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Some New Results on Policy Limit Allocations

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Abstract. Suppose that a policyholder faces n risks $X_1, \ldots, X_n$ which are insured under the policy limit with the total limit of l. Usually, the policyholder is asked to protect each X_i with an arbitrary limit of l_i such that $\sum_{i=1}^{n} l_i = l$. If the risks are independent and identically distributed with log-concave cumulative distribution function, using the notions of majorization and stochastic orderings, we prove that the equal limits provide the maximum of the expected utility of the wealth of policyholder. If the risks with log-concave distribution functions are independent and ordered in the sense of the reversed hazard rate order, we show that the equal limits is the most favourable allocation among the worst allocations. We also prove that if the joint probability density function is arrangement increasing, then the best arranged allocation maximizes the utility expectation of policyholder's wealth. We apply the main results to the case when the risks are distributed according to a log-normal distribution.

Keywords. Arrangement Increasing Function, Log-Normal Distribution, Majorization, Schur-Convex Function, Stochastic Orders, Utility Function.

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1 Introduction

Some of the unforeseen random events like fire, flood, accident and medical issue might cause a large random amount of loss which is called risk. There are several types of insurance policies designed to protect a policyholder and cover either full or a partial amount of the risk. The insurance company might not be interested in covering the whole risk, since the risk may comes up as a huge amount and the policyholder also might not be interested in a full coverage, since he/she needs to pay a large amount of premium. A common partial coverage that extensively offered by the insurance companies is policy limit that policyholder's risk is covered up to a pre-specific value, known as limit and shown by l. In other words, it covers the left tail of the risk. If X is a risk faced by the policyholder, the covered risk by the insurance company is $(X \wedge l) = \min(X, l)$ and the retained risk self-insured by the policyholder is $(X - l)_+$ where $x_+ = \max(x, 0)$. This observation follows from the fact that the risk X can be expressed as $X = (X - a)_+ + (X \wedge a)$ where a is an arbitrary real value. For more details on the policy limits coverage, the reader is referred to Klugman et al. (2004).

Usually a policyholder faces a set of n risks $X_1, \ldots, X_n$ and wants to be protected by a policy limit coverage with total limit of l. A policy limit that protect the total risk of $\sum_{i=1}^{n} X_i$ with the total limit of l has a covered risk equal to $(\sum_{i=1}^{n} X_i \wedge l)$. This is called global insurance. The global insurance for the insurer is less beneficial than the case when the risk X_i is protected with the limit of l_i, $i = 1, \ldots, n$ (cf. Hua and Cheung , 2008, Lemma 4). Therefore, the policyholder is asked to protect each X_i with an arbitrary limit of l_i such that $\sum_{i=1}^{n} l_i = l$ (cf. Cheung , 2007). Thus, the total risk $\sum_{i=1}^{n} X_i$ decomposes in two parts of $\sum_{i=1}^{n}(X_i \wedge l_i)$ and $\sum_{i=1}^{n}(X_i - l_i)_+$ which are covered and retained risks, respectively.

The policyholder needs to choose an allocation that has the least retained risk based on some criteria. The well known optimization criteria in the literature are:

- maximizing the expected value of a utility function of the wealth residual,
- maximizing the expected value of the covered risk with respect to a distortion function,
- minimizing ruin probability and

- minimizing variance.

For more details, see for example Van Heerwaarden et al. (1989) and Denuit and Vermandele (1998). In this paper, we use the utility function as well as distortion function criteria to evaluate the retained risk for different allocations. Let

$$s_n(l) = \{\mathbf{l} = (l_1, \ldots, l_n) \in \mathbb{R}^{+n} \mid \sum_{i=1}^{n} l_i = l\}.$$

and w be the initial wealth of the policyholder after paying the required premium which is independent of l_i's. Now, the best allocation is $\mathbf{l}^* = (l_1^*, \ldots, l_n^*) \in s_n(l)$ that maximizes the expected utility, that is,

$$E\left[u(w - \sum_{i=1}^{n}(X_i - l_i^*)_+)\right] = \max_{\mathbf{l}\in s_n(l)} E\left[u(w - \sum_{i=1}^{n}(X_i - l_i)_+)\right], \tag{1.1}$$

where u is the policyholder's utility function.

Utility function modifies the potential value of the actual wealth based on the risk aversion status of the policyholder. Other way to change the potential value of the wealth is through distorting the survival function of the wealth based on the policyholder's risk aversion level. Let V be a non-negative continuous random variable with survival function $\bar{H}$ and g be an increasing function over the interval $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$. Then the distorted survival function with respect to function g is $g(\bar{H}(.))$. If the function g is concave, then for all x, $g(\bar{H}(x)) > \bar{H}(x)$ and if the function g is convex, then $g(\bar{H}(x)) < \bar{H}(x)$. Therefore, potential value of V is always greater (less) than actual value of V if the function g is concave (convex). The expected value of V with respect to distorted survival function is

$$H_g[V] = \int_0^\infty g(\bar{H}(x))dx.$$

$H_g[V]$ is called distorted expected value of V with respect to g. That is, the distorted tail probability is used to compute the distorted expected value. For more details on the theory of the distorted expectation see Denuit et. al (2005).

Under the theory of the distorted expectation, a decision maker with initial wealth w and a faced risk X tries to make a decision that maximizes the distorted expectation. In other word, the policyholder prefers the contract with retained risk X over the retained risk Y if and only if $H_g[w-X] \geq H_g[w-Y]$. According to this preference order, the policyholder is looking for an allocation $\mathbf{l}^* = (l_1^*, \ldots, l_n^*) \in s_n(\mathbf{l})$ that satisfies

$$H_g\left[w - \sum_{i=1}^{n}(X_i - l_i^*)_+\right] = \max_{\mathbf{l}\in s_n(l)} H_g\left[w - \sum_{i=1}^{n}(X_i - l_i)_+\right]. \tag{1.2}$$

Next, we review the required definitions of various stochastic orders, majorization order and arrangement increasing function that are used to prove the main results in this paper. Throughout this paper, we assume that all the faced risks are continuous random variables and all the expectations of the random variables considered exist. We use increasing for non-decreasing and decreasing for non-increasing.

Let X and Y be two univariate random variables with cumulative distribution functions (cdf's) F and G, survival functions $\bar{F}$ and $\bar{G}$, probability density functions (pdf's) f and g, hazard rates r_F $(= f/\bar{F})$ and r_G and reversed hazard rates $\tilde{r}_F$ $(= f/F)$ and $\tilde{r}_G$, respectively. Let l_X, l_Y and u_X, u_Y be the (finite or infinite) left and right endpoints of the support of X and Y, respectively. The random variable X is said to be smaller than random variable Y in the

- stochastic dominance order (denoted by $X \leq_{st} Y$), if $\bar{F}(x) \leq \bar{G}(x)$ for all x which is equivalent to that $E\left[\phi(X)\right] \leq E\left[\phi(Y)\right]$ for all increasing function ϕ,
- increasing concave (convex) order (denoted by $X \leq_{icv\,(icx)} Y$), if $E\left[\phi(X)\right] \leq E\left[\phi(Y)\right]$ for all increasing concave (convex) function ϕ,
- hazard rate order (denoted by $X \leq_{hr} Y$), if $\bar{G}(x)/\bar{F}(x)$ is increasing in $x \in (-\infty, \max(u_X, u_Y))$,
- reversed hazard rate order (denoted by $X \leq_{rh} Y$), if $G(x)/F(x)$ is increasing in $x \in (-\infty, \max(u_X, u_Y))$,
- likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in $x \in (-\infty, \max(u_X, u_Y)$.

For more details on stochastic orders, see e.g. Müller and Stoyan (2002), Denuit et al. (2005) and Shaked and Shanthikumar (2007).

The notion of majorization is used to derive the various interesting inequalities in statistics and probability. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $x_{(i)}$ and $x_{[i]}$ denote the ith smallest and the ith largest of x_i's, receptively.

Definition 1.1. A vector $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$ is said to be majorized by another vector $\mathbf{y} \in \mathbb{R}^{\mathbf{n}}$, denoted by $\mathbf{x} \leq_m \mathbf{y}$, if $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

Definition 1.2. A real valued function ϕ defined on set $\mathbb{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on $\mathbb{A}$, if $\phi(\mathbf{x}) \leq (\geq)\phi(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{A}$ such that $\mathbf{x} \leq_{\mathbf{m}} \mathbf{y}$.

For more details on majorization and its properties the reader is referred to Marshall et al. (2011).

Next, we define the concept of arrangement increasing (AI) and arrangement decreasing (AD) functions which are used in this paper. Let S be the set of all permutations of $(1, \ldots, n)$. Assume that $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_i, \ldots, \tau_j, \ldots, \tau_n)$ and $\boldsymbol{\tau}' = (\tau_1, \ldots, \tau_j, \ldots, \tau_i, \ldots, \tau_n)$ are two elements of S such that for $i < j$, we have that $\tau_i < \tau_j$. Then, it is said that $\boldsymbol{\tau}'$ is a simple transposition of $\boldsymbol{\tau}$ and denoted by $\boldsymbol{\tau}' \leq_t \boldsymbol{\tau}$. Suppose $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ are two arbitrary elements of S. It is said that $\boldsymbol{\tau}$ is better arranged than $\boldsymbol{\tau}'$ and denoted by $\boldsymbol{\tau}' \leq_a \boldsymbol{\tau}$ if $\boldsymbol{\tau}$ is obtained from $\boldsymbol{\tau}'$ by a finite number of simple transpositions.

Definition 1.3. A function $h(\mathbf{x}, \boldsymbol{\lambda}) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be $AI(AD)$ in $(\mathbf{x}, \boldsymbol{\lambda})$ if the following conditions are hold.

1. $h(\mathbf{x}, \boldsymbol{\lambda}) = h(\mathbf{x} o \boldsymbol{\tau}, \boldsymbol{\lambda} o \boldsymbol{\tau})$.
2. If $x_1 \leq \ldots \leq x_n$, then

$$\boldsymbol{\tau}' \leq_a \boldsymbol{\tau} \Longrightarrow h(\mathbf{x} o \boldsymbol{\tau}', \boldsymbol{\lambda} \uparrow) \leq (\geq) h(\mathbf{x} o \boldsymbol{\tau}, \boldsymbol{\lambda} \uparrow).$$

where $\mathbf{x} o \boldsymbol{\tau} = (x_{\tau_1}, \ldots, x_{\tau_n})$ and $\boldsymbol{\lambda} \uparrow = (\lambda_{(1)}, \ldots, \lambda_{(n)})$.

Definition 1.4. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be AI (AD), if for $x_i \leq x_j$, $1 \leq i < j \leq n$,

$$f(x_1, \ldots, x_i, \ldots x_j, \ldots x_n) \geq (\leq) f(x_1, \ldots, x_j, \ldots x_i, \ldots x_n).$$

For more details, one may refer to Marshal et al. (2011) and Hollander et al.(1977).

Definition 1.5. A real valued function ϕ defined on set $\mathbb{A} = \{\mathbf{x} \in \mathbb{R}^n : \phi(\mathbf{x}) \geq 0\}$ is said to be log-concave, if for any $\mathbf{x}, \mathbf{y} \in \mathbb{A}$ and $\alpha \in [0, 1]$,

$$\phi(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) \geq [\phi(\mathbf{x})]^{\alpha} [\phi(\mathbf{y})]^{1-\alpha}.$$

In Section 2, we prove that if $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d) risks with a log-concave cdf, then the best and worse allocations in Problems (1.1) and (1.2) are $(\bar{l}, \ldots, \bar{l})$ and $(l, 0, \ldots, 0)$, respectively (Corollaries 2.2 and 2.5). These are extensions of the results of Theorem 2.4 and 2.8 in Manesh and Khaledi (2015).

The above optimization problems don't have an intuitive solution for the cases when the risks are independent but not identically distributed. However, stochastic behaviour of the retained risk with respect to different arrangements of the given $(l_1, \ldots, l_n)$ is of interest. Intuitively, the ith smallest value of l_i's is to be assign to the ith smallest of X_i's. We show that (Theorem 2.1) if $X_1, \ldots, X_n$ are independent risks with log-concave cdf, then $X_1 \geq_{rh} X_2 \geq_{rh} \ldots \geq_{rh} X_n$ implies that

$$\mathbf{l} \geq_m \mathbf{l}^* \Longrightarrow \sum_{i=1}^{n} (X_i - l_{(i)})_+ \geq_{st} \sum_{i=1}^{n} (X_i - l^*_{(i)})_+.$$

Hu and Wang (2014) and Manesh and Khaledi (2015) proved the same result under the stronger condition that the pdf is either log-concave or decreasing.

Furthermore, we prove that (Corollary 2.4) if the joint pdf of the risks is an AI function, then, for a given vector $(l_1, \ldots, l_n)$ and an increasing utility function, allocation $(l_{(1)}, \ldots, l_{(n)})$ maximizes and allocation $(l_{[1]}, \ldots, l_{[n]})$ minimizes the expected utility function of the policyholder's wealth. Cheung (2007) obtained the same result for the case when the utility function is increasing concave and $X_i \leq_{hr} X_j$, for $i < j$.

2 Main Results

We use the following lemmas to prove the main results.

Lemma 2.1. *(Marshall et al. , 2011). Let $\mathbb{A}$ be a set with the property*

$$\mathbf{y} \in \mathbb{A} \quad \textit{and} \quad \mathbf{x} \leq_m \mathbf{y} \quad \textit{implies} \quad \mathbf{x} \in \mathbb{A}.$$

A continuous function ϕ defined on $\mathbb{A}$ is Schur-convex on $\mathbb{A}$ if and only if ϕ is symmetric and $\phi(x_1, s - x_1, x_3, \ldots, x_n)$ is decreasing in $x_1 \leq \frac{s}{2}$ for each fixed $s, x_3, \ldots, x_n$.

Lemma 2.2. *(Manesh and Khaledi , 2015). Let X_1 and X_2 be two random risks with joint density function $f_{X_1,X_2}(x_1, x_2)$. If for all $c > 0$ and $0 \leq a \leq \frac{c}{2}$,*

$$\int_0^{c-a} f_{X_1,X_2}(t + a, x)dx \geq \int_0^{a} f_{X_1,X_2}(x, t + c - a)dx, \tag{2.1}$$

then,

$$X_{a,c} \geq_{st} X_{a^*,c},$$

where $X_{a,c} = (X_1 - a)_+ + (X_2 - (c - a))_+$ and $0 \leq a \leq a^* \leq \frac{c}{2}$.

Stochastic behavior of the retained risk with respect to the various arrangements of the given $(l_1, \ldots, l_n)$ is of interest for the cases when the risks are independent but not identically distributed. Intuitively, the ith smallest value of l_i's is to be assigned to the ith smallest of X_i's. This kind of problems have been studied by Lu and Meng (2011), Hu and Wang (2014) and Manesh and Khaledi (2015). The next theorem is an extension of Proposition 3.2.a of Hu and Wang (2014) and also Theorem 2.11 of Manesh and Khaledi (2015).

Theorem 2.1. *Let $X_1, \ldots, X_n$ be independent risks with log-concave cdf. If $X_1 \geq_{rh} X_2 \geq_{rh} \ldots \geq_{rh} X_n$, then*

$$\mathbf{l} \geq_m \mathbf{l}^* \Longrightarrow \sum_{i=1}^{n} (X_i - l_{(i)})_+ \geq_{st} \sum_{i=1}^{n} (X_i - l^*_{(i)})_+.$$

Proof. From Lemma 2 of Hua and Cheung (2008) and the fact that the stochastic dominance order is closed under convolution of independent random variables, it is enough to show that

$$(l_1, l_2) \geq_m (l_1^*, l_2^*) \Longrightarrow (X_1 - l_{(1)})_+ + (X_2 - l_{(2)})_+ \geq_{st} (X_1 - l^*_{(1)})_+ + (X_2 - l^*_{(2)})_+.$$

Now, let $l_{(1)} = a$ and $l_{(2)} = c - a$, where $0 \leq a \leq \frac{c}{2}$ and $\tilde{r}_1$ and $\tilde{r}_2$ be the reversed hazard rate functions of X_1 and X_2, respectively. Then, for $t \geq 0$, we have that

$$\begin{aligned} f_{X_1}(t+a)F_{X_2}(c-a) &= \tilde{r}_1(t+a)F_{X_1}(t+a)F_{X_2}(c-a) \\ &\geq \tilde{r}_2(t+a)F_{X_1}(t+a)F_{X_2}(c-a) \\ &\geq \tilde{r}_2(t+c-a)F_{X_1}(t+a)F_{X_2}(c-a) \\ &= f_{X_2}(t+c-a)\frac{F_{X_1}(t+a)}{F_{X_2}(t+c-a)}F_{X_2}(c-a), \end{aligned} \tag{2.2}$$

where the first inequality follows from the assumption $X_1 \geq_{rh} X_2$ and the second inequality holds since the log-concavity of cdf implies that $\tilde{r}$ is decreasing. Now, from the assumptions $X_1 \geq_{rh} X_2$ and log-concavity of F, we obtain

$$\frac{f_{X_1}(t+a)}{F_{X_1}(t+a)} \geq \frac{f_{X_2}(t+a)}{F_{X_2}(t+a)} \geq \frac{f_{X_2}(t+c-a)}{F_{X_2}(t+c-a)},$$

which implies $f_{X_1}(t+a)F_{X_2}(t+c-a) \geq f_{X_2}(t+c-a)F_{X_1}(t+a)$. That is $\frac{F_{X_1}(t+a)}{F_{X_2}(t+c-a)}$ is an increasing function of t. Combining this observation with (2.2) results

$$f_{X_1}(t+a)F_{X_2}(c-a) \geq f_{X_2}(t+c-a)F_{X_1}(a),$$

which is equivalent to the inequality (2.1). Now, the required result follows from Lemma 2.2. □

Intuitively, the worst case of an allocation is to assign the smaller limit to the larger risk. The result of Theorem 2.1 is useful to evaluate the retained risk for this kind of allocation. It shows that less dispersed allocation creates less retained risk. In particular, since $(\bar{l},\ldots,\bar{l}) \leq_m (l_1,\ldots,l_n) \leq_m (l,0,\ldots,0)$, we conclude that assigning the equal limits is the most favourable and assigning the total limit to the smallest risk is the least favourable than any other worst allocation.

The following corollaries are particular cases of Theorem 2.1.

Corollary 2.1. Let $X_1,\ldots,X_n$ be iid risks with X_1 having density function f and log-concave distribution function F. Then, for all $\mathbf{l},\mathbf{l}^* \in S_n(l)$,

$$\mathbf{l} \geq_m \mathbf{l}^* \Longrightarrow \sum_{i=1}^{n}(X_i - l_i)_+ \geq_{st} \sum_{i=1}^{n}(X_i - l_i^*)_+.$$

Corollary 2.2. Let $X_1,\ldots,X_n$ be iid risks with X_1 having a log-concave cdf faced by a policyholder with arbitrary utility function u. Then, $(\bar{l},\ldots,\bar{l})$ and $(l,0,\ldots,0)$ are the best and worst allocations, respectively, with respect to the utility expectation criterion.

It is known that if a pdf is log-concave or decreasing then its cdf is log-concave. Thus, Corollary 2.1 shows that the result of Theorem 2.4 and 2.8 of Manesh and Khaledi (2015) holds true under the weaker condition of log-concavity of cdf when the risks are iid.

Log-normal is a commonly used distribution for modelling loss data, specially in auto-insurance and fire insurance. It has neither log-concave nor decreasing pdf, but has a log-concave cdf. Thus, from Corollary 2.2, if a policyholder faces the iid log-normal risks $X_1,\ldots,X_n$, then $(\bar{l},\ldots,\bar{l})$ and $(l,0,\ldots,0)$ are the best and worst allocations, respectively.

Let X_1, X_2, X_3 and X_4 be independent risks having the common log-normal distribution with parameters $\mu = 0$ and $\sigma^2 = 1$. Suppose that the total risk is insured with the total limit of $l = 140$. In Table 1, we simulate the upper tail quantiles of $\sum_{i=1}^{4}(X_i - l_i)_+$ for the vectors of limits $(3,3,3,3) \leq_m (0,2,4,6) \leq_m (12,0,0,0)$. In Figure 1, we plot the survival function of the retained risks for the given allocations. Figure 1 and Table 1 describe the result of Theorem 2.1. It is clear that from the entries of the table, the more dispersed limits, the larger retained risk quantile. That is, the smallest quantile is achieved by the equal limits allocation. As a result, the policyholder has to cover the least amount of 2.12, 3.02, 4.38 and 7.01 with probability 0.8, 0.85, 0.9 and 0.95, respectively.

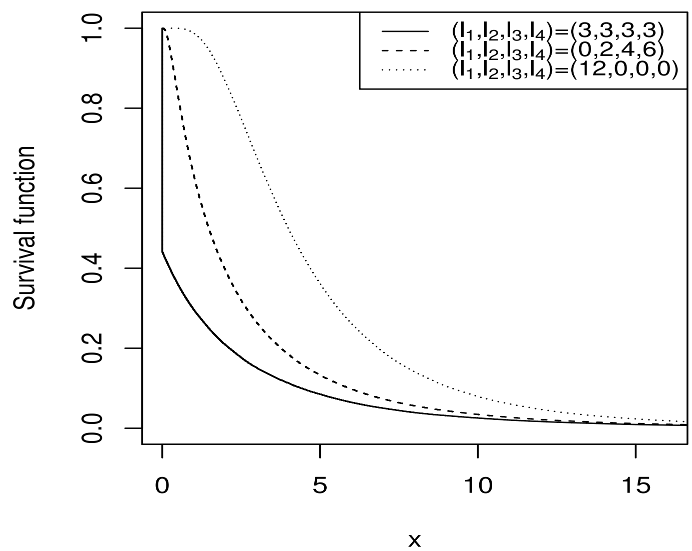

Figure 1: Survival function of retained risk for several allocations.

Table 1: Retained risk values for i.i.d log-normal risks

(l_1, l_2, l_3, l_4)	p-quantile				mean
	$p = 0.8$	$p = 0.85$	$p = 0.9$	$p = 0.95$	
(3, 3, 3, 3)	2.12	3.02	4.38	7.01	1.40
(0, 2, 4, 6)	3.74	4.59	5.90	8.46	2.56
(12, 0, 0, 0)	6.84	7.78	9.16	11.74	4.98

Now, suppose that an insured with capital w has the utility function $u()$. As given in Kass et al. (2008), it is common to find a proper premium P such that

$$E[u(w - \sum_{i=1}^{n}(X_i - l_i)_+ - P)] \geq E[u(w - \sum_{i=1}^{n} X_i)].$$

The policyholder is interested to determine the appropriate premium P for the case when $l_i = \bar{l}, i = 1, \dots n$, that maximizes the utility expectation of the wealth. Since u is a non-decreasing function, then the maximum premium P^+ paid by the policyholder is the solution of the following equation with respect to P.

$$E[u(w - \sum_{i=1}^{n}(X_i - \bar{l})_+ - P)] = E[u(w - \sum_{i=1}^{n} X_i)].$$

Next, we consider $w = 300$, $\bar{l} = 3$ and some utility functions which are ordered according to the risk aversion to obtain the solution of the above equation for the random risks X_1, X_2, X_3 and X_4 described above. The simulated results are given in Table 2. It is observed that the maximum premium is increasing with respect to the risk aversion property. We tried the similar simulations for various values of w and observed that the maximum premium is slightly decreasing with respect to w.

Table 2: Policy limit premium upper bound for iid log-normal risks

	$u(x)$				
(l_1, l_2, l_3, l_4)	x^2	x	$\sqrt[2]{x}$	$\sqrt[3]{x}$	$\log(x)$
$(3, 3, 3, 3)$	5.172	5.187	5.195	5.199	5.200

Next, we discuss the stochastic comparison of the retained risks with respect to different permutations of a given limit.

Lemma 2.3. *Let ϕ be a convex function and $\boldsymbol{l} = (l_1, \dots, l_n)$ and $\mathbf{x} = (x_1, \dots, x_n)$ be two vectors in $\mathbb{R}^n$. Then, $\sum_{i=1}^{n} \phi(x_i - l_i)$ is an AD function in $(\mathbf{x}, \boldsymbol{l})$.*

Proof. It is known that $\sum_{i=1}^{n} \phi(a_i)$ is Schur-convex if $\phi(.)$ is convex. Now, the required result follows from Lemma 2.2 of Hollander et al. (1977). □

Lemma 2.4. *(Pecaric , 1992, p. 395). Let $\{f_{\boldsymbol{\lambda}}(\mathbf{x}), \boldsymbol{\lambda} \in \boldsymbol{\Lambda} \subseteq \mathbb{R}^n\}$ be a family of AI joint pdf in $(\mathbf{x}, \boldsymbol{\lambda})$. If $\mathbf{X}$ is a random vector with joint pdf $f_{\boldsymbol{\lambda}}(\mathbf{x})$ and $h : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is an AI function, then,*

$$P_{\boldsymbol{\lambda},\mathbf{a}}(\mathbf{X}) = P\left[h(\mathbf{a}, \mathbf{X}) \geq c\right],$$

is an AI function in $(\boldsymbol{\lambda}, \mathbf{a})$, for all constant c.

Theorem 2.2. *Let $X_1, \ldots, X_n$ be n risks with AI joint pdf f. Then, for any vector $(l_1, \ldots, l_n)$ and convex function ϕ,*

$$\sum_{i=1}^{n} \phi(X_i - l_{(i)}) \leq_{st} \sum_{i=1}^{n} \phi(X_i - l_{\tau_i}) \leq_{st} \sum_{i=1}^{n} \phi(X_i - l_{[i]}),$$

where $(\tau_1, \ldots, \tau_n)$ is an arbitrary permutation of $(1, \ldots, n)$.

Proof. Let $h(\mathbf{l}, \mathbf{x}) = -\sum_{i=1}^{n} \phi(x_i - l_i)$ which from Lemma 2.3 is an AI function and for every $\boldsymbol{\lambda} \in \boldsymbol{\Lambda} \subseteq \mathbb{R}^n$, $f_{\boldsymbol{\lambda}}(\mathbf{x}) = f(\mathbf{x})$. Combining this observation with $\mathbf{l} \downarrow \leq_a \mathbf{l} o \boldsymbol{\tau} \leq_a \mathbf{l} \uparrow$, it follows from Lemma 2.4

$$P(\sum_{i=1}^{n} \phi(X_i - l_{(i)}) > t) \leq P(\sum_{i=1}^{n} \phi(X_i - l_{\tau_i}) > t) \leq P(\sum_{i=1}^{n} \phi(X_i - l_{[i]}) > t),$$

which is the required result. □

If $X_1, \ldots, X_n$ are independent and $X_1 \leq_{lr} \ldots \leq_{lr} X_n$, then the joint pdf is AI. Thus, the subsequent result is a consequence of Theorem 2.2.

Corollary 2.3. If $X_1, \ldots, X_n$ are independent such that $X_1 \leq_{lr} \ldots \leq_{lr} X_n$, then, for any vector $(l_1, \ldots, l_n)$ and convex function ϕ,

$$\sum_{i=1}^{n} \phi(X_i - l_{(i)}) \leq_{st} \sum_{i=1}^{n} \phi(X_i - l_{\tau_i}) \leq_{st} \sum_{i=1}^{n} \phi(X_i - l_{[i]}).$$

Since $\phi(x) = (x)_+$ is a convex function, the next corollary can be concluded from Theorem 2.2.

Corollary 2.4. Let $X_1, \ldots, X_n$ be n risks protected by the policy limits coverage and limits $(l_1, \ldots, l_n)$. If the joint pdf of the risks is an AI function, then the allocations $(l_{(1)}, \ldots, l_{(n)})$ and $(l_{[1]}, \ldots, l_{[n]})$ are the best and worst allocations, respectively, with respect to the utility expectation criterion.

Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ has a multivariate log-normal distribution with parameter $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma} = \sigma^2 \mathbf{R}$ where $\mathbf{R}$ is a correlation matrix such that

$$\rho_{i,j} = \rho \in (-\frac{1}{n-1}, 1) \text{ for all } \quad i \neq j, \tag{2.3}$$

for arbitrary but fixed $\sigma^2 > 0$. Hollander et al. (1977) showed that under the conditions of $\boldsymbol{\Sigma} = \sigma^2 \mathbf{R}$ and (2.3), $f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$, a multivariate normal density function is AI in $(\mathbf{x}, \boldsymbol{\mu})$.

Using this observation and the fact that $log(x)$ is an increasing function, it can be shown that $\mathbf{X}$ has also an AI density function in $(\mathbf{x}, \boldsymbol{\mu})$.

Now, consider the risks (X_1, X_2, X_3, X_4) has the above multivariate log-normal distribution wit parameter $\boldsymbol{\mu} = (0.1, 0.6, 1, 1.5)$, $\sigma^2 = 1$ and $\rho = 0.5$. In Figure 2, we plot the survival function of $\sum_{i=1}^{4}(X_i - l_i)_+$ for the vector of limits $(10, 7, 3, 0) \leq_a (7, 0, 10, 3) \leq_a (0, 3, 7, 10)$. In Table 3, we also simulate the upper tail quantiles of the retained risks for the given allocations of limits. Figure 2 and Table 3 describe the result of Corollary 2.4. It is clear that from the entries of the table, assigning the smaller limit to the smaller risk induces the smaller quantile of the retained risk.

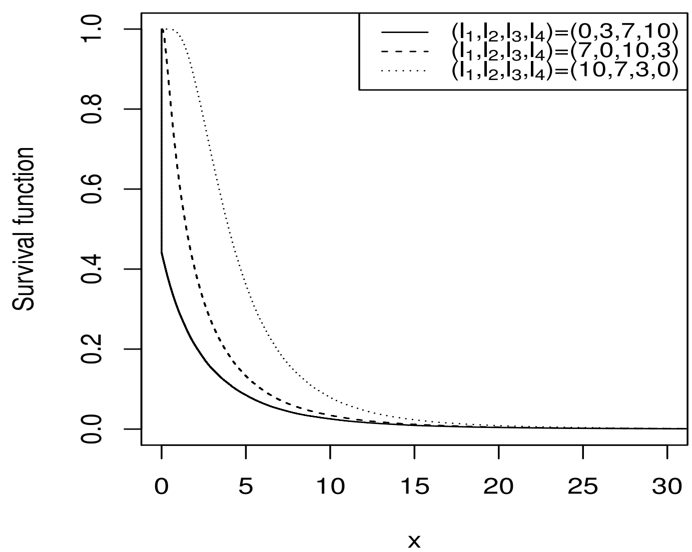

Figure 2: Survival function of retained risk of multivariate log-normal risks for several allocations.

The next results discuss the possible solutions of Problem (1.2) which can be derived from Theorems 2.1 and 2.1 and Corollary 2.4 using the implication

$$X \leq_{st} Y \Longrightarrow H_g[X] \leq H_g[Y].$$

Table 3: Retained risk values of multivariate log-normal risks

(l_1, l_2, l_3, l_4)	p-quantile				mean
	$p = 0.8$	$p = 0.85$	$p = 0.9$	$p = 0.95$	
(0, 3, 7, 10)	8.09	11.30	16.59	27.50	6.25
(7, 0, 10, 3)	12.68	15.96	21.15	31.59	8.69
(10, 7, 3, 0)	14.74	18.17	23.47	34.14	10.23

Corollary 2.5. Let $X_1, \ldots, X_n$ be n risks faced by a policyholder with the policy limits coverage and g be a distortion function.

a. If the risks are iid with log-concave cdf, then

$$\mathbf{l} \geq_m \mathbf{l}^* \Longrightarrow H_g\left[w - \sum_{i=1}^{n}(X_i - l_i)_+\right] \leq H_g\left[w - \sum_{i=1}^{n}(X_i - l_i^*)_+\right].$$

b. If the risks are independent with log-concave cdfs and ordered as $X_1 \geq_{rh} X_2 \ldots \geq_{rh} X_n$, then

$$\mathbf{l} \geq_m \mathbf{l}^* \Longrightarrow H_g\left[w - \sum_{i=1}^{n}(X_i - l_{(i)})_+\right] \leq H_g\left[w - \sum_{i=1}^{n}(X_i - l_{(i)}^*)_+\right].$$

c. If the joint pdf of the risks is AI, then

$$H_g\left[w - \sum_{i=1}^{n}(X_i - l_{[i]})_+\right] \leq H_g\left[w - \sum_{i=1}^{n}(X_i - l_i)_+\right] \leq H_g\left[w - \sum_{i=1}^{n}(X_i - l_{(i)})_+\right].$$

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