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Signature-based Reliability and Maintenance Planning for a Load-sharing Coherent System Operating in a Random Environment

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Abstract. This paper, using the signature technique and a generalized Farlie-Gumbel-Morgenstern (FGM) copula function, presents a generic mean residual lifetime (MRL) model for the reliability analysis of a load-sharing coherent system. The present approach differs from earlier models in that in addition to load-sharing phenomenon it simultaneously considers the effect of operating conditions on the system. Further, using the developed model and the renewal-reward argument, an age replacement policy is investigated. The proposed MRL model and the behavior of the optimal solution as the model parameters change are illustrated through numerical examples.

Keywords. Coherent System, Degradation Model, Average Cost Rate, Maintenance, Aging, Load-sharing, Signature.

MSC: 90B25, 60K10.

1 Introduction

In the reliability literature, the age-dependent MRL model is presented as a key tool for failure prediction and defined as the expected time elapsed from some fixed time t until the next failure. Although it is computationally tractable, its formulation does allow either (i) incorporating some important aspect of degradation, or (ii) reliability

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modeling of systems whose degradation is influenced by load-sharing phenomenon. When considering systems functioning in a random environment, it is typically assumed that their aging process is independent of the operating environment. This flaw leads to overestimating reliability, tardiness of maintenance and so increasing resulting costs. Age-based modeling is a general approach (Huynh et al. (2012); Cha et al. (2022); Eryilmaz and Pekalp (2020)) that is extensively discussed in the literature due to its ease of use. Despite covering a wide range of systems, a drawback of such an approach is its limited versatility in effectively adapting to environmental changes.

During past years, authors have attempted to fix this shortcoming through incorporating a supplementary variable reflecting the effect of the operating environment. For example, Jardine et al. (2006) considered the age factor and operating environment simultaneously to present a method based on MRL, and also Huynh et al. (2012) use their method for the reliability analysis of a system subject to competing failure modes. In addition, Patra and Kundu (2021) focused on stochastic comparisons and analyzing the aging features within residual lifetime mixture models. More recentlly, Ahmadi (2022) proposes a generic MRL model through the use of an n-variate Farlie-Gumbel-Morgenstern (FGM) copula function within a bivariate state process framework for analyzing multi-component systems.

As systems consist of multi-component working in a common environment, it is not realistic to assume that components work independently and their intrinsic failure mechanisms have no interdependence. In practical situations, however, the model should incorporate stochastic dependencies among the systems components. According to Navara Navarro and Rychlik (2007), if the components are positively exchangeable Eryilmaz (2011); Eryilmaz et al. (2016) and dependent Ansell and Walls (1991); Ye et al. (2014); Shen et al. (2021), the lifetimes of the components can be modeled using exchangeable examples that are presumed to be dependent because the failure of any component leads to the failure of other components. Thus, this problem is typically addressed with two tools: copula models Ahmadi (2020); Fang et al. (2020); Li et al. (2016) or joint probability distribution Rykov et al. (2022) of correlated variables. For instance, Li et al. (2016) studied copula-based reliability modeling for the dependency of components (identical and nonidentical) in a multi-state k-out-of-n system. More recently, Fang et al. (2020) use a copula-based reliability analysis of degrading systems with dependent failures. The works cited above shows that although taking into account the lifetime of independent components makes calculations simpler, there are circumstances where the components are dependent on each other. Moreover, Ahmadi (2020) suggests a decision model based on MRL for maintaining a load-sharing k-out-of-n system.

Apart from the degradation phenomenon, the system configuration is an important feature in reliability modeling. In the absence of load-sharing phenomenon or operating conditions, this problem is well addressed in literature through the signature technique. For example, Da et al. (2018) proposed an efficient algorithm for computing the signatures of a system with exchangeable components, and also Suárez Llorens et al.

(2022) has studied the critical advantages of using signatures by taking into account the cost and other relevant criteria. Bhattacharya and Samaniego (2010), Navarro et al. (2012), and Zhang et al. (2015) all investigated statistical inferences of the component lifetime distribution based on the signature utilizing system failure data. Moreover, the signature is a useful tool for estimating a complicated network's dependability based on Monte Carlo simulations; for example, see Gertsbakh and Shpungin (2012) and Ross (2008). The signature has also a strong connection to other well-known reliability engineering techniques. Navarro and Rychlik (2007) generalize the concept of signature used in systems that incorporate interchangeable components. In addition, the scenario of the joint signature of a coherent system was analyzed by Navarro et al. (2010). Marichal and Mathonet (2011) discussed in the event that the dependent components in question are subject to arbitrary definitions of system signature. Additionally, most of the improvements in reliability metrics such as MRL Asadi and Bayramoglu (2006); Sadegh (2008) and hazard rate function of the coherent systems are defined based on the signature Samaniego (2007); Li et al. (2020); Burkschat and Navarro (2013). More recently, Asadi et al. (2023) review the age-based maintenance based on the signature of multi-component coherent systems. Consequently, both the signature and survival signature of the coherent system are two significant tools for calculating the reliability of different systems.

This paper shares some features with earlier models cited above, but it contains in a unifying model some features that have not been addressed or previously studied in isolation. More specifically, the similarities and differences between our model and the models cited above are as follows:

1. With the same approach as Ahmadi and Amirhossein (2022) and Asadi et al. (2023) , we study a coherent system consisting of multi-components. However our model differs from both models in that it considers the load-sharing phenomenon of components by the use of the FGM copula function. Further, in contrast to the Asadi's model, our model allows the consideration of a damage process reflecting the effect of operating environment.

2. Ahmadi (2022, 2020) consider the effect of the operating environment as an important factor of deterioration process. In that sense our model is similar to that of Ahmadi (2022, 2020), but herein we develop the previous models by considering a coherent system with dependent components.

3. Although, Ahmadi (2022), Asadi and Bayramoglu (2006)], incorporate the history of failed components, they fails to account for the load-sharing phenomenon.

The rest of this paper is organized as follows: The second section present assumptions and models the failure mechanism through considering the joint effect of the age factor and the operating environment. The section is developed by formulating a generic reliability index (MRL) at both component and system level. The latter is implemented through using the FGM copula function and the signature technique. In section 3 a maintenance model and numerical example are given. Section 4 concludes the paper with some final remarks.

2 Degradation Model

2.1 Assumptions

we generalize the previous models by considering the following assumptions:

1. The system studied is coherent and its components lifetimes are positively dependent, exchangeable, and modeled through a generic FGM copula function Ahmadi (2020).

2. The effect of the operating environment on components is reflected by a two-state non-homogeneous Markov (damage) process Ahmadi (2022).

3. In the absence of environmental factors, the lifetimes of the components follow a Weibull distribution with the shape and scale parameters α, β.

2.2 MRL Model

Before proceeding with the development of the MRL model, we model the failure mechanism of a component via the link between its failure rate and the joint effect of the age factor and operating environment, as follows:

$$r(t, Y(t)) = r_0(t)1(Y(t) = 0) + r_1(t)1(Y(t) = 1), \tag{2.1}$$

where $(Y(t) : t \geq 0)$ is a non-homogenous Markov (NHM) damage process describing the state of the operating environment at time t taking its values in Ω={0, 1}(normal state(0) and severe state (1)), $1(\cdot)$ is an indicator function, and $r_i(t) = f_i(t)/\bar{F}_i(t)$ $(i = 0, 1)$ is a failure rate of components associated with damage state i with corresponding failure density function $f_i(t)$ and reliability function $\bar{F}_i(t)$:

$$r_0(t) \leq r_1(t). \tag{2.2}$$

The assumption (2.2) implies that as the damage process $Y(t)$ shifts to the higher state, the system becomes more susceptible to failure. One can note that in terms of the disruption time τ_d at which the damage process shifts to the higher state. In other words,

$$\tau_d = \inf\{t > 0 : Y(t) = 1\}, \tag{2.3}$$

the state-dependent failure rate (2.1) can be expressed as:

$$r(t, Y(t)) = r_0(t)1(\tau_d > t) + r_1(t)1(\tau_d < t)). \tag{2.4}$$

In the following we present an auxiliary Lemma contributing to the MRL model development.

Lemma 2.1. *Let $R_i(w,t)$ ($i \in \Omega$) denote the conditional survival function of a component at age w given that the operating state at age t ($w > t$) is observed in state $Y(t) = i$, that is,*

$$R_i(w,t) = P(T > w|Y(t) = i). \tag{2.5}$$

Then, $R_i(w,t)$ for $i = 0, 1$ is respectively given by:

$$R_0(w,t) = \bar{F}_0(w)\frac{\bar{F}_{\tau_d}(w)}{\bar{F}_{\tau_d}(t)} + \frac{\bar{F}_1(w)}{\bar{F}_{\tau_d}(t)}\int_t^w \frac{\bar{F}_0(u)}{\bar{F}_1(u)} f_{\tau_d}(u)du, \tag{2.6}$$

and

$$R_1(w,t) = \frac{\bar{F}_1(w)}{F_{\tau_d}(t)}\int_0^t \frac{\bar{F}_0(u)}{\bar{F}_1(u)} f_{\tau_d}(u)du, \tag{2.7}$$

where $f_{\tau_d}(t)$ and $F_{\tau_d}(t)$ respectively indicate the density function and the cumulative distribution function of the disruption time τ_d.

The proof is given by Ahmadi (2022).

Corollary 2.1. *In the absence of operating environment factor, the state-dependent survival functions (2.6) and (2.7) reduce to $R(w) = R_i(w)(i = 0, 1)$. It follows from the fact that $\bar{F}_{\tau_d}(u) = 1$, $\bar{F}_0(u) = \bar{F}_1(u)$ and $\bar{F}_{\tau_d}(t) = \bar{F}_{\tau_d}(w)$.*

2.2.1 MRL at Component Level

For a component whose performance is subject to the age factor and operating environment, a quantity of interest is to measure the mean remaining time to failure. Let $m_i(t)$ denote the mean residual lifetime of a component given the operating state $Y(t) = i$ ($i \in \Omega$). In terms of the conditional survival functions (2.6) and (2.7), $m_i(t)$ can be expressed as:

$$m_i(t) = E(T - t|T > t, Y(t) = i) = \int_t^\infty \frac{R_i(w,t)}{R_i(t,t)}dw, \tag{2.8}$$

where $R_i(\cdot)$ is the conditional survival function that is defined in Lemma 2.1. It is observed that in the absence of the operating environment factor, (2.8) turns to

$$m(t) = \int_t^\infty \frac{R(w)}{R(t)}dw, \tag{2.9}$$

referring to the ordinary MRL.

Examples 2.1. To illustrate an evolution of MRL functions at the component level, let the disruption time τ_d conform to the Weibull distribution with parameters (α, β_{τ_d}) = (2, 1) and the failure rate associated with the state i ($i \in \Omega$) be

$$r_i(t) = \frac{\alpha}{\beta_i}(\frac{t}{\beta_i})^{(\alpha-1)},$$

with known parameters (β_0, β_1)=(1.5, 1).

Figure 1 indicates that as the operating state shifts to the higher state at the disruption time ($\hat{\tau}_d$= 0.88), the likelihood of failure of the component increases.

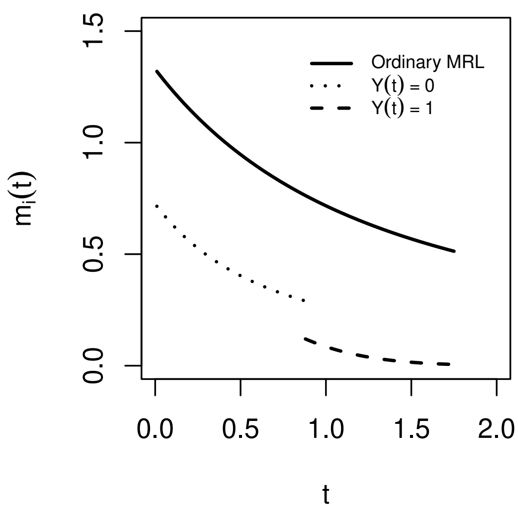

Figure 1: Ordinary MRL and MRL given operating state i ($i = 0, 1$)

2.2.2 MRL at System Level

In this section we aim at formulating the MRL at system level. As stated in the introductory section, the main contribution of this paper is to model an MRL at coherent system level.
This problem is addressed with two tools; an n-variate FGM copula function and signature technique. Before proceeding to the model development, we present some preliminary definitions and auxiliary propositions facilitating the formulation of a generic MRL model at a load-sharing coherent system level.

Definition 2.1. Consider the vector $T = (T_1, T_2, \ldots, T_n)$ representing the lifetimes of components in an n-component system. The dependency structure of component is modeled through using an n-variate FGM copula function, $C(u_1, u_2, \ldots, u_n; \theta)$: see Ahmadi (2020).

$$\mathrm{C}\left(u_1, u_2, \ldots, u_\mathrm{n}; \theta\right) = \prod_{j=1}^{n} u_j \times (1 + \theta \sum_{1 \le j < k \le n} \left(1 - u_j\right)(1 - u_k) + \tag{2.10}$$
$$\theta \sum_{1 \le j < k < l \le n} \left(1 - u_j\right)(1 - u_k)(1 - u_l) + \ldots + \theta\,(1 - u_1)(1 - u_2)\ldots(1 - u_n)).$$

Proposition 2.1. *Assume that $\bar{F}_{i:n}(t; \theta)$ denotes the survival function of the i-order statistics*

$T_{i:n}$ of lifetimes $T_1, T_2, \cdots, T_n$ given the dependence factor $\theta \in [0,1]$, then

$$\bar{F}_{i:n}(t;\theta) = \sum_{r=0}^{i-1} \binom{n}{r} F^r(t) \bar{F}^{(n-r)}(t) \left(1 + \theta \psi_n(r;t)\right), \tag{2.11}$$

where

$$\psi_n(r;t) = \sum_{v=2}^{n} \sum_{A(v)} \binom{n-r}{x_1}\binom{r}{x_2}(-1)^{x_1} F^{x_1}(t) \bar{F}^{x_2}(t),$$

and $A(v)$ is the set of non-negative integer solutions to the equation $x_1 + x_2 = v$:

$$A(v) = \{(x_1, x_2) : x_1 + x_2 = v;\ x_1,\ x_2 \geq 0\}.$$

For proof see Ahmadi (2020).

Proposition 2.2. *Consider a coherent system of the n-order statistics, in which T denotes the system's reliability lifetime and T_i represents i.i.d. random variables with a continuous distribution function F, and $T_{i:n}$ shows the i-th order lifetime. Then, the probability vector $\mathbf{s} = (s_1, s_2, ..., s_n)$ is said to be the signature vector of the system if $s_i = P(T = T_{i:n}),\ i = 1, 2, \ldots, n$. Therefore, the survival function of the system's lifetime T can be described by the following equation:*

$$P(T > t) = \sum_{i=1}^{n} s_i \cdot P(T_{i:n} > t).$$

For proof see Samaniego (2007).

In the following theorem, we present a formula facilitating the calculation of signature vectors.

Theorem 2.1. *Let the random variable vector $T = (T_1, T_2, \cdots, T_n)$ be exchangeable, which means the joint probability of its components does not change under any permutation π, ie., $P(T_1 \leq t_1, \cdots, T_n \leq t_n)$= $P(T_{\pi(1)} \leq t_1, \cdots, T_{\pi(n)} \leq t_n)$, implying that the components have identical distributions but are not independent. Therefore, the survival signature of a coherent system with exchangeable components can be reformulated using minimal path sets $(P_1, P_2, \cdots, P_r)$ as follows:*

$$\bar{S}_i = \sum_{V \subseteq (1,\ldots,r), |V| \neq 0} (-1)^{|V|+1} \frac{\binom{n-i}{m_v}}{\binom{n}{m_v}},$$

where, V be a subset of $1, ..., r$ including the empty subset, and $m_v =\mid U_{i \in V} P_i \mid$.
This allows the computation of the system signature vector $\mathbf{s}$ with respect to the elements $s_i = \bar{S}_{i-1} - \bar{S}_i$ for $i = 1, ..., n$, and $\bar{S}_0 = 1$.

For proof see Da et al. (2018).

Remark 1. According to the Navarro et al. (2008), the independency assumption given in proposition 2.2 can be relaxed if lifetimes are exchangeable.

Therefore, with respect to the proposition 2.1 and proposition 2.2 the lifetime distribution of the system can be expressed as

$$R(t;\theta) = \sum_{i=1}^{n} s_i \sum_{r=0}^{i-1} \binom{n}{r} F^r(t)\bar{F}^{(n-r)}(t)\left(1 + \theta\psi_n(r;t)\right). \tag{2.12}$$

The reliability model (2.12) encompasses some familiar models as special cases. They are recovered by an appropriate choice of the signature vector **s** and the dependency parameter θ. It is noted that

- in the absence of the dependency factor ($\theta = 0$), the reliability function for a coherent system becomes

$$R(t;0) = \sum_{i=1}^{n} s_i \sum_{r=0}^{i-1} \binom{n}{r} F^r(t)\bar{F}^{(n-r)}(t). \tag{2.13}$$

- the reliability function of a parallel system with dependent components is recovered by $\mathbf{s} = (0, 0, \cdots, 1)$. In this case (2.12) turns into (2.14):

$$R_p(t;\theta) = 1 - (F^n(t)(1 + \theta\mathfrak{I}(n;t))), \tag{2.14}$$

  where
  $\mathfrak{I}(n;t) = \sum_{v=2}^{n} \sum_{A(v)} \binom{n}{x_2} (-1)^{x_1} F^{x_1}(t)\bar{F}^{x_2}(t)$.

- under the independence assumption ($\theta = 0$) the reliability function (2.12) for a parallel system reduces to

$$R_p(t;0) = 1 - F^n(t). \tag{2.15}$$

Examples 2.2. Consider the 10-component coherent system given in Figure 2. By using the methodology presented in theorem 2.1, the signature vector associated with the system is given by: $\mathbf{s} = (0, 0, 0/50, 1/20, 13/100, 11/50, 7/25, 21/100, 9/100, 0)$. With known parameter values given in the example 2.1, we examined the both sensitivity and the effect of environmental process on the reliability function (2.12) with respect to $\theta = \{0, 0.5, 1\}$.

To get insight into dependency and operating environment, Figure 3 on the left side illustrates that in the early phase of the system's life cycle, a higher level of dependency θ makes the system more susceptible to failure. Figure 3 on the right side differ in that the reliability function is regulated by both the operating environment factor and the age factor. As shown, the dependency factor influences on the reliability function in the same manner. However, it is not the case as time progresses.

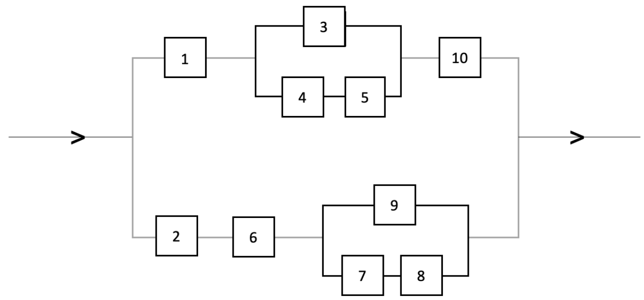

Figure 2: Coherent System with 10-component

Now we are in a position to formulate a generic MRL model for a load-sharing coherent system subject to operating conditions. For this, let $M_i(t,\theta)(i=0,1)$ stand for the MRL function at system level given the operating state i, in which the $R_i(t,\theta)$ for $i=0,1$ are defined in equation (2.6) and (2.7) . Then, on the basis of the developed reliability model (2.12), $M_i(t,\theta)$ for $i=0,1$ can be respectively expressed as:

$$M_0(t,\theta)=\frac{\displaystyle\int_t^{\infty}\sum_{i=1}^{n}s_i\sum_{r=0}^{i-1}\binom{n}{r}(1-R_0(w,t))^r R_0(w,t)^{(n-r)}(A_0)dw}{\displaystyle\sum_{i=1}^{n}s_i\sum_{r=0}^{i-1}\binom{n}{r}(1-[\bar{F}_0(t)])^r\bar{F}_0(t))^{(n-r)}(A_1)}, \tag{2.16}$$

where

$$A_0=1+\theta\sum_{v=2}^{n}\sum_{A(v)}\binom{n-r}{x_1}\binom{r}{x_2}(-1)^{x_1}(1-R_0(w,t))^{x_1}R_0(w,t)^{x_2},$$

$$A_1=1+\theta\sum_{v=2}^{n}\sum_{A(v)}\binom{n-r}{x_1}\binom{r}{x_2}(-1)^{x_1}(1-\bar{F}_0(t))^{x_1}\bar{F}_0(t)^{x_2},$$

and

$$M_1(t,\theta)=\frac{\displaystyle\int_t^{\infty}\sum_{i=1}^{n}s_i\sum_{r=0}^{i-1}\binom{n}{r}(1-R_1(w,t))^r R_1(w,t)^{(n-r)}(B_0)dw}{\displaystyle\sum_{i=1}^{n}s_i\sum_{r=0}^{i-1}\binom{n}{r}(1-[R_1(t,t)])^r R_1(t,t))^{(n-r)}(B_1)}, \tag{2.17}$$

where

$$B_0=1+\theta\sum_{v=2}^{n}\sum_{A(v)}\binom{n-r}{x_1}\binom{r}{x_2}(-1)^{x_1}(1-R_1(w,t))^{x_1}R_1(w,t)^{x_2},$$

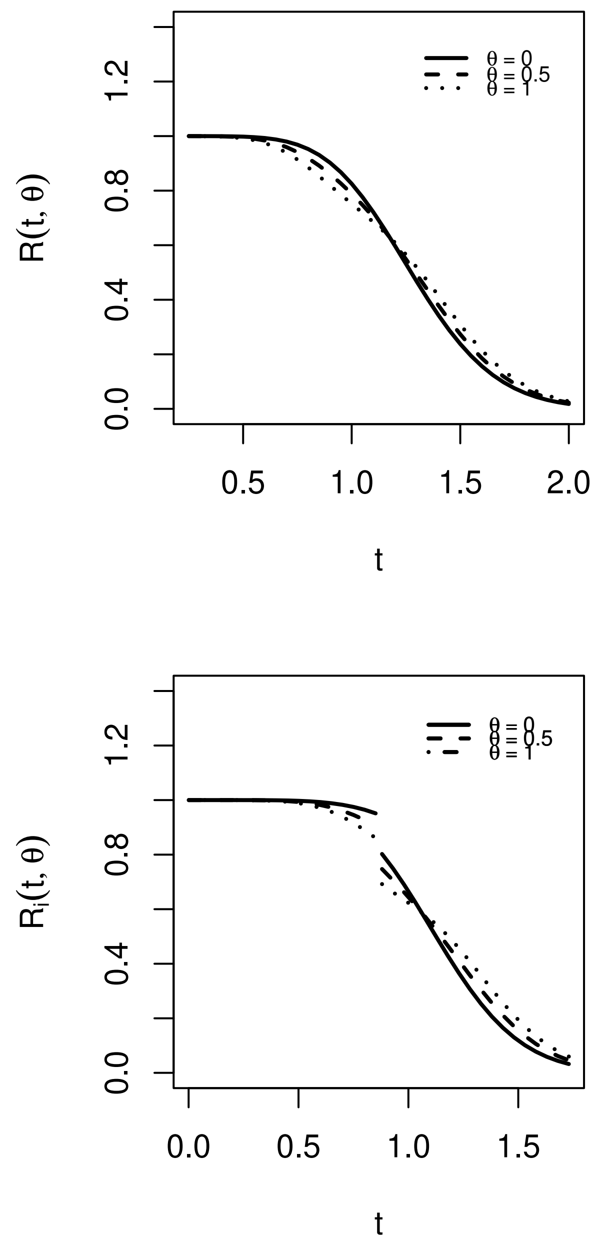

Figure 3: The Figure (R(t, θ)) represents the reliability function without the operating environment factor, and the Figure ($R_i(t,\theta)$) illustrates the reliability function with operating environment factor.

and

$$B_1 = 1 + \theta \sum_{v=2}^{n} \sum_{A(v)} \binom{n-r}{x_1}\binom{r}{x_2}(-1)^{x_1}(1 - R_1(t,t))^{x_1} R_1(t,t)^{x_2}.$$

Here, we indicate the generality of our model and demonstrate how several existing MRL models emerge as special cases.

2.2.3 Specific MRL Models at System Level

1. Ignoring load-sharing factor. If components are dependent and the environmental process has no impact on the degradation process, then the MRL model reduces to

$$M(t,\theta) = \frac{\displaystyle\int_t^{\infty} \sum_{i=1}^{n} s_i \sum_{r=0}^{i-1} \binom{n}{r}(F(s))^r \bar{F}(s)^{(n-r)}(1+\theta \sum_{v=2}^{n}\sum_{A(v)} \binom{n-r}{x_1}\binom{r}{x_2}(-1)^{x_1}(F(s))^{x_1}\bar{F}(s)^{x_2})ds}{\displaystyle\sum_{i=1}^{n} s_i \sum_{r=0}^{i-1} \binom{n}{r}(F(t))^r \bar{F}(t))^{(n-r)}(1+\theta(\sum_{v=2}^{n}\sum_{A(v)} \binom{n-r}{x_1}\binom{r}{x_2}(-1)^{x_1}(F(t))^{x_1}\bar{F}(t)^{x_2}]))}.$$

2. Ignoring dependency factor. If components are independent and the environmental process has no impact on the degradation process, then the MRL model becomes as follows:

$$M(t;0) = \frac{\displaystyle\int_t^{\infty} \sum_{i=1}^{n} s_i \sum_{r=0}^{i-1} \binom{n}{r}(F(s))^r \bar{F}(s)^{(n-r)} ds}{\displaystyle\sum_{i=1}^{n} s_i \sum_{r=0}^{i-1} \binom{n}{r}(F(t))^r \bar{F}(t)^{(n-r)}}.$$

3. Parallel system with independent components. This corresponds to $\theta = 0$ and $\mathbf{s} = (0, 0, \cdots, 1)$. Under these assumptions, the MRL model associated with the operating state $i = 0, 1$ becomes

$$M_i(t;0) = \frac{\displaystyle\int_t^{\infty} 1-(1-R_i(w,t))^n ds}{1-(1-R_i(w,t))^n}.$$

4. Further to the assumption 3,let n = 1. Then, the MRL equation without the environmental process for a single-component system becomes

$$M_p(t;0) = \frac{\displaystyle\int_t^{\infty} 1-F(s)ds}{1-F(t)}.$$

Examples 2.3. To examine the behavior of the MRL model at system level, assume that it consists of 10 dependent components (see Figure 2) whose lifetimes conform to the Weibull distribution with parameters $\alpha = 2$, $(\beta_0, \beta_1) = (1.5, 1)$. We set $\theta = 0.09$ and $\beta_{\tau_d} = 1$, and use the same signature vector given in example 2.2.

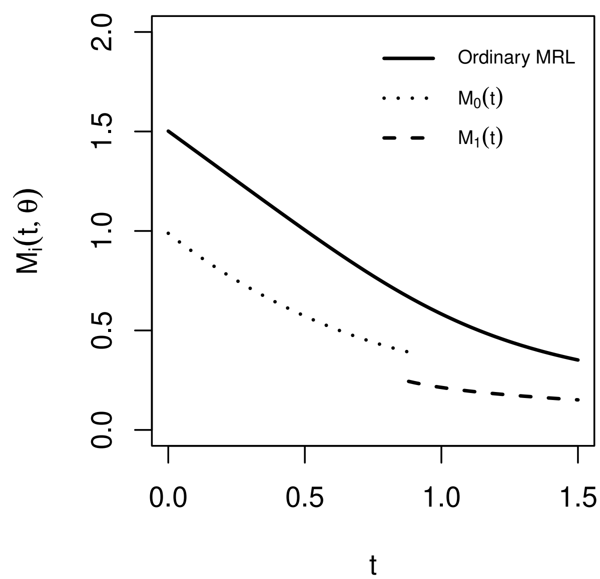

Figure 4: Effect of environmental process on MRL

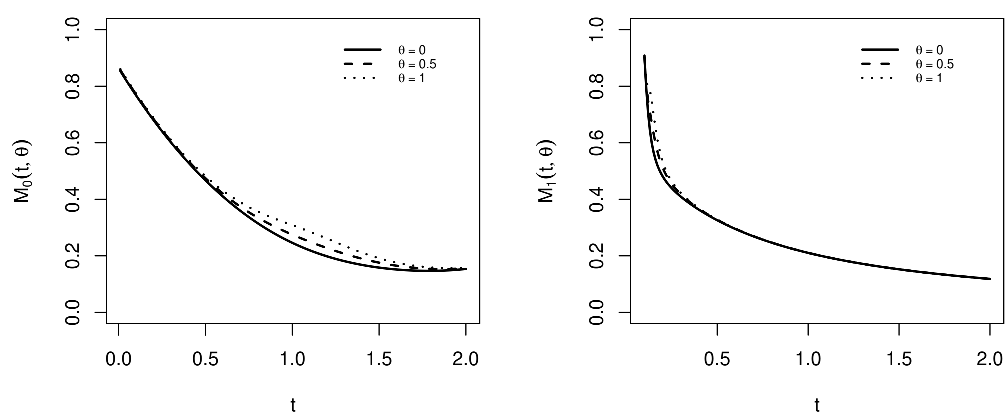

Figure 5: Effect of both operating environment ($Y(t) = 0, 1$) and dependence factor on MR

Figure 4 illustrates the behavior of an ordinary MRL (solid line) and MRL functions subject to the age factor and also operating environment. The results reveal that in the absence of operating environment, the system is less susceptible to failure (MRL is overestimated), whereas the inclusion of the operating environment makes an abrupt reduction in MRL implying that the system deteriorates more rapidly. As the damage process shifts to the higher state at the disruption time $\hat{\tau}_d$= 0.88, the system becomes more prone to failure. Furthermore, Figure 5 describes the joint effect of the environmental and dependency factor on MRL. Observe that the lower level of dependency (θ) in MRL leads to more increased tendency of the coherent system to fail.

3 Maintenance Model

For deteriorating systems suffering from failures, an appropriate maintenance strategy is essential to increase availability against increasing costs due to downtime. In the present paper, we adopt an age-based replacement policy for maintaining a load-sharing coherent system operating in a random environment. For this, let τ refer to the age at which the system is preventively replaced by a new one, and C_F (C_P) be the cost associated with the C_M (P_M). Since the time between two consecutive replacements is a renewal cycle, using the standard renewal-reward theorem argument Ross (1979), the average cost rate becomes

$$\mathbb{C}_{(\tau)} = \frac{E(C)}{E(L)},$$

where C and $L = min(T, \tau)$ respectively refer to the cost per cycle and the cycle length:

$$C = C_F \cdot 1(T < \tau) + C_P \cdot 1(T > \tau) = C_F + (C_P - C_F) \cdot 1(T > \tau), \tag{3.1}$$

and

$$L = \tau \cdot 1(T > \tau) + T \cdot 1(T < \tau), \tag{3.2}$$

with respective expected values:

$$E(C) = C_F + (C_P - C_F) \sum_{i=0,1} R_i(\tau, \theta)\pi_i(\tau),$$

with respective expected value $E(L)$:

$$E(L) = \sum_{i=0,1} \int_0^\tau R_i(t, \theta)\pi_i(\tau)dt,$$

where $\pi_i(\tau) = P(Y(t) = i)(i = 0, 1)$ and $R_i(t, \theta)$ is given by (2.5).

The objective is to determine an optimal replacement time τ^*:

$$\tau^* = argmin_{\tau \in (0,\infty)} \mathbb{C}(\tau).$$

Using the set of known parameters given in example 2.1 and $(\theta, \beta_0, C_F, C_P) = (0.5, 1.5, 4, 2)$, we obtain an optimal solution to the replacement time τ. The model postulate that a preventive replacement should be scheduled at $\tau^* = 0.9$. This policy incurs the minimum cost $\mathbb{C}(\tau^*) = 2.68$.

Table 1 and Figure 6 (on left) show the effect of both the dependency factor θ and the degradation parameter β_0 on the optimal solutions. Observe that decreasing $\beta_0(\beta_0>\beta_1)$ (lower level of reliability) induces an increase in average cost rate and the model suggests that the system should be replaced sooner. For a fixed value of β_0, the behavior of optimal solutions to θ is examined. The results show that increasing dependency level among components leads to a slight decrease in replacement time implying that the replacements should be scheduled more frequently. Consequently, this leads to an increase in the expected cost rate. The behavior of the average cost rate for different β_{τ_d} is also examined (see Table 2 and Figure 6 on right). The results reveal that as the sojourn time of the environmental process in normal state (state 0) decreases ($\beta_{\tau d}$ decreases), the replacements should be scheduled earlier, on the other hand decreasing $\beta_{\tau d}$ results in a reduction in expected cycle length and so increasing in the average cost rate. Table 3 and Figure 7 graphically show the behavior of the average cost rate and the optimal replacement time for different C_F and θ. Observe that for a fixed value of the dependency degree (θ), how increasing in C_F affects on replacement time (τ^*) and average cost rate of the coherent system. Specifically, as C_F increases, the optimal replacement time of the system decreases indicating that system should be replaced more often. Also, increasing unplanned replacement cost leads to an increase in the average cost rate. In addition, for a fixed value of C_F the sensitivity of optimal solutions to θ is examined. The results demonstrate that increasing θ makes replacements more frequent, resulting a slight increase in the expected cost rate.

Table 1: Optimal solutions for different $\theta \in \{0, 0.5, 1\}$ and $\beta_0 \in \{1.5, 3, 4.5\}$.

β_0	$(\tau^*,\ \mathbb{C}(\tau^*))$		
	$\theta = 0$	$\theta = 0.5$	$\theta = 1$
1.5	(0.9, 2.61)	(0.9, 2.68)	(0.85, 2.74)
3	(0.95, 2.56)	(0.9, 2.63)	(0.9, 2.69)
4.5	(0.95, 2.56)	(0.9, 2.62)	(0.9, 2.68)

Table 2: Optimal solutions for different $\theta \in \{0, 0.5, 1\}$ and $\beta_{\tau_d} \in \{1.1, 1.5, 3\}$.

β_{τ_d}	$(\tau^*,\ \mathbb{C}(\tau^*))$		
	$\theta = 0$	$\theta = 0.5$	$\theta = 1$
1.1	(0.95, 2.56)	(0.9, 2.63)	(0.9, 2.69)
1.5	(1.1, 2.056)	(1.15, 2.066)	(1.2, 2.068)
3	(2.15, 1.22)	(2.05, 1.24)	(2, 1.26)

Table 3: Optimal solutions for different $\theta \in \{0, 0.5, 1\}$ and $C_F \in \{3, 4, 5\}$.

C_F	$(\tau^*,\ \mathbb{C}(\tau^*))$		
	$\theta = 0$	$\theta = 0.5$	$\theta = 1$
3	(0.95, 2.53)	(0.95, 2.57)	(0.95, 2.62)
4	(0.9, 2.61)	(0.9, 2.68)	(0.85, 2.74)
5	(0.9, 2.66)	(0.85, 2.75)	(0.8, 2.82)

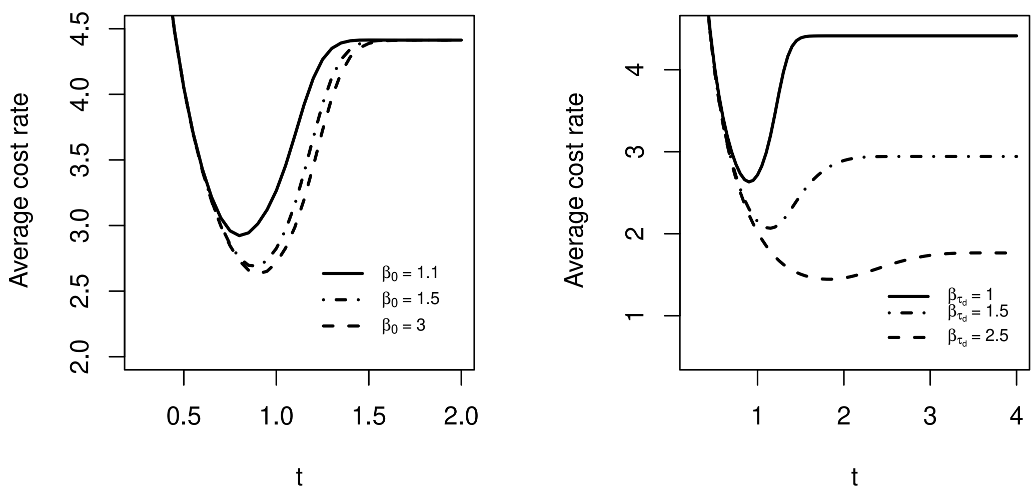

Figure 6: Average cost rate for different β_0 and β_{τ_d}

4 Conclusion

In this paper, we explore a MRL modeling approach for reliability analysis and maintenance planning of a coherent system. In contrast to existing models, it simultaneously accommodates the age factor, operating environmental factor, and load-sharing phenomenon.

The results of the model give insight into the effect of the dependency factor and the degradation parameter on the optimal solutions. It shows that decreasing the degradation parameter induces an increase in the average cost rate and the system should be replaced sooner. Increasing the dependency level among components leads to a slight decrease in replacement time implying that the replacements should be scheduled more frequently. The paper also examines the behavior of the average cost rate for different sojourn times of the environmental process in the normal state, and shows that decreasing the sojourn time results in earlier replacements and an increase in the average cost rate. The behavior of the average cost rate and the optimal replacement time for different unplanned replacement costs and dependency degrees

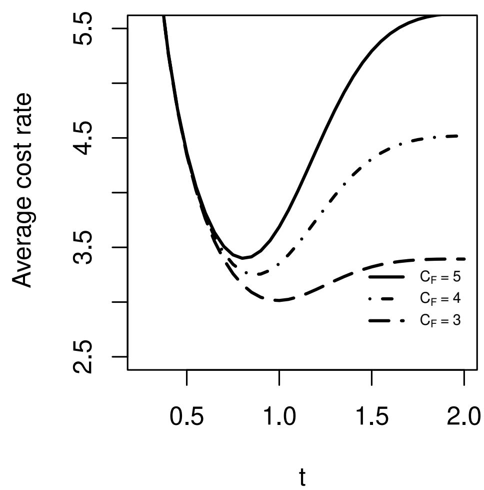

Figure 7: Effect of different amount of failure cost on average cost rate

is also analyzed, showing that increasing the unplanned replacement cost and the dependency degree make replacements more frequent, resulting in a slight increase in the expected cost rate.

This research can be extended in several directions that would enhance its applicability. Possible extensions of practical interests include the case of imperfect repair and the consideration of the explored MRL model as maintenance decision variable.

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