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Bivariate Extension of Past Entropy

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Abstract. Di Crescenzo and Longobardi (2002) has been proposed a measure of uncertainty related to past life namely past entropy. The present paper addresses the question of extending this concept to bivariate set-up and study some properties of the proposed measure. It is shown that the proposed measure uniquely determines the distribution function. Characterizations for some bivariate lifetime models are obtained using the proposed measure. Further, we define new classes of life distributions based on this measure and properties of the new classes are also discussed. We also proposed a non-parametric kernel estimator for the proposed measure and illustrated performance of the estimator using a numerical data.

Keywords. Bivariate Reversed Hazard Rate, Bivariate Mean Inactivity Time, Non-parametric Kernel Estimation, Past Entropy.

MSC: 62G30; 62E10, 62B10.

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1 Introduction

In information theory, in order to quantify the lost information in communication channels a new concept was introduced by Shannon (1948) known as Shannons entropy, similar to the entropy discussed in statistical thermodynamics. If X is a non-negative random variable having an absolutely continuous distribution function $F(x)$ with probability density function $f(x)$, then the entropy of X is defined as

$$H(X) = -\int_0^\infty f(x) \log f(x)\, dx. \tag{1.1}$$

$H(X)$ measures the expected uncertainty in f about the predictability of an observation of X. This measure finds application in various fields such as financial analysis, data compression, molecular biology, hydrology, meteorology, computer science, and information theory.

In life testing experiments, if a system has survived up to time t, then obviously Shannon entropy does not suit well in measuring uncertainty as far as the remaining lifetime of a system is considered. Accordingly, Ebrahimi (1996) introduced a measure of uncertainty, known as residual entropy and is defined as

$$H(X;t) \quad = \quad -\int_{\mathbf{t}}^\infty \frac{f(x)}{\overline{F}(t)} \log \frac{f(x)}{\overline{F}(t)}\, dx, \tag{1.2}$$

where $\overline{F}(x)$ is the survival function of X. That is, for the component which has survived up to time t, (1.2) measures the expected uncertainty contained in the density of residual lifetime about the predictability of the remaining lifetime of the component.

The measure of uncertainty associated with past lifetime plays an important role in the context of reliability theory. There are many real situations in which uncertainty is not necessarily related to the future but can also refer to the past. For instance, consider a system whose state is observed only at certain preassigned inspection times. At time t, if the system is inspected for the first time and is found to be 'down', then the uncertainty relies on the past, that is, it has failed somewhere on an instant in $(0,t)$.Thus it seems natural to introduce a notion of uncertainty that is dual to (1.2) in the sense that it refers to the past time, not to the future time. Based on this idea, Di Crescenzo and Longobardi (2002) introduced past entropy over $(0,t)$, namely dynamic past entropy (DPE) and is defined as

$$\overline{H}(X;t) \quad = \quad -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)}\, dx. \tag{1.3}$$

DPE defined in (1.3) measures the uncertainty about the past life given that the item has been found failing at time t and can also be interpreted as the entropy of the inactivity time.

Even though a lot of works have been carried out in the univariate case on the past entropy function in reference to the residual or future lifetimes, very few works were seen in higher dimensions. It is therefore of interest to study the reliability aspects of the multi component system where the lifetimes of individual components are assumed to be dependent on each other. Hence in such situations, we have to employ multivariate lifetime distributions such that the reliability characteristics in the univariate case has to be extended to the corresponding multivariate set up. Accordingly, Ebrahimi et al. (2007) introduced a measure of uncertainty for the bivariate random variable, known as joint residual entropy and is defined as

$$H(f, t_1, t_2) = -\int_{t_1}^{\infty} \int_{t_2}^{\infty} \frac{f(x_1, x_2)}{\overline{F}(t_1, t_2)} \log \frac{f(x_1, x_2)}{\overline{F}(t_1, t_2)} dx_1 dx_2. \tag{1.4}$$

The residual entropy (1.4) is the direct extension of (1.2) to the bivariate random variable and it measures the uncertainty of the remaining lifetimes of the random variable when the ages of components are t_1 and t_2. The authors studied several important properties such as monotonicity of the residual entropy of a system, transformations that preserve the monotonicity and the order of entropies between two systems. The results also include the study of information properties of well-known bivariate lifetime models discussed in Balakrishnan and Lai (2009) such as Marshall Olkin, McKay bivariate gamma, bivariate Gumbel model, BEC model, and bivariate gamma.

However, when we consider bivariate measures, it should be noted that measurement based on one component is not affected by the missing or unreliable data on the other component and hence it is necessary to consider component wise measures subject to the condition that both of the components exceed the threshold value respectively. Such a measure will be more reliable as the unreliable data are omitted. With this motivation, Rajesh et al. (2009) proposed an alternative measure of uncertainty for the bivariate random variable, known as bivariate vector valued residual entropy and is defined as

$$H(X, t_1, t_2) = (H_1(X_1, t_1, t_2), H_2(X_2, t_1, t_2)), \tag{1.5}$$

where $H_1(X_1, t_1, t_2)$ and $H_2(X_2, t_1, t_2)$ are given respectively as

$$H_i\left(X_i, t_i, t_j\right) = -\int_{t_i}^{\infty} \frac{f\left(x_i \mid X_j > t_j\right)}{\overline{F}\left(t_i \mid X_j > t_j\right)} \log \frac{f\left(x_i \mid X_j > t_j\right)}{\overline{F}\left(t_i \mid X_j > t_j\right)} dx_i dx_j, i \neq j = 1, 2.$$

Similarly, Abdul-Sathar et al. (2009) proposed a more generalized measure of uncertainty namely bivariate vector valued generalized residual entropy and is defined as

$$H^{\beta}(X, t_1, t_2) = \left(H_1^{\beta}(X, t_1, t_2), H_2^{\beta}(X, t_1, t_2)\right), \tag{1.6}$$

where

$$H_i^{\beta}\left(X, t_i, t_j\right) = \frac{1}{1-\beta} \log \int_{t_i}^{\infty} \left(\frac{f\left(x_i \mid X_j > t_j\right)}{\overline{F}\left(t_i \mid X_j > t_j\right)} \right)^{\beta} dx_i, i \neq j = 1, 2.$$

The authors have discussed uniquely determined property of the measure and established several characterization results for some well-known bivaraite distribtuions such as, Gumbels bivariate exponential, bivariate Pareto and bivariate beta.

However, some inefficiencies inherited by (1.1) motivated various authors to introduce other suitable measures of information. Recently, Rao et al. (2004) introduced an alternative measure of uncertainty called cumulative residual entropy (CRE). Asadi and Zohrevand (2007) extended the concept of CRE for residual life, namely the dynamic cumulative residual entropy (DCRE) and studied various properties of it. Recently, Rajesh et al. (2014a), extended DCRE to bivariate setup and studied its different properties. Some well-known bivariate distributions were also characterized. Several generalizations to the concept of bivariate DCRE can also be found in Sunoj and Linu (2012) and Rajesh et al. (2014b). Recently, Kundu and Kundu (2017) have considered the extension of CPE, a dual measure of CRE to bivariate setup and obtained some of its properties. The generalized version of cumulative past entropy can be seen in Kundu and Kundu (2018).

Although the concepts of CRE and CPE have been extended to bivariate set-up, to the best of our knowledge, it seems that past entropy (1.3) has not been extended yet to bivariate set-up. In this paper our main aim is to extend DPE defined in (1.3) to bivariate set-up and study its useful various properties in reliability.

The rest of the paper is organized as follows: in section 2, we provide two definitions of bivariate past entropy and discussed the relationships of this measure with some well-known reliability measures. We have also computed the measures for some bivariate lifetime models. In section 3, we have derived several characterization results for some bivariate lifetime models. In section 4, we define new classes of life distributions based on the proposed measure and study various properties of the new classes. We also proposed a non-parametric kernel estimator for the proposed measure and illustrated the performance of the estimator using a numerical data in section 5.

2 Definition and Properties

In this section, we look into the problem of extending (1.3) to the bivariate setup. One of the main problems encountered while extending a univariate concept to the higher dimensions is that it cannot be done in a unique way. Accordingly, several extensions are possible for (1.3) in the bivariate set-up. A direct extension of (1.3) to the bivariate set-up is given below.

Definition 2.1. Let $X = (X_1, X_2)$ be an absolutely continuous random vector with joint distribution function $F(x_1, x_2)$,where, for some real numbers a_1, b_1 and a_2, b_2 the support of (X_1, X_2) is $(a_1, b_1) \times (a_2, b_2)$, then bivariate DPE is defined as

$$\overline{H}(X; t_1, t_2) = -\int_{a_1}^{t_1}\int_{a_2}^{t_2} \frac{f(x_1, x_2)}{F(t_1, t_2)} \log \frac{f(x_1, x_2)}{F(t_1, t_2)} dx_2 dx_1, \tag{2.1}$$

where $t_1 \leq b_1$ and $t_2 \leq b_2$. Alternatively, it can be shown that

$$\begin{aligned}\overline{H}(X; t_1, t_2) &= \log F(t_1, t_2) - \tfrac{1}{F(t_1,t_2)} \\ &\times \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(x_1, x_2) \log f(x_1, x_2) dx_2 dx_1.\end{aligned}$$

Differentiating (2.1) with respect to t_1 and t_2 on both sides and simplifying, we have

$$\begin{aligned}\frac{\partial^2 \overline{H}(X; t_1, t_2)}{\partial t_1 \partial t_2} &= \overline{h}(X; t_1, t_2)\left(1 - \overline{H}(X; t_1, t_2) - \log \overline{h}(X; t_1, t_2)\right) \\ &- \left[\overline{h}_1(X_1; t_1, t_2)\tfrac{\partial \overline{H}(X;t_1,t_2)}{\partial t_2} + \overline{h}_2(X_2; t_1, t_2)\tfrac{\partial \overline{H}(X;t_1,t_2)}{\partial t_1}\right] \\ &+ \overline{h}_1(X_1; t_1, t_2)\overline{h}_2(X_2; t_1, t_2),\end{aligned}$$

where $\overline{h}(X; t_1, t_2) = \frac{f(t_1,t_2)}{F(t_1,t_2)}$ is the bivariate reversed hazard rate proposed by Bismi (2005), $\overline{h_i}(X_i; t_1, t_2) = \frac{\partial}{\partial t_i} \log F(t_1, t_2), i = 1, 2$ are the components of the vector valued reversed hazard rate proposed by Roy (2002). Given that at times t_1 and t_2, the components are found to be down, $\overline{H}(X; t_1, t_2)$ measures the uncertainty about the past lifetimes of the components. The next theorem shows that (2.1) is not invariant under non-singular transformations.

Theorem 2.1. *Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be bivariate random vectors. For a differentiable one-to-one transformations, $Y_i = \phi_i(X_i), i = 1, 2,$*

$$\overline{H}(Y;\phi_1(t_1),\phi_2(t_2)) = \overline{H}(X;\phi_1^{-1}(t_1),\phi_2^{-1}(t_2)) \\ - E\left[\log J(X_1,X_2)|X_1 < t_1, X_2 < t_2\right],$$

where $J(X_1,X_2) = \left|\frac{\partial}{\partial x_1}\phi_1(x_1)\frac{\partial}{\partial x_2}\phi_2(x_2)\right|$ *is the absolute value of the Jacobian of transformation.*

It is to be noted that (2.1) does not determine the joint distribution function uniquely and has only limited applications. Hence, an alternative measure in the bivariate set-up is proposed here. Analogous to the vector valued residual entropy function of Rajesh et al. (2009), we have given below an alternative definition of the vector valued past entropy function.

Definition 2.2. If $X = (X_1, X_2)$ denotes the random vector in the support $(a_1, b_1) \times (a_2, b_2)$, admitting an absolutely continuous distribution function, then vector valued past entropy function is defined as

$$\underline{H}(X;t_1,t_2) = \left(\overline{H}_1(X_1;t_1,t_2), \overline{H}_2(X_2;t_1,t_2)\right), \tag{2.2}$$

where

$$\overline{H}_1(X_1;t_1,t_2) = -\int_{a_1}^{t_1} \frac{\frac{\partial}{\partial x_1}F(x_1,t_2)}{F(t_1,t_2)}\log\left(\frac{\frac{\partial}{\partial x_1}F(x_1,t_2)}{F(t_1,t_2)}\right)dx_1, \tag{2.3}$$

and

$$\overline{H}_2(X_2;t_1,t_2) = -\int_{a_2}^{t_2} \frac{\frac{\partial}{\partial x_2}F(t_1,x_2)}{F(t_1,t_2)}\log\left(\frac{\frac{\partial}{\partial x_2}F(t_1,x_2)}{F(t_1,t_2)}\right)dx_2, \tag{2.4}$$

where $t_1 \leq b_1$ and $t_2 \leq b_2$. It is to be noted that, $\overline{H}_i(X_i;t_1,t_2), i = 1,2$ denote the components of (2.2) and can be interpreted as the marginal past entropy functions of the conditional random variable $(X_i|X_1 < t_1, X_2 < t_2)$. In other words, if the random vector X denotes the lifetimes of components in a two component system, then (2.3) and (2.4) measure the uncertainty contained in the conditional distributions of X_i subject to the condition that the failure of first component has been occured at any time during $(0,t_1)$ and the second, during $(0,t_2)$.

Assume X_i, $i = 1,2$ are left-tail decreasing in X_j and if the univariate marginal distribution function $F_j(x_j)$, $i \neq j = 1,2$, have decreasing mean inactivity lifetime, then it can be easily shown that past entropy function of X_j given $X_i < t_i$, $i \neq j = 1,2$ does not exceed marginal past entropy function of X_j. That is

$$\overline{H}_j(X_j;t_1,t_2) \leq \overline{H}_j(X_j;t_j),$$

where

$$\overline{H}_j(X_j;t_j) = -\int_0^{t_j} \frac{f_j(x_j)}{F_j(t_j)} \log \frac{f_j(x_j)}{F_j(t_j)} dx_j,$$

are the univariate past entropy functions of X_j, $j = 1, 2$.

From (2.3) and (2.4), we have

$$\overline{H}_1(X_1;t_1,t_2) = 1 - \int_{a_1}^{t_1} \frac{\frac{\partial}{\partial x_1} F(x_1,t_2)}{F(t_1,t_2)} \log \overline{h}_1(x_1;t_1,t_2)\, dx_1, \tag{2.5}$$

and

$$\overline{H}_2(X_2;t_1,t_2) = 1 - \int_{a_2}^{t_2} \frac{\frac{\partial}{\partial x_2} F(t_1,x_2)}{F(t_1,t_2)} \log \overline{h}_2(x_2;t_1,t_2)\, dx_2, \tag{2.6}$$

where, $\overline{h_i}(X_i;t_1,t_2) = \frac{\partial}{\partial t_i} \log F(t_1,t_2)$, $i = 1, 2$ are the components of the vector valued reversed hazard rate proposed by Roy (2002). Differentiating (2.5) with respect to t_1, we get

$$\frac{\partial}{\partial t_1} \overline{H}_1(X_1;t_1,t_2) = \overline{h}_1(X_1;t_1,t_2)\left(1 - \overline{H}_1(X_1;t_1,t_2) - \log \overline{h}_1(X_1;t_1,t_2)\right). \tag{2.7}$$

Similarly from (2.6), we also get

$$\frac{\partial}{\partial t_2} \overline{H}_2(X_2;t_1,t_2) = \overline{h}_2(X_2;t_1,t_2)\left(1 - \overline{H}_2(X_2;t_1,t_2) - \log \overline{h}_2(X_2;t_1,t_2)\right). \tag{2.8}$$

If $X = (X_1, X_2)$ denote the random vector in the support $(a_1, b_1) \times (a_2, b_2)$, admitting an absolutely continuous distribution function $F(x_1, x_2)$, the bivariate reversed mean residual life function is defined as a vector,

$$\overline{m}(X;t_1,t_2) = \left(\overline{m}_1(X_1;t_1,t_2), \overline{m}_2(X_2;t_1,t_2)\right),$$

where

$$\overline{m}_i(X_i;t_1,t_2) = E\left[t_i - X_i | X_1 < t_1, X_2 < t_2\right], i = 1, 2; t_i > 0.$$

For $i = 1$, we have

$$\overline{m}_1\left(X_1; t_1, t_2\right) = \frac{1}{F(t_1,t_2)} \int\limits_{a_1}^{t_1} F\left(x_1, t_2\right) dx_1. \tag{2.9}$$

(2.9) represents the expected inactivity time of the first component under the assumption that both the components were failed before times t_1 and t_2 respectively. In the similar way we can interpret the other component too.

The fundamental relationship between bivariate reversed hazard rate and reversed mean residual life function is given by

$$\overline{h}_i\left(X_i; t_1, t_2\right) = \frac{1 - \frac{\partial}{\partial t_i} \overline{m}_i\left(X_i; t_1, t_2\right)}{\overline{m}_i\left(X_i; t_1, t_2\right)}, i = 1, 2. \tag{2.10}$$

The computation of vector valued past entropy function, defined above, for some well-known bivariate distributions are given through the following examples.

Example 2.1 Let X be distributed as bivariate uniform distribution with joint distribution function

$$F\left(t_1, t_2\right) = \frac{t_1 t_2}{bd}, 0 \leq t_1 \leq b, 0 \leq t_2 \leq d.$$

Direct integration of (2.3) and (2.4) gives

$$\overline{H}_i\left(X_i; t_1, t_2\right) = \log t_i, i = 1, 2.$$

Example 2.2 Let X be distributed as bivariate uniform with joint distribution function,

$$F\left(t_1, t_2\right) = t_1^{1+\theta \log t_2} t_2, 0 < t_1, t_2 < 1.$$

Using (2.3) and (2.4) it follows from straight forward calculations that

$$\overline{H}_i\left(X_i; t_1, t_2\right) = \frac{\theta \log t_j}{1 + \theta \log t_j} + \log\left[\frac{t_i}{1 + \theta \log t_j}\right], i \neq j = 1, 2.$$

Example 2.3 Let X follows bivariate power distribution with distribution function,

$$F\left(t_1, t_2\right) = t_2^{2k_2 - 1} t_1^{2k_1 - 1 + \theta \log t_2}, 0 < t_1, t_2 < 1, k_1, k_2 > 0.$$

Direct integration using (2.3) and (2.4) gives

$$\overline{H_i}\left(X_i;t_1,t_2\right)=\frac{2k_i-2}{2k_i-1+\theta\log t_j}+\log\left[\frac{t_i}{\left(2k_i-1+\theta\log t_j\right)\left(t_i^{2k_i-2+\theta\log t_j}\right)}\right],$$
$$i\neq j=1,2.$$

Example 2.4 Let X follows bivariate logistic distribution with distribution function,

$$F\left(t_1,t_2\right)=\left(1+e^{-t_1}+e^{-t_2}\right)^{-1},-\infty<t_1,t_2<\infty. \tag{2.11}$$

The component wise residual past entropy defined in (2.3) and (2.4) is given by,

$$\overline{H}_i\left(X_i;t_1,t_2\right)=1+e^{-t_i}\left(1+e^{-t_i}+e^{-t_j}\right)^{-2},i\neq j=1,2.$$

Example 2.5 Let X follow bivariate extreme value distribution with distribution function,

$$F\left(t_1,t_2\right)=\exp\left[-\exp\left[-t_1\right]-\exp\left[-t_2\right]\right],-\infty<t_1<\infty, \\ -\infty<t_1<\infty. \tag{2.12}$$

The component wise residual past entropy defined in (2.3) and (2.4) is given by,

$$\overline{H}_i\left(X_i;t_1,t_2\right)=1+t_i-e^{-t_i}+e^{-1+t_i},i\neq j=1,2.$$

3 Characterizations

In this section, we give characterization of distributions in terms of bivariate past entropy function defined in (2.2). The following theorem characterizes the vector valued past entropy function in the sense that under certain conditions the vector valued past entropy function uniquely determines the corresponding distribution function.

Theorem 3.1. *If* $\overline{H}_i(X_i;t_1,t_2)$ *is non-decreasing in* $t_i, i=1,2$, *then* $\underline{H}(X;t_1,t_2)$ *defined in* (2.2) *uniquely determines the corresponding* $F(t_1,t_2)$.

Proof. Using equations (2.7) and (2.8), for $t_1, t_2 > 0$, we have $\overline{h}_1(X_1; t_1, t_2)$ and $\overline{h}_2(X_2; t_1, t_2)$ are respectively the positive solutions of the following equations,

$$g(y_1) = y_1 \left[1 - \overline{H}_1(X_1; t_1, t_2) - \log y_1\right] - \frac{\partial}{\partial t_1} \overline{H}_1(X_1; t_1, t_2) = 0,$$

and

$$g(y_2) = y_2 \left[1 - \overline{H}_2(X_2; t_1, t_2) - \log y_2\right] - \frac{\partial}{\partial t_2} \overline{H}_2(X_2; t_1, t_2) = 0.$$

Proceeding in similar arguments as Belzunce et al. (2004), the above two equations have positive solutions $\overline{h}_1(X_1; t_1, t_2)$ and $\overline{h}_2(X_2; t_1, t_2)$ respectively for all t_1 and t_2. Hence $(\overline{H}_1(X_1; t_1, t_2), \overline{H}_2(X_2; t_1, t_2))$ uniquely determines $(\overline{h}_1(X_1; t_1, t_2), \overline{h}_2(X_2; t_1, t_2))$ and thereby we get the desired result. □

The following theorem, characterizes bivariate reversed exponential distribution by using the local constancy of vector valued past entropy function with support $a_i = -\infty$ and $b_i < \infty, i = 1, 2$.

Theorem 3.2. *If X is a random vector in the support $(-\infty, b_1) \times (-\infty, b_2)$ with $b_i < \infty$, possessing absolutely continuous distribution function, then*

$$\overline{H}_i(X_i; t_1, t_2) + \log\left(c_i + c_3(t_j - b_j)\right) = 1, i, j = 1, 2, i \neq j, \tag{3.1}$$

if and only if

$$F(t_1, t_2) = \exp\left[c_1(t_1 - b_1) + c_2(t_2 - b_2) + c_3(t_1 - b_1)(t_2 - b_2)\right], c_i > 0.$$

Proof. For $i = 1$, when (3.1) holds in the light of (2.5), we have

$$F(t_1, t_2) \log(c_1 + c_3(t_2 - b_2)) = \int_{a_1}^{t_1} \frac{\partial}{\partial x_1} F(x_1, t_2) \log \overline{h}_1(x_1; t_1, t_2)\, dx_1. \tag{3.2}$$

Differentiating (3.2) with respect to t_1 and rearranging the terms, we get

$$\overline{h}_1(X_1; t_1, t_2) = c_1 + c_3(t_2 - b_2).$$

The rest of the proof follows from Nair and Asha (2008).

□

The following theorem is a characterization of bivariate logistic distribution, using the relation between bivariate reversed mean residual life function and vector valued past entropy with support $a_i = -\infty$ and $b_i < \infty, i = 1, 2$.

Theorem 3.3. *If $X = (X_1, X_2)$ is a random vector in the support R^2 with absolutely continuous distribution function $F(t_1, t_2)$, and with $E(X_i) < \infty, i = 1, 2$. Then the relation*

$$\overline{H}_i\left(X_i; t_1, t_2\right) - \frac{\partial}{\partial t_i}\overline{m}_i\left(X_i; t_1, t_2\right) = 1 \tag{3.3}$$

holds if and only if X has the bivariate logistic distribution defined in (2.11).

Proof. For $i = 1$, when (3.3) holds in the light of (2.9), we have

$$\overline{H}_1\left(X_1; t_1, t_2\right) + \overline{h}_1\left(X_1; t_1, t_2\right)\overline{m}_1\left(X_1; t_1, t_2\right) = 2. \tag{3.4}$$

Differentiating (3.4) with respect to t_1 on both sides and using (2.7), we get

$$\frac{\partial}{\partial t_1}\log\int_{a_1}^{t_1} F\left(x_1, t_2\right) dx_1 = \frac{\partial}{\partial t_1}\left(\log\left(-\log\overline{h}_1\left(X_1; t_1, t_2\right)\right)\right). \tag{3.5}$$

Integrating (3.5) with respect to t_1, we get

$$\frac{\partial}{\partial t_1}F\left(t_1, t_2\right) = K_1 F\left(t_1, t_2\right)\left(K_2 - F\left(t_1, t_2\right)\right),$$

where K_1 and K_2 are suitable constants of integration. Solving this differential equation, the required result is founded. For the if part, suppose X follows (2.11), for $i = 1$, we have

$$\overline{h}_1(X_1; t_1, t_2) = e^{-t_1}\left(1 + e^{-t_1} + e^{-t_2}\right)^{-1},$$

$$\overline{m}_1\left(X_1; t_1, t_2\right) = \left(1 + e^{-t_1} + e^{-t_2}\right)\left(1 + e^{-t_2}\right)^{-1}\left(t_1 + \log\left(1 + e^{-t_1} + e^{-t_2}\right)\right),$$

and

$$\overline{H}_1\left(X_1; t_1, t_2\right) = 2 + t_1 - \overline{m}_1\left(X_1; t_1, t_2\right) + \log\left(1 + e^{-t_1} + e^{-t_2}\right). \tag{3.6}$$

Using (2.10), we get

$$\begin{aligned} \tfrac{\partial\overline{m}_1(X_1; t_1, t_2)}{\partial t_1} &= 1 - \overline{h}_1\left(X_1; t_1, t_2\right)\overline{m}_1\left(X_1; t_1, t_2\right) \\ &= 1 - e^{-t_1}\left(1 + e^{-t_2}\right)^{-1}\left(t_1 + \log\left(1 + e^{-t_1} + e^{-t_2}\right)\right). \end{aligned} \tag{3.7}$$

Using (3.6) and (3.7), we get

$$\overline{H}_1(X_1; t_1, t_2) - \frac{\partial}{\partial t_1}\overline{m}_1(X_1; t_1, t_2) = 1.$$

Proceeding along the same way we get the result for $i = 2$. □

The following theorem is a characterization of bivariate extreme value distribution, using the relation between bivariate reversed mean residual life function and vector valued past entropy function with support $a_i = -\infty$ and $b_i < \infty, i = 1, 2$.

Theorem 3.4. *If $X = (X_1, X_2)$ is a random vector in the support R^2 with absolutely continuous distribution function $F(t_1, t_2)$, and with $E(X_i) < \infty, i = 1, 2$. Then the relation*

$$\overline{H}_i(X_i; t_1, t_2) + \overline{m}_i(X_i; t_1, t_2) = 1 + t_i, i = 1, 2, \tag{3.8}$$

holds if and only if X has the bivariate extreme value distribution defined in (2.12).

Proof. When (3.8) holds with $i = 1$, we get

$$\overline{H}_1(X_1; t_1, t_2) + \overline{m}_1(X_1; t_1, t_2) = 1 + t_1.$$

Using (2.9), we get

$$\int_{-\infty}^{t_1} F(x_1, t_2)\, dx_1 = F(t_1, t_2)\Big((1 + t_1) - \overline{H}_1(X_1; t_1, t_2)\Big).$$

Differentiating the above equation with respect to t_1 on both sides, we get

$$\begin{aligned} F(t_1, t_2) = F(t_1, t_2)&\Big(1 - \tfrac{\partial}{\partial t_1}\overline{H}_1(X_1; t_1, t_2)\Big) \\ &+ \tfrac{\partial}{\partial t_1}F(t_1, t_2)\Big(1 + t_1 - \overline{H}_1(X_1; t_1, t_2)\Big). \end{aligned} \tag{3.9}$$

Simplifying (3.9), we get

$$\frac{\partial}{\partial t_1}\overline{H}_1(X_1; t_1, t_2) = \overline{h}_1(X_1; t_1, t_2)\Big(1 - \overline{H}_1(X_1; t_1, t_2) - \log \overline{h}_1(X_1; t_1, t_2)\Big). \tag{3.10}$$

Dividing (3.9) by $F(t_1, t_2)$, we have

$$1 = 1 - \frac{\partial}{\partial t_1}\overline{H}_1(X_1; t_1, t_2) + \overline{h}_1(X_1; t_1, t_2)\Big(1 + t_1 - \overline{H}_1(X_1; t_1, t_2)\Big). \tag{3.11}$$

Substituting (3.10) in (3.11) and simplifying, we get

$$\overline{h}_1 \left(X_1; t_1, t_2\right) = e^{-t_1}.$$

Similarly for $i = 2$, we get

$$\overline{h}_2 \left(X_2; t_1, t_2\right) = e^{-t_2}.$$

Hence, we have

$$\log F\left(t_1, t_2\right) = -e^{-t_1} + K_1\left(t_2\right).$$

Similarly for $i = 2$, we obtain

$$\log F\left(t_1, t_2\right) = -e^{-t_2} + K_2\left(t_1\right).$$

After performing some algebraic calculations, the above expression reduces to

$$K_i\left(t_j\right) = -e^{-t_j}, i \neq j = 1, 2.$$

Therefore,

$$F(t_1, t_2) = exp(-e^{-t_1} - e^{-t_2}).$$

To prove the if part, suppose X follows (2.12), we have

$$\frac{\partial}{\partial t_1} F\left(t_1, t_2\right) = e^{-e^{-t_2}} \frac{\partial}{\partial t_1}\left(e^{-e^{-t_1}}\right) = e^{-t_1}\left(e^{-e^{-t_1} - e^{-t_2}}\right) = e^{-t_1} F\left(t_1, t_2\right).$$

Hence, we get

$$\overline{h}_1 \left(X_1; t_1, t_2\right) = \frac{\frac{\partial}{\partial t_1} F\left(t_1, t_2\right)}{F\left(t_1, t_2\right)} = e^{-t_1}.$$

Also, it is obvious that

$$\begin{aligned}
\overline{H}_1\left(X_1; t_1, t_2\right) &= 1 - \int_{a_1}^{t_1} \frac{\frac{\partial}{\partial x_1} F(x_1, t_2)}{F(t_1, t_2)} \log \overline{h}_1\left(x_1; t_1, t_2\right) dx_1 \\
&= 1 - \int_{a_1}^{t_1} \frac{\frac{\partial}{\partial x_1} F(x_1, t_2)}{F(t_1, t_2)} \log\left(e^{-x_1}\right) dx_1 \\
&= 1 + t_1 - \frac{1}{F(t_1, t_2)} \int_{a_1}^{t_1} F\left(x_1, t_2\right) dx_1 \\
&= 1 + t_1 - \overline{m}_1\left(t_1, t_2\right).
\end{aligned}$$

Hence, we have

$$\overline{H}_1\left(X_1;t_1,t_2\right)+\overline{m}_1\left(t_1,t_2\right)=1+t_1.$$

Proceeding along the same way, we can also get the result for $i=2$. $\square$

In the context of analysis of lifetime data, Nair and Rajesh (2000) introduced the concept of geometric vitality function based on the geometric mean of the residual lifetime. Abdul-Sathar et al. (2010) extends the same to the bivariate set-up in the residual life, namely vector valued geometric vitality function. The authors have characterised some important bivariate models such as bivariate type-I distribution, Gumbels bivariate exponential distribution, bivariate Pareto type-II, bivariate finite range distribution and bivariate Weibull distribution using this measure. In a similar approach, we define the vector valued reversed geometric vitality function for the random vector X considered in Definition 2.2, as

$$\log G(X;t_1,t_2)=(\log G_1(X_1;t_1,t_2),\log G_2(X_2;t_1,t_2)),$$

where

$$\log G_1\left(X_1;t_1,t_2\right)=\int_{a_1}^{t_1}\log x_1\frac{\frac{\partial F(x_1,t_2)}{\partial x_1}}{F\left(t_1,t_2\right)}dx_1,$$

and

$$\log G_2\left(X_2;t_1,t_2\right)=\int_{a_2}^{t_2}\log x_2\frac{\frac{\partial F(t_1,x_2)}{\partial x_2}}{F\left(t_1,t_2\right)}dx_2.$$

Let $X=(X_1,X_2)$ be a bivariate random vector, representing the lifetimes of a two component system, then $G\left(X_2;t_1,t_2\right)$ defines the geometric mean of the components of the random vector X subject to the condition that the first component has failed at any time during (a_1,t_1) and the second one during (a_2,t_2). In survival analysis and reliability engineering, this measure play a significant role in studying the various characteristics of a system or component.

In the following theorem, we establish a relationship of vector valued past entropy with the vector valued bivariate reversed geometric vitality function defined above. The proof is straight forward and hence is omitted.

Theorem 3.5. *If $X=(X_1,X_2)$ denote the random vector in the support R^2 with absolutely continuous distribution function $F(t_1,t_2)$, and with $E(X_i)<\infty, i=1,2$. Then the relation*

$$\overline{H}_i\left(X_i;t_1,t_2\right)+\log G_i\left(X_i;t_1,t_2\right)=k_i\left(t_j\right), i\neq j=1,2,$$

holds if and only if X has the bivariate power distribution given in example 2.3.

4 New Class of Bivariate Life Time Distributions

In this section, we define stochastic orders between two bivariate random vectors based on bivariate vector valued past entropy function. In the following, we recall some of the stochastic orders discussed by Shaked and Shanthikumar (2007), which will be useful in the sequel.

Definition 4.1 F is said to have bivariate decreasing (increasing) reversed hazard rate BDRHR (BIRHR) if $\overline{h}_i(X_i; t_1, t_2)$ is decreasing (increasing) in $t_i, i = 1, 2$.

Definition 4.2 The random variables X_1 and X_2 are said to be left corner set increasing (LCSI) if $P(X_1 < x_1, X_2 < x_2 | X_1 < t_1, X_2 < t_2)$ is increasing in t_1 and t_2 for every choice of x_1 and x_2.

Definition 4.3 Let X and Y be two non-negative random variables with cumulative distribution functions F and G and with probability density functions f and g respectively, then X is said to be smaller than Y in the dispersive order, denoted by $Y_i \overset{D}{\leq} X_i$ if $G^{-1}F(x) - x$ is increasing in $x \geq 0$.

Di Crescenzo and Longobardi (2002) introduced the univariate increasing uncertainty of life class (IUL), Nanda and Paul (2006) has given an interesting result that decreasing reversed hazard rate (DRHR) implies, IUL. In the following, we define stochastic orders between two bivariate random vectors based on bivariate vector valued past entropy function.

Definition 4.4 F is said to have bivariate decreasing (increasing) uncertainty past life BDUPL (BIUPL) if $\overline{H}_i(X_i; t_1, t_2)$ is decreasing (increasing) in $t_i, i = 1, 2$.

The following theorem gives an upper bound to bivariate past entropy in terms of bivariate reversed hazard rate, which is the extension of proposition 2.3 in Di Crescenzo and Longobardi (2002).

Theorem 4.1. *If F is BIUPL,*

$$\overline{H}_i\left(X_i; t_1, t_2\right) \leq 1 - \log \overline{h}_i\left(X_i; t_1, t_2\right).$$

Proof. The proof follows from using (2.7) and (2.8) respectively. □

In the following theorem, we extend the Proposition 2.2 of Di Crescenzo and Longobardi (2002).

Theorem 4.2. *If F is BDRHR, then F is BIUPL.*

Proof. From (2.5) and (2.7) we have,

$$\frac{\partial}{\partial t_1}\overline{H_1}(X_1;t_1,t_2) = \overline{h_1}(X_1;t_1,t_2) \times\left[\frac{1}{F(t_1,t_2)}\int_0^{t_1}\frac{\partial}{\partial x_1}F(x_1,t_2)\log\overline{h_1}(x_1;t_1,t_2) - \log\overline{h_1}(X_1;t_1,t_2)\right].$$

If F is BDRHR it is obvious that,

$$\overline{h_1}(x_1;t_1,t_2) < \overline{h_1}(X_1;t_1,t_2)\,, for\ x_1 < t_1,$$

Then,

$$\begin{aligned}\frac{\partial}{\partial t_1}\overline{H_1}(X_1;t_1,t_2) &\geq \overline{h_1}(X_1;t_1,t_2)\\ &\times\left[\frac{\log\overline{h_1}(X_1;t_1,t_2)}{F(t_1,t_2)}\int_{a_1}^{t_1}\frac{\partial}{\partial x_1}F(x_1,t_2)\,dx_1 - \log\overline{h_1}(X_1;t_1,t_2)\right]\\ &= 0.\end{aligned}$$

Hence F is BIUPL. □

In Figures 1-4, we plot vector valued residual past entropy for the distributions given in Examples 2.1-2.4. Intuitively, the graphs identify the BIUPL (BDUPL) classes of life distributions. Figures 1,4 are examples of BIUPL, classes and Figures 2,3 are examples of BDUPL classes.

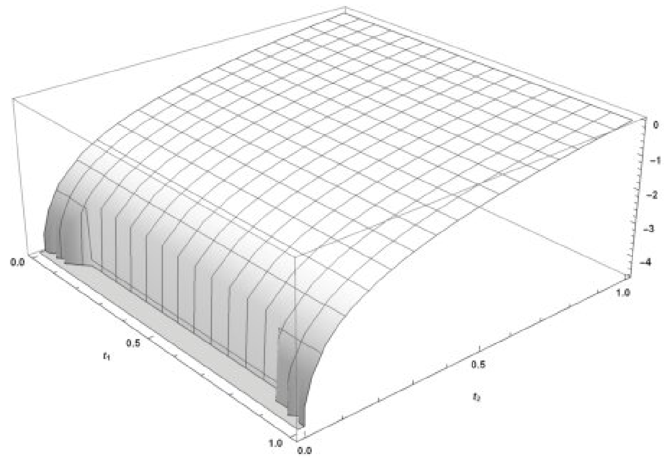

Figure 1: Graph of vector valued residual past entropy of bivariate uniform distribution given in Example 2.1 with $b = 1$ and $d = 0.6$ for BIUPL.

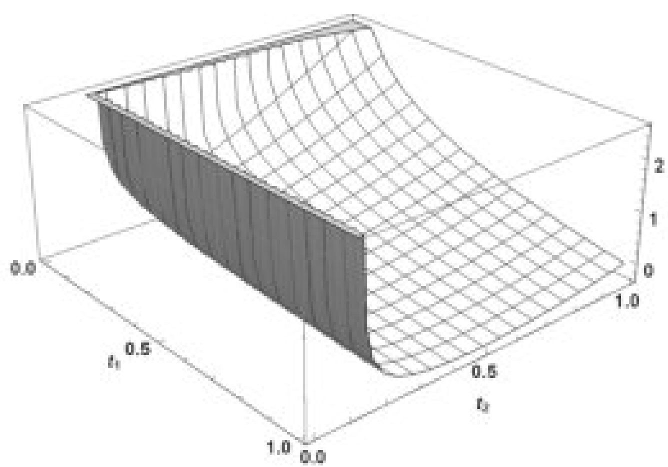

Figure 2: Graph of vector valued residual past entropy of bivariate uniform distribution of Example 2.2. with $\theta = .5$ for BDUPL.

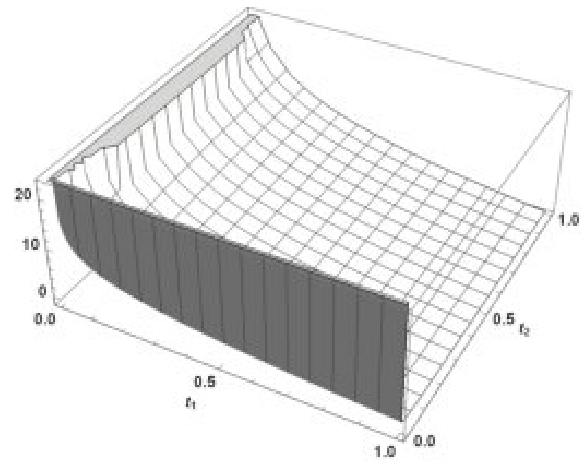

Figure 3: Graph of vector valued past entropy of bivariate power distribution with $k_1 = 5$ with $\theta < .5$ given in Example 2.3 for BDUPL.

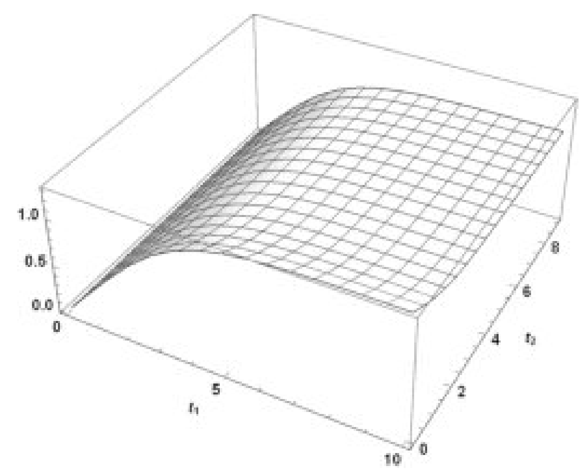

Figure 4: Graph of vector valued residual past entropy of bivariate logistic distribution given in (2.11) for BIUPL.

Ebrahimi et al. (2007) used the positive dependence in terms of right corner set increasing (RCSI) to identify the transformation that preserves monotonicity of the bivariate residual entropy. Similarly, we use left corner set increasing (LCSI) to the preserve monotonicity of the bivariate past entropy function respectively.

Theorem 4.3. *Let X_1 and X_2 be LCSI and $Y_i = \phi_i(X_i), i = 1,2$ be the non-negative increasing transformations. Assume $\phi_i(X_i), i = 1,2$ are concave (convex) and $\overline{H}_i(X_i; t_1, t_2), i = 1,2$ to be decreasing (increasing) functions of $t_i, i = 1,2$ then $\overline{H}_i\left(Y_i; \phi_1(t_1), \phi_2(t_2)\right)$ is decreasing (increasing) in $\phi_i(t_i), i = 1,2$.*

Proof. Using the definition of bivariate past entropy for Y_i,

$$\overline{H}_i\left(Y_i; \phi_1(t_1), \phi_2(t_2)\right) = \overline{H}_i(X_i; t_1, t_2) + E[\log \frac{\partial}{\partial x_i}\left(\phi_i^{-1}(X_i) | X_1 < t_1, X_2 < t_2\right)]. \quad (4.1)$$

In (4.1), it has been observed that the second term on the RHS is decreasing as X_1 and X_2 are LCSI and $\phi_i(X_i)$ are increasing and convex functions. Furthermore by incorporating the assumption of $\overline{H_i}(X_i; t_1, t_2)$ is decreasing one can easily obtain the desired result. Proceeding in the similar way, we can prove the remaining part also.

If $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ are the components of two systems, where $Y_i = \phi_i(X_i), i = 1,2$. Using the notion of dispersion ordering between X_i and Y_i, i=1,2 we show that this transformation either increases or decreases the uncertainty between these random variables. □

The following theorem gives sufficient condition for holding the relation between $\overline{H}_i(Y_i; t_1, t_2)$ and $\overline{H}_i(X_i; t_1, t_2)$.

Theorem 4.4. *(a)If $Y_i \overset{D}{\geq} X_i, i = 1,2$ and if $\overline{H}_i(X_i; t_1, t_2)$ is decreasing in $t_i, i = 1,2$, then $\overline{H}_i(Y_i; t_1, t_2) \geq \overline{H}_i(X_i; t_1, t_2)$. (b)If $Y_i \overset{D}{\leq} X_i, i = 1,2$ and if $\overline{H}_i(X_i; t_1, t_2)$ is decreasing in $t_i, i = 1,2$, then $\overline{H}_i(Y_i; t_1, t_2) \leq \overline{H}_i(X_i; t_1, t_2)$.*

Proof. (a)We have

$$\begin{aligned}\overline{H}_i(Y_i; t_1, t_2) &= \overline{H}_i\left(X_i; \phi_1^{-1}(t_1), \phi_2^{-1}(t_2)\right) \\ &\quad + E \log \tfrac{\partial}{\partial x_i}\left[\phi_i(X_i) | X_1 < \phi_1^{-1}(t_1), X_2 < \phi_2^{-1}(t_2)\right].\end{aligned}$$

If $Y_i \overset{D}{\geq} X_i \Rightarrow \phi_i(t_i) \geq 1 \Rightarrow t_i > \phi_i^{-1}(t_i), i = 1,2$
or

$$\begin{aligned}\overline{H}_i(Y_i; t_1, t_2) &\geq \overline{H}_i\left(X_i; \phi_1^{-1}(t_1), \phi_2^{-1}(t_2)\right) \\ &\geq \overline{H}_i(X_i; t_1, t_2)\, for\ t_i > \phi_i^{-1}(t_i).\end{aligned}$$

(b) Proof follows from similar arguments as in (a). □

Corollary 4.1 If $Y = a_iX_i + b_i, 0 \leq a_i \leq 1, i = 1, 2$ and if $\overline{H}_i(X_i; t_1, t_2)$ is decreasing in $t_i, i = 1, 2$, then $\overline{H}_i(Y_i; t_1, t_2) \geq \overline{H}_i(X_i; t_1, t_2)$ for $a_i \leq 1$, and if $\overline{H}_i(X_i; t_1, t_2)$ is increasing then $\overline{H}_i(Y_i; t_1, t_2) \leq \overline{H}_i(X_i; t_1, t_2)$.

The closure of bivariate past entropy ordering under increasing convex transformation is shown in the following theorem.

Theorem 4.5. *Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be two bivariate random vectors with distribution function $F(x_1, x_2)$, $F(y_1, y_2)$; reversed hazard rate $\overline{h}_i(X_i; t_1, t_2)$, $\overline{h}_i(Y_i; t_1, t_2)$ and past entropies $\overline{H}_i(X_i; t_1, t_2)$, $\overline{H}_i(Y_i; t_1, t_2)$ respectively. Let $\phi_i(X_i)$ and $\phi_i(Y_i)$, $i = 1, 2$ where $\phi_i(.), i = 1, 2$ are non-negative increasing convex functions,*
If (a)$\overline{h}_i(X_i; t_1, t_2) \leq \overline{h}_i(Y_i; t_1, t_2), \forall t_1, t_2$,
and (b)$\overline{H}_i(X_i; t_1, t_2) \leq \overline{H}_i(Y_i; t_1, t_2)$, then
$\overline{H}_i\left(\phi_i(X_i); \phi_1^{-1}(t_1), \phi_2^{-1}(t_2)\right) \leq \overline{H}_i\left(\phi_i(Y_i); \phi_1^{-1}(t_1), \phi_2^{-1}(t_2)\right)$.

Proof. Using (4.1), we have

$$\begin{aligned}
&\overline{H}_i\left(\phi_i(Y_i); \phi_1^{-1}(t_1), \phi_2^{-1}(t_2)\right) - \overline{H}_i\left(\phi_i(X_i); \phi_1^{-1}(t_1), \phi_2^{-1}(t_2)\right) \\
&\quad = \overline{H}_i(Y_i; t_1, t_2) - \overline{H}_i(X_i; t_1, t_2) \\
&\quad + \begin{bmatrix} E\left(\log \frac{\partial}{\partial y_i}\phi_i^{-1}(Y_i) \,|Y_1 < t_1, Y_2 < t_2\right) \\ -E\left(\log \frac{\partial}{\partial x_i}\phi_i^{-1}(X_i) \,|X_1 < t_1, X_2 < t_2\right) \end{bmatrix}.
\end{aligned}$$

If (a) holds, then conditional distribution function of $[Y_i|Y_1 < t_1, Y_2 < t_2]$ is stochastically larger than the conditional distribution function of $[X_i|X_1 < t_1, X_2 < t_2]$. If $\phi_i(.), i = 1, 2$ are non-negative convex and increasing then the terms inside the bracket are clearly non-negative and finally by using the condition (b) the result is direct. □

5 Non-parametric Estimation

Let $(X_{1i}, X_{2i}); i = 1, \ldots, n$ be n independent and identically distributed pair of lifetimes with joint distribution function $F(x_1, x_2)$, respectively. Based on these observations and using the kernel density $k_i(.), i = 1, 2$, a non-parametric estimate of $F(x_1, x_2)$ is defined as

$$\hat{F}(x_1, x_2) = \frac{1}{na_n^2} \sum_{j=1}^{n} K_1\left(\frac{x_1 - X_{1j}}{a_n}\right) K_2\left(\frac{x_2 - X_{2j}}{a_n}\right), \tag{5.1}$$

where

$$K_i(z) = a_n \int_0^z k_i(v)\,dv, i = 1, 2, \tag{5.2}$$

and $\{a_n\}$ is a non-increasing sequence of positive real numbers such that $a_n \to 0$ and $na_n \to \infty$, as $n \to \infty$. Also, kernel estimates of $\frac{\partial}{\partial x_1} F(x_1, x_2)$ and $\frac{\partial}{\partial x_2} F(x_1, x_2)$ are respectively given as

$$\hat{G}_1(x_1, x_2) = \frac{1}{na_n^2} \sum_{j=1}^{n} K_1\left(\frac{x_1 - X_{1j}}{a_n}\right) K_2\left(\frac{x_2 - X_{2j}}{a_n}\right), \tag{5.3}$$

$$\hat{G}_2(x_1, x_2) = \frac{1}{na_n^2} \sum_{j=1}^{n} K_2\left(\frac{x_2 - X_{2j}}{a_n}\right) K_1\left(\frac{x_1 - X_{1j}}{a_n}\right), \tag{5.4}$$

where $K_i(.), i = 1, 2$ are defined in (5.2). From (2.3) and (2.4), we propose a non-parametric kernel estimator for vector valued past entropy function as

$$\overset{\wedge}{\overline{H}}_1(X_1; t_1, t_2) = -\int_0^{t_1} \frac{\hat{G}_1(x_1, t_2)}{\hat{F}(t_1, t_2)} \log \frac{\hat{G}_1(x_1, t_2)}{\hat{F}(t_1, t_2)} dx_1, \tag{5.5}$$

and

$$\overset{\wedge}{\overline{H}}_2(X_2; t_1, t_2) = -\int_0^{t_1} \frac{\hat{G}_2(t_1, x_2)}{\hat{F}(t_1, t_2)} \log \frac{\hat{G}_2(t_1, x_2)}{\hat{F}(t_1, t_2)} dx_2, \tag{5.6}$$

where $\hat{F}(t_1, t_2)$ is given in (5.1), $\hat{G}_1(t_1, x_2)$ and $\hat{G}_2(x_1, t_2)$ are obtained from (5.3) and (5.4).

5.1 Numerical Illustration

In this section, we illustrate the usefulness of the proposed estimators given in (5.5) and (5.6). Consider the data-set reported by Kim and Kvam (2004), which consists of the failure times of 20 sample units from a system consisting of three components. We consider only the failure times of first two components. At each value of (t_1, t_2), we calculate the bias and the mean-squared error of $\overset{\wedge}{\overline{H}}_i(X_i; t_1, t_2)\, i = 1, 2$ using 100 bootstrap samples of size 20. Table 1 presents the bootstrap estimates of the bias and the mean-squared error for $\hat{\overline{H}}_i\left(X_i; t_1, t_2\right), i = 1, 2$. In column 2 of Table 1, we provide the absolute values of bias of $\hat{\overline{H}}_i\left(X_i; t_1, t_2\right), i = 1, 2$ and in column 3, we provide the mean-squared error for $\hat{\overline{H}}_i\left(X_i; t_1, t_2\right), i = 1, 2$.

Table 1: Bootstrap estimates of the bias and the mean-squared error for $\hat{\overline{H}}_i\left(X_i; t_1, t_2\right), i = 1, 2$.

Bias and MSE of $\hat{\overline{H}}_i\left(X_i; t_1, t_2\right), i = 1, 2$		
(t_1, t_2)	Bias	MSE
(2.50, 2.64)	(0.1971, 0.1684)	(0.0424, 0.0838)
(3.25, 3.28)	(0.1828, 0.3295)	(0.0369, 0.1461)
(2.89, 3.88)	(0.2149, 0.2672)	(0.0479, 0.1069)
(1.53, 2.08)	(0.1399, 0.1398)	(0.0238, 0.0423)
(1.30, 2.17)	(0.1243, 0.2176)	(0.0172, 0.0871)

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