.288	47.925	−0.038	0.865
			CMLE	−2.211	24.797	−7.781	43.049	3.192	8.922
10	7	[6]	MLE	5.956	20.458	−1.071	40.851	−0.036	0.857
			CMLE	4.822	19.275	−10.881	39.541	1.863	8.050
10	7	[7]	MLE	5.139	20.811	0.039	40.413	−0.008	0.970
			CMLE	4.287	19.478	−13.415	40.815	2.193	8.216
10	7	[8]	MLE	5.211	21.097	−0.358	40.103	0.013	1.204
			CMLE	0.912	21.202	−15.292	40.808	4.126	10.639
20	7	MRC	MLE	0.928	15.468	−9.656	41.371	0.211	1.613
			CMLE	−1.249	8.220	−13.860	44.689	0.177	3.203
20	7	MFC	MLE	1.900	12.527	−7.611	42.392	0.202	1.540
			CMLE	−1.860	8.261	−13.800	44.195	0.200	3.199
20	7	uniform	MLE	2.574	11.564	−6.586	38.873	0.069	1.024
			CMLE	2.369	8.501	−8.797	39.914	−0.331	2.750
20	7	binomial	MLE	3.950	33.683	−8.077	37.437	0.171	2.269
		with $p = 0.4$	CMLE	3.424	9.183	−10.410	40.593	0.093	4.164
20	7	binomial	MLE	2.053	16.119	−1.335	41.479	−0.042	1.077
		with $p = 0.7$	CMLE	3.054	10.188	−7.387	41.250	0.225	4.260
20	7	[9]	MLE	1.449	16.167	4.117	46.259	−0.050	0.958
			CMLE	−0.163	16.800	−4.763	42.247	1.284	6.593
20	7	[10]	MLE	2.671	12.110	−4.789	39.415	−0.052	0.786
			CMLE	4.488	7.692	−7.598	40.853	−0.429	3.013
20	7	[11]	MLE	1.001	16.338	−6.946	39.503	0.183	1.434
			CMLE	3.956	8.103	−12.092	41.532	−0.100	3.934
20	7	[12]	MLE	1.390	15.328	−5.329	45.185	0.260	1.903
			CMLE	2.847	8.614	−18.540	50.773	1.124	5.893
20	10	MRC	MLE	2.796	10.226	−3.238	34.200	−0.063	0.738
			CMLE	−1.060	7.642	−5.61	35.190	−0.425	0.757
20	10	MFC	MLE	2.650	10.620	−3.275	34.650	−0.032	0.796
			CMLE	−0.687	7.965	−6.995	32.998	−0.446	0.733
20	10	uniform	MLE	4.132	7.951	−1.113	33.044	−0.116	0.414
			CMLE	3.671	11.007	−1.940	33.927	−0.073	0.554
20	10	binomial	MLE	3.103	12.456	−1.684	33.251	−0.067	0.657
		with $p = 0.4$	CMLE	3.964	8.497	−3.973	31.694	−0.690	1.796
20	10	binomial	MLE	3.610	10.678	−1.451	34.368	−0.105	0.524
		with $p = 0.7$	CMLE	3.884	9.036	−3.317	33.244	−0.402	3.068
20	10	[13]	MLE	3.273	10.845	2.642	37.265	−0.105	0.488
			CMLE	3.903	8.956	−3.118	33.358	−0.391	3.017
20	10	[14]	MLE	3.385	14.203	−2.411	36.980	−0.101	0.485
			CMLE	4.893	7.446	−3.119	32.992	−0.763	1.543
20	10	[15]	MLE	3.149	10.655	−4.227	33.008	−0.038	0.638
			CMLE	4.737	7.631	−5.546	33.211	−0.632	2.215
20	10	[16]	MLE	2.951	10.814	−5.020	34.345	−0.017	0.882
			CMLE	3.703	8.475	−4.595	38.635	0.012	4.060

Table 5: Simulated results under different approaches of generating progressive censoring schemes with $\alpha = 300$, $\theta = 100$ and $\beta = 1$

				α		θ		β	
n	m	Removal plan	method	EB	ERMSE	EB	ERMSE	EB	ERMSE
20	15	MRC	MLE	3.945	7.288	−1.052	27.352	−0.102	0.327
			CMLE	−0.754	7.637	−0.922	26.710	−0.486	0.557
20	15	MFC	MLE	4.039	7.979	−1.025	26.528	−0.088	0.394
			CMLE	−0.307	7.458	−2.169	26.398	−0.512	0.576
20	15	uniform	MLE	4.361	8.234	0.589	27.902	−0.096	0.287
			CMLE	2.006	7.224	−1.506	26.978	−0.647	0.698
20	15	binomial	MLE	4.239	7.665	0.291	27.116	−0.098	0.310
		with $p = 0.4$	CMLE	2.469	8.065	0.074	27.515	−0.678	1.130
20	15	binomial	MLE	4.151	7.415	−0.778	28.484	−0.095	0.310
		with $p = 0.7$	CMLE	4.361	7.736	−1.666	28.480	−0.755	1.414
20	15	[17]	MLE	3.833	8.149	1.251	28.592	−0.082	0.348
			CMLE	4.868	7.226	−1.187	27.504	−0.801	1.209
20	15	[18]	MLE	4.046	7.905	1.082	28.836	−0.093	0.292
			CMLE	4.868	6.934	−2.727	26.448	−0.828	1.112
20	15	[19]	MLE	4.136	7.528	−0.195	26.330	−0.099	0.318
			CMLE	4.837	7.111	−1.523	27.127	−0.080	1.203
20	15	[20]	MLE	3.742	9.030	0.419	27.550	−0.104	0.418
			CMLE	4.685	7.288	−3.584	27.287	−0.751	1.650
40	20	MRC	MLE	2.306	3.535	−2.117	22.520	−0.087	0.206
			CMLE	2.128	3.308	−2.314	22.581	−0.821	0.827
40	20	MFC	MLE	2.347	3.822	−1.524	23.353	−0.078	0.229
			CMLE	2.545	3.730	−1.028	39.907	−0.831	0.835
40	20	uniform	MLE	2.251	3.382	−2.529	22.602	0.079	0.193
			CMLE	2.444	3.770	−1.744	22.141	−0.823	0.828
40	20	binomial	MLE	2.240	3.555	−1.227	23.943	−0.080	0.206
		with $p = 0.4$	CMLE	2.429	3.459	−1.141	24.057	−0.840	0.843
40	20	binomial	MLE	2.377	3.640	−1.719	23.714	−0.095	0.199
		with $p = 0.7$	CMLE	2.514	3.612	−0.408	23.477	−0.842	0.843
40	20	[21]	MLE	2.194	3.463	0.514	24.217	−0.097	0.210
			CMLE	2.631	3.819	1.175	23.618	−0.840	0.842
40	20	[22]	MLE	2.278	4.028	0.139	23.880	−0.082	0.219
			CMLE	2.445	3.531	0.190	24.202	−0.839	0.841
40	20	[23]	MLE	2.253	3.524	−3.893	23.260	−0.075	0.203
			CMLE	2.559	3.733	−2.552	22.986	−0.837	0.839
40	20	[24]	MLE	2.206	3.864	−1.858	25.264	−0.083	0.295
			CMLE	2.451	3.396	−3.048	25.184	−0.838	0.840
40	35	MRC	MLE	2.262	3.435	−0.383	17.549	−0.060	0.160
			CMLE	2.154	3.689	0.070	17.428	−0.800	0.808
40	35	MFC	MLE	2.359	3.510	−0.159	18.094	−0.064	0.152
			CMLE	2.070	3.604	−0.611	17.823	−0.799	0.808
40	35	uniform	MLE	2.402	3.563	−0.775	17.688	−0.064	0.149
			CMLE	1.976	3.710	0.227	18.410	−0.776	0.788
40	35	binomial	MLE	2.344	3.461	−1.476	17.972	−0.065	0.150
		with $p = 0.4$	CMLE	2.251	3.505	0.884	18.032	−0.813	0.819
40	35	binomial	MLE	2.245	3.457	−1.221	17.705	−0.054	0.162
		with $p = 0.7$	CMLE	2.460	3.486	0.773	17.668	−0.834	0.836
40	35	[25]	MLE	2.414	3.627	−0.636	17.619	−0.068	0.157
			CMLE	2.488	3.545	0.890	17.932	−0.841	0.843
40	35	[26]	MLE	2.031	3.222	−0.910	17.835	−0.060	0.159
			CMLE	2.532	3.563	0.012	17.847	−0.837	0.839
40	35	[27]	MLE	2.410	3.595	−0.981	16.868	0.062	0.156
			CMLE	2.293	3.462	0.099	17.095	−0.054	0.158
40	35	[28]	MLE	2.415	3.635	−0.461	17.004	−0.071	0.169
			CMLE	2.146	3.321	−0.362	17.764	−0.074	0.170

Table 6: Simulated results under different approaches of generating progressive censoring schemes with $\alpha = 300$, $\theta = 100$ and $\beta = 2$

				α		θ		β	
n	m	Removal plan	method	EB	ERMSE	EB	ERMSE	EB	ERMSE
10	4	MRC	MLE	10.149	39.785	−13.190	44.634	0.510	3.428
			CMLE	7.344	24.970	−37.225	49.967	6.993	11.615
10	4	MFC	MLE	10.965	40.742	−11.721	44.085	0.375	3.562
			CMLE	8.050	25.286	−38.126	50.514	7.343	12.603
10	4	uniform	MLE	6.480	42.156	−11.839	47.176	0.593	3.480
			CMLE	8.820	26.853	−40.190	50.006	7.535	12.527
10	4	binomial	MLE	8.445	42.348	−10.900	46.264	0.392	3.308
		with $p = 0.4$	CMLE	6.198	27.380	−39.787	48.996	8.477	12.817
10	4	binomial	MLE	8.445	45.423	−12.281	52.258	0.140	3.090
		with $p = 0.7$	CMLE	2.624	29.550	−41.853	49.800	10.646	14.475
10	4	[1]	MLE	3.251	50.113	−7.702	58.963	0.241	2.988
			CMLE	10.171	30.981	−37.539	47.136	11.043	15.519
10	4	[2]	MLE	6.742	41.932	−17.278	49.510	0.549	3.588
			CMLE	11.788	30.134	−47.872	54.075	8.299	13.200
10	4	[3]	MLE	5.721	43.567	−12.092	47.938	0.776	3.789
			CMLE	10.518	29.707	−46.415	53.648	9.738	14.163
10	4	[4]	MLE	11.221	3.906	−10.639	44.0126	0.405	3.506
			CMLE	9.826	27.677	−41.790	56.139	10.335	15.184
10	7	MRC	MLE	9.981	40.700	−10.187	42.650	−0.018	2.239
			CMLE	7.216	25.020	−30.700	38.091	3.573	9.429
10	7	MFC	MLE	7.657	43.463	−9.254	47.651	0.073	2.334
			CMLE	8.544	26.063	−31.300	38.668	2.915	7.963
10	7	uniform	MLE	8.920	41.732	−11.235	46.508	0.066	2.250
			CMLE	10.820	28.532	−39.107	44.374	5.915	12.003
10	7	binomial	MLE	8.158	41.930	−10.138	44.810	0.161	2.371
		with $p = 0.4$	CMLE	7.735	27.049	−32.219	40.089	4.464	10.064
10	7	binomial	MLE	10.318	41.180	−14.217	47.752	−0.060	2.016
		with $p = 0.7$	CMLE	7.147	26.748	−34.407	40.976	5.731	11.315
10	7	[5]	MLE	7.838	42.862	−11.340	49.603	0.040	2.162
			CMLE	5.487	29.553	−41.360	46.373	9.560	14.201
10	7	[6]	MLE	11.354	37.258	−15.921	43.473	−0.055	2.013
			CMLE	11.683	31.438	−43.401	47.814	7.758	13.649
10	7	[7]	MLE	5.435	41.532	−7.416	46.075	0.169	2.125
			CMLE	9.105	29.977	−42.857	47.6322	8.808	14.349
10	7	[8]	MLE	9.226	39.168	−8.089	40.625	0.069	2.349
			CMLE	11.103	31.058	−40.755	46.495	8.260	13.996
20	7	MRC	MLE	5.707	30.299	−3.595	33.086	0.145	2.489
			CMLE	5.429	18.180	−22.017	37.004	0.876	5.597
20	7	MFC	MLE	5.114	31.324	−2.551	33.675	0.234	2.543
			CMLE	6.421	18.548	−22.483	36.192	0.744	5.498
20	7	uniform	MLE	4.558	31.453	−10.423	36.188	0.159	2.119
			CMLE	10.634	22.059	−30.997	38.643	1.906	7.794
20	7	binomial	MLE	4.771	32.917	−7.839	37.753	0.074	2.106
		with $p = 0.4$	CMLE	10.026	23.352	−33.716	40.480	3.470	9.478
20	7	binomial	MLE	4.384	32.289	−10.223	39.631	0.035	2.018
		with $p = 0.7$	CMLE	8.026	23.941	−34.719	40.698	4.524	10.185
20	7	[9]	MLE	2.721	34.992	−7.410	44.673	−0.025	2.026
			CMLE	1.400	25.793	−33.166	39.568	6.565	11.691
20	7	[10]	MLE	8.060	28.949	−15.854	37.776	−0.156	1.642
			CMLE	13.649	23.968	−35.336	40.892	2.150	8.675
20	7	[11]	MLE	2.273	32.848	−5.491	39.243	0.364	2.398
			CMLE	12.083	23.769	−33.0412	41.278	3.022	9.515
20	7	[12]	MLE	5.155	31.687	0.883	35.452	0.264	2.801
			CMLE	11.100	22.370	−28.911	45.878	4.403	10.687
20	10	MRC	MLE	5.636	34.082	−4.730	34.542	0.040	2.141
			CMLE	6.730	18.029	−20.942	27.751	−0.563	3.121
20	10	MFC	MLE	5.697	29.154	−5.833	30.488	0.018	1.822
			CMLE	6.933	17.671	−20.711	28.691	−0.500	3.213
20	10	uniform	MLE	7.332	26.121	−11.868	33.458	−0.127	1.333
			CMLE	8.097	20.310	−26.490	32.522	0.811	7.029
20	10	binomial	MLE	4.693	29.752	−9.388	35.393	0.018	1.503
		with $p = 0.4$	CMLE	11.422	23.433	−30.660	36.246	1.951	8.149
20	10	binomial	MLE	4.693	29.752	−9.389	35.394	0.018	1.503
		with $p = 0.7$	CMLE	10.227	24.1864	−32.308	37.384	2.963	9.374
20	10	[13]	MLE	4.048	29.788	−8.560	37.929	−0.041	1.481
			CMLE	8.961	25.131	−34.367	38.517	3.903	10.800
20	10	[14]	MLE	4.577	30.859	−9.959	37.950	0.018	1.512
			CMLE	14.194	23.545	−31.798	37.126	1.495	7.954
20	10	[15]	MLE	3.631	30.896	−7.501	74.086	0.222	1.867
			CMLE	14.192	23.150	−31.837	37.630	1.602	8.003
20	10	[16]	MLE	5.784	30.400	−2.267	30.298	0.048	2.150
			CMLE	11.859	23.455	−26.986	37.089	2.841	9.527

Table 7: Simulated results under different approaches of generating progressive censoring schemes with $\alpha = 300$, $\theta = 100$ and $\beta = 2$

				α		θ		β	
n	m	Removal plan	method	EB	ERMSE	EB	ERMSE	EB	ERMSE
20	15	MRC	MLE	6.197	28.487	−8.669	31.584	−0.050	1.328
			CMLE	7.587	17.636	−21.140	25.833	−1.081	1.555
20	15	MFC	MLE	6.080	25.900	−8.714	32.832	−0.017	1.234
			CMLE	7.504	17.030	−20.657	26.045	−0.983	1.600
20	15	uniform	MLE	5.116	26.728	−9.043	32.632	−0.020	1.157
			CMLE	9.286	18.911	−24.047	28.949	−0.361	4.376
20	15	binomial	MLE	5.072	25.764	−8.805	31.678	0.061	1.165
		with $p = 0.4$	CMLE	9.536	20.315	−24.362	29.450	0.041	5.195
20	15	binomial	MLE	6.085	26.307	−9.845	32.937	−0.087	1.124
		with $p = 0.7$	CMLE	13.077	22.475	−28.609	33.180	0.771	6.957
20	15	[17]	MLE	5.133	28.401	−7.654	34.785	0.013	1.253
			CMLE	12.166	23.259	−31.368	35.503	1.930	8.098
20	15	[18]	MLE	6.233	26.675	−9.220	32.189	−0.027	1.167
			CMLE	15.469	24.073	−31.128	35.388	0.853	7.171
20	15	[19]	MLE	3.237	30.660	-5.576	35.580	0.100	1.347
			CMLE	15.037	24.027	−29.043	33.517	1.056	7.583
20	15	[20]	MLE	5.386	28.453	−6.310	31.020	−0.050	1.394
			CMLE	14.887	23.830	−28.758	33.813	1.186	7.597
40	20	MRC	MLE	3.604	21.827	−6.352	25.482	−0.007	1.045
			CMLE	10.335	14.898	−17.185	22.042	−1.473	2.533
40	20	MFC	MLE	3.561	19.193	−6.457	24.361	−0.019	0.890
			CMLE	12.005	15.517	−17.238	22.213	−1.579	2.577
40	20	uniform	MLE	4.274	19.303	−7.815	25.970	−0.060	0.896
			CMLE	11.687	15.327	−18.309	22.919	−1.551	2.447
40	20	binomial	MLE	4.258	18.320	−8.230	25.019	−0.064	0.856
		with $p = 0.4$	CMLE	13.545	16.200	−18.917	23.059	−1.578	2.755
40	20	binomial	MLE	4.229	17.456	−7.421	23.948	−0.055	0.857
		with $p = 0.7$	CMLE	12.798	16.466	−20.124	24.670	−1.174	4.014
40	20	[21]	MLE	4.118	17.630	−7.878	25.213	−0.069	0.831
			CMLE	12.167	16.749	−20.875	25.334	−0.936	4.280
40	20	[22]	MLE	4.084	21.109	−7.353	27.483	−0.054	0.943
			CMLE	13.564	16.266	−18.532	22.820	−1.606	2.750
40	20	[23]	MLE	4.172	17.133	−6.721	21.499	−0.007	0.809
			CMLE	13.634	16.284	−0.18.380	23.139	−1.529	2.822
40	20	[24]	MLE	3.837	20.480	−2.310	20.408	−0.011	1.341
			CMLE	12.966	16.379	−15.062	23.446	−1.147	3.745
40	35	MRC	MLE	4.848	15.034	−6.795	19.978	−0.070	0.586
			CMLE	9.832	14.164	−15.831	19.198	−1.581	1.696
40	35	MFC	MLE	5.395	15.909	−7.364	20.346	−0.095	0.576
			CMLE	8.969	13.720	−15.590	19.094	−1.491	1.798
40	35	uniform	MLE	4.818	16.086	−6.527	21.310	−0.056	0.582
			CMLE	7.617	13.114	−15.420	19.116	−1.419	1.709
40	35	binomial	MLE	5.280	13.749	−6.670	18.520	−0.070	0.515
		with $p = 0.4$	CMLE	9.670	14.533	−15.821	19.303	−1.559	1.759
40	35	binomial	MLE	4.924	14.767	−6.899	19.598	−0.058	0.550
		with $p = 0.7$	CMLE	13.014	15.839	−17.626	20.988	−1.681	2.161
40	35	[25]	MLE	4.427	14.176	−5.964	19.527	−0.056	0.550
			CMLE	13.449	15.855	−17.631	21.114	−1.639	2.487
40	35	[26]	MLE	4.770	16.075	−6.925	20.900	−0.075	0.614
			CMLE	13.914	15.757	−16.673	19.505	−1.799	2.013
40	35	[27]	MLE	4.683	13.921	−6.226	18.403	−0.058	0.518
			CMLE	4.324	14.585	−5.540	18.658	−0.033	0.567
40	35	[28]	MLE	4.703	14.861	−6.278	18.194	−0.066	0.624
			CMLE	5.458	15.860	−6.943	19.743	−0.094	0.639

Table 8: Expected experiment time under different approaches of generated progressive Type-II censoring schemes

			$\mathbf{E(X_{m:m:n})}$		
n	m	Removal plan	$\beta = 0.5$	$\beta = 1$	$\beta = 2$
10	4	MRC	356.080	363.151	374.706
		MFC	355.299	364.734	376.180
		Uniform	497.303	408.674	397.905
		Binomial(p=0.4)	463.269	406.384	397.197
		Binomial(p=0.7)	669.709	466.734	424.418
		[1]	781.656	495.468	433.785
		[2]	707.890	465.146	421.672
		[3]	587.044	434.410	411.821
		[4]	329.248	348.047	366.399
10	7	MRC	651.240	475.283	426.586
		MFC	753.095	487.354	434.570
		Uniform	955.172	538.548	448.415
		Binomial(p=0.4)	878.416	518.352	442.960
		Binomial(p=0.7)	1035.433	545.427	452.001
		[5]	1103.180	552.993	452.734
		[6]	1044.580	546.752	452.946
		[7]	1041.490	527.404	447.689
		[8]	436.278	409.362	402.570
20	7	MRC	334.671	353.006	370.763
		MFC	345.933	357.684	373.826
		Uniform	827.640	493.810	433.293
		Binomial(p=0.4)	701.849	426.943	425.493
		Binomial(p=0.7)	982.950	532.837	450.312
		[9]	1016.348	546.497	453.291
		[10]	1062.248	535.338	449.885
		[11]	767.510	489.198	433.989
		[12]	320.214	342.583	363.809
20	10	MRC	410.464	394.852	394.812
		MFC	482.854	411.247	403.464
		Uniform	1127.344	573.670	461.949
		Binomial(p=0.4)	1110.874	554.830	456.548
		Binomial(p=0.7)	1237.709	581.485	456.548
		[13]	1209.657	588.746	464.840
		[14]	1232.962	574.884	463.405
		[15]	1018.197	541.502	452.786
		[16]	350.317	365.866	380.963
20	15	MRC	775.406	491.260	430.781
		MFC	1086.530	554.101	455.605
		Uniform	1549.909	630.137	475.711
		Binomial(p=0.4)	1451.309	610.993	473.502
		Binomial(p=0.7)	1516.957	629.082	479.125
		[17]	1518.176	630.795	478.706
		[18]	1514.279	631.487	477.461
		[19]	1413.236	614.028	472.247
		[20]	490.251	432.480	413.435
40	20	MRC	1064.499	543.692	453.425
		MFC	1374.991	597.729	471.898
		Uniform	1659.950	649.519	484.944
		Binomial(p=0.4)	1648.895	647.705	484.460
		Binomial(p=0.7)	1698.632	657.280	486.996
		[21]	1720.893	1706.614	485.601
		[22]	1713.309	674.461	487.499
		[23]	1460.633	614.642	473.990
		[24]	349.124	368.171	381.533
40	35	MRC	1714.758	659.135	487.209
		MFC	1763.142	676.524	492.604
		Uniform	2155.902	718.283	500.882
		Binomial(p=0.4)	1992.756	697.873	498.134
		Binomial(p=0.7)	2195.691	710.061	501.578
		[25]	2160.797	718.529	502.505
		[26]	2168.771	712.161	501.898
		[27]	2135.444	697.238	497.583
		[28]	710.161	499.590	440.574

Table 9: Progressive Type-II censored data obtained from the breakdown data by using the presented four approaches for determining the observed r's

Ball bearing data	17.88	28.92	33.00	41.52	42.12	45.60	48.40	51.84	51.96	54.12	55.56
	67.80	68.64	68.88	84.12	93.12	98.64	105.12	105.84	127.92	128.04	173.40
binomial p=0.4	17.88	28.92	33.00	41.52	45.60	48.40	54.12	68.88	84.12	105.12	
r	3	1	4	4	0	0	0	0	0	0	
binomial p=0.7	17.88	45.60	54.12	55.56	67.80	68.64	105.84	127.92	128.04	173.40	
r	7	3	2	0	0	0	0	0	0	0	
Uniform	17.88	28.92	33.00	41.52	42.12	45.60	48.40	67.80	84.12	105.84	
r	2	6	3	1	0	0	0	0	0	0	
MRC	17.88	28.92	33.00	41.52	42.12	45.60	48.40	51.84	51.96	67.80	
r	1	0	1	0	0	1	0	0	1	8	
MFC	17.88	33.00	41.52	42.12	45.60	48.40	54.12	55.56	67.80	68.64	
r	1	1	1	1	0	0	2	1	0	5	

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A Appendix

In this paper, we presented the corrected maximum likelihood procedure for estimating the parameters. At first, in order to simplify of derivative calculations, let us consider the following definitions:

$$c_1=\left(\frac{x_{1:m:n}-\alpha}{\theta}\right),\ \ c_2=\left(\frac{x_{1:m:n}-\alpha}{\theta}\right)^{\beta},\ \ c_3=\left(\frac{x_{2:m:n}-\alpha}{\theta}\right),\ \ c_4=\left(\frac{x_{2:m:n}-\alpha}{\theta}\right)^{\beta},$$
$$c_5=\exp\left[-(\frac{x_{1:m:n}-\alpha}{\theta})^{\beta}\right]-\exp\left[-(\frac{x_{2:m:n}-\alpha}{\theta})^{\beta}\right],$$
$$c_{5.1}=\exp\left(-(\frac{x_{1:m:n}-\alpha}{\theta})^{\beta}\right),\ \ c_{5.2}=\exp\left(-(\frac{x_{2:m:n}-\alpha}{\theta})^{\beta}\right),\quad c_5=c_{5.1}-c_{5.2},$$
$$\ell_{-1}=\log\left(\prod_{i=2}^{m}f(x_{i:m:n};\alpha,\theta,\beta)(1-F(x_{i:m:n};\alpha,\theta,\beta))^{r_i}\right)=(m-1)\log(\beta)-\beta(m-1)\log(\theta)$$
$$+(\beta-1)\sum_{i=2}^{m}\log\left(x_{i:m:n}-\alpha\right)-\sum_{i=2}^{m}(r_i+1)\left(\frac{x_{i:m:n}-\alpha}{\theta}\right)^{\beta}.$$

The first-order partial derivatives of $c_1, c_2, c_3, c_{5.1}, c_{5.2}$ and c_4 w.r.t. the model parameters can be written as

$$\frac{\partial c_1}{\partial\alpha}=\frac{-1}{\theta},\quad \frac{\partial c_1}{\partial\theta}=\frac{-c_1}{\theta},\quad \frac{\partial c_1}{\partial\beta}=0.$$

$$\frac{\partial c_2}{\partial\alpha}=-\frac{\beta}{\theta}c_1^{\beta-1},\quad \frac{\partial c_2}{\partial\theta}=-\frac{\beta}{\theta}c_2,\quad \frac{\partial c_2}{\partial\beta}=c_2\log(c_1),$$

$$\frac{\partial c_3}{\partial\alpha}=\frac{-1}{\theta},\quad \frac{\partial c_3}{\partial\theta}=\frac{-c_3}{\theta},\quad \frac{\partial c_3}{\partial\beta}=0,$$

$$\frac{\partial c_4}{\partial\alpha}=-\frac{\beta}{\theta}c_3^{\beta-1},\quad \frac{\partial c_4}{\partial\theta}=-\frac{\beta}{\theta}c_4,\quad \frac{\partial c_4}{\partial\beta}=c_4\log(c_3),$$

$$\frac{\partial c_{5.1}}{\partial\alpha}=\frac{\beta}{\theta}c_1^{\beta-1}c_{5.1},\quad \frac{\partial c_{5.2}}{\partial\alpha}=\frac{\beta}{\theta}c_3^{\beta-1}c_{5.2},$$

$$\frac{\partial c_{5.1}}{\partial\beta}=-c_2c_{5.1}\log(c_1),\quad \frac{\partial c_{5.2}}{\partial\beta}=-c_4c_{5.2}\log(c_3),$$

$$\frac{\partial c_{5.1}}{\partial\theta}=\frac{\beta}{\theta}c_2c_{5.1},\quad \frac{\partial c_{5.2}}{\partial\theta}=\frac{\beta}{\theta}c_4c_{5.2}.$$

Moreover, the second-order partial derivatives of $c_{5.1}$, $c_{5.2}$, c_2 and ℓ_{-1} w.r.t. the model parameters can be obtained, respectively, by the following expressions:

$$\frac{\partial^2 c_{5.1}}{\partial\alpha^2} = \frac{\beta}{\theta^2} c_1^{\beta-2} c_{5.1}\Big[-(\beta-1)+\beta c_1^{\beta}\Big], \quad \frac{\partial^2 c_{5.2}}{\partial\alpha^2} = \frac{\beta}{\theta^2} c_3^{\beta-2} c_{5.2}\Big[-(\beta-1)+\beta c_3^{\beta}\Big],$$

$$\frac{\partial^2 c_{5.1}}{\partial\theta^2} = \frac{\beta}{\theta^2} c_2 c_{5.1}\Big[-1+\beta(c_2-1)\Big], \quad \frac{\partial^2 c_{5.2}}{\partial\theta^2} = \frac{\beta}{\theta^2} c_4 c_{5.2}\Big[-1+\beta(c_4-1)\Big],$$

$$\frac{\partial^2 c_{5.1}}{\partial\beta^2} = c_2 c_{5.1}[\log(c_1)]^2\Big[c_2-1\Big], \quad \frac{\partial^2 c_{5.2}}{\partial\beta^2} = c_4 c_{5.2}[\log(c_3)]^2\Big[c_4-1\Big],$$

$$\frac{\partial^2 c_{5.1}}{\partial\beta\partial\alpha} = \frac{\beta}{\theta} c_1^{\beta-1} c_{5.1}\log(c_1)\Big[1-c_2\Big] + \frac{1}{c_1\theta} c_2 c_{5.1}, \quad \frac{\partial^2 c_{5.2}}{\partial\beta\partial\alpha} = \frac{\beta}{\theta} c_3^{\beta-1} c_{5.2}\log(c_3)\Big[1-c_4\Big] + \frac{1}{c_3\theta} c_4 c_{5.2},$$

$$\frac{\partial^2 c_{5.1}}{\partial\beta\partial\theta} = \frac{1}{\theta} c_2 c_{5.1}\Big[1+\beta\log(c_1)\Big(1-c_2\Big)\Big], \quad \frac{\partial^2 c_{5.2}}{\partial\beta\partial\theta} = \frac{1}{\theta} c_4 c_{5.2}\Big[1+\beta\log(c_3)\Big(1-c_4\Big)\Big],$$

$$\frac{\partial^2 c_2}{\partial\alpha^2} = \frac{\beta(\beta-1)}{\theta^2} c_1^{\beta-2}, \quad \frac{\partial^2 c_2}{\partial\theta^2} = \frac{\beta c_2}{\theta^2}\Big[1+\beta\Big], \quad \frac{\partial^2 c_2}{\partial\beta^2} = c_2\,[\log(c_1)]^2,$$

$$\frac{\partial^2 c_2}{\partial\alpha\partial\beta} = -\frac{\beta}{\theta} c_1^{\beta-1}\log(c_1) - \frac{c_2}{c_1\theta}, \quad \frac{\partial^2 c_2}{\partial\alpha\partial\theta} = \frac{\beta^2}{\theta^2} c_1^{\beta-1}, \quad \frac{\partial^2 c_2}{\partial\beta\partial\theta} = \frac{-c_2}{\theta}\Big[\beta\log(c_1)+1\Big],$$

and

$$\frac{\partial^2\ell_{-1}}{\partial\alpha^2} = -(\beta-1)\sum_{i=2}^{m}\frac{1}{(x_{i:m:n}-\alpha)^2} - \frac{\beta(\beta-1)}{\theta^2}\sum_{i=2}^{m}(r_i+1)\left(\frac{x_{i:m:n}-\alpha}{\theta}\right)^{\beta-2},$$

$$\frac{\partial^2\ell_{-1}}{\partial\theta^2} = \frac{(m-1)\beta}{\theta^2} - \frac{\beta(\beta+1)}{\theta^2}\sum_{i=2}^{m}(r_i+1)\left(\frac{x_{i:m:n}-\alpha}{\theta}\right)^{\beta},$$

$$\frac{\partial^2\ell_{-1}}{\partial\beta^2} = \frac{-m+1}{\beta^2} - \sum_{i=2}^{m}(r_i+1)\left(\frac{x_{i:m:n}-\alpha}{\theta}\right)^{\beta}\left(\log(\frac{x_{i:m:n}-\alpha}{\theta})\right)^2,$$

$$\frac{\partial^2\ell_{-1}}{\partial\alpha\partial\theta} = \frac{\partial^2\ell_{-1}}{\partial\theta\partial\alpha} = -\frac{\beta^2}{\theta^2}\sum_{i=2}^{m}(r_i+1)\left(\frac{x_{i:m:n}-\alpha}{\theta}\right)^{\beta-1},$$

$$\frac{\partial^2\ell_{-1}}{\partial\alpha\partial\beta} = \sum_{i=2}^{m}\frac{-1}{x_{i:m:n}-\alpha} + \sum_{i=2}^{m}\frac{(r_i+1)}{\theta}\left(\frac{x_{i:m:n}-\alpha}{\theta}\right)^{\beta-1}\left[1+\beta\log(\frac{x_{i:m:n}-\alpha}{\theta})\right],$$

$$\frac{\partial^2\ell_{-1}}{\partial\theta\partial\beta} = \frac{-m+1}{\theta} + \sum_{i=2}^{m}\frac{(r_i+1)}{\theta}\left(\frac{x_{i:m:n}-\alpha}{\theta}\right)^{\beta}\left[1+\beta\log(\frac{x_{i:m:n}-\alpha}{\theta})\right].$$