Sequential-Based Approach for Estimating the Stress-Strength Reliability Parameter for Exponential Distribution

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Abstract. In this paper, two-stage and purely sequential estimation procedures are considered to construct fixed-width confidence intervals for the reliability parameter under the stress-strength model when the stress and strength are independent exponential random variables with different scale parameters. The exact distribution of the stopping rule under the purely sequential procedure is approximated using the law of large numbers and Monte Carlo integration. For the two-stage sequential procedure, explicit formulas for the distribution of the total sample size, the expected value and mean squared error of the maximum likelihood estimator of the reliability parameter under the stress-strength model are provided. Moreover, it is shown that both proposed sequential procedures are finite, and in exceptional cases, the exact distribution of stopping times is degenerate distribution at the initial sample size. The performances of the proposed methodologies are investigated with the help of simulations. Finally using real data, the procedures are clearly illustrated.

Keywords. Law of Large Numbers, Purely Sequential Sampling, Stopping Rule, Two-stage Sequential Sampling.

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1   Introduction

As mentioned in reliability literature, stress-strength models have been described as an assessment of the reliability of a component or system in terms of variables X indicating stress undertaken by the system and Y representing the system's strength used to endure the stress. Concerning this scenario, the system fails if the stress surpasses the strength. Thus, $R = P(X < Y)$ is the reliability of a system under the stress-strength model. Birnbaum (1956) has introduced the main idea associated with this model. Birnbaum and McCarty (1958) developed this idea using a numerical procedure. Stress-strength models have received notable attention for many years due to their applicability in diverse areas such as medicine, physics, genetics, mechanics as well as in quality control, aerospace engineering, and civil (see, e.g., Domma and Giordano (2013) and Chapter 7 in Kotz *et al.* (2003)).

The problem of estimating R has been widely studied under various distributional assumptions on stress and strength for a variety of data types such as complete data, censored data, and data with explanatory variables, using Monte-Carlo simulation, parametric, nonparametric and Bayesian approaches. For instance, when the independent random variables X and Y follow the two-parameter exponential distribution, Beg (1980) obtained uniformly minimum variance unbiased estimators (UMVUE) of R. Kundu and Gupta (2006) proposed the maximum likelihood estimator (MLE), two bootstrap confidence intervals and Bayesian estimate of R when the Weibull model holds for stress and strength. When stress and strength are normal or lognormal random variables, Chiodo (2014) discussed a Bayesian inference method for the estimation of the reliability parameter under the stress-strength model. Based on two independent samples from Weinman multivariate exponential distributions with unknown scale parameters, Cramer and Kamps (1997) obtained UMVUE of $P(X < Y)$ for both, unknown and known common location parameter, under type-II censoring. Awad *et al.* (1981) and Nadarajah and Kotz (2006) considered a bivariate exponential model for stress and strength.

When stress and strength follow exponential distributions, Tong (1974) derived minimum variance unbiased estimator (MVUE) of R, also several authors discussed evaluation of the mean square error (MSE) of MLE of R (see, e.g., Kelley *et al.* (1976), and Chao (1982)), Moreover Enis and Geisser (1971) applied Bayesian approach to estimate R. Mahmoudi *et al.* (2018) considered minimum risk sequential point estimation of R and utilized the expected sample size and risk of the sequential procedure. Comprehensive reviews of frequentist inference for stress-strength models are given in Johnson (1988). Kotz *et al.* (2003) have provided a comprehensive review

of the development of the stress-strength model up to the year 2003. Another review of stress-strength models and interference models are considered by Patowary *et al*. (2013).

Govindarajulu (1974) has considered the sequential procedure for the problem of obtaining the confidence interval for the parameter R in three cases, the first of which assumes that (X, Y) follow the bivariate normal distribution and the next two apply the nonparametric approaches. Bandyopadhyay *et al*. (2003) considered a nonparametric fixed-width confidence interval estimation for the parameter R by adopting a partial sequential sampling scheme. Recently, Mukhopadhyay and Zhuang (2016) have considered a fixed-accuracy interval for the reliability parameter under a bivariate exponential model, namely, the Fisher's Nile example. Bapat (2018) has constructed a purely sequential estimation methodology to find a fixed-accuracy confidence interval for the unknown parameter R in two separate cases, where (X, Y) have a bivariate exponential distribution due to Marshall and Olkin (1967) and Freund (1961).

As far as we are concerned, there has been no research on sequential analysis for constructing a fixed width confidence interval of the stress-strength reliability parameter, in which the stress and strength distributions are supposed to be exponential. A real example in the composites industry is studied, using the stress-strength model to illustrate the useful application of our proposed method. Referring to Xia *et al*. (2009), by using the strengths of two jute fibers possessing two different diameters, say, 10 mm and 20 mm, the reliability parameter under the stress-strength model is analyzed.

In order to construct an asymptotic, fixed-width $2d$ confidence interval for the stress-strength reliability parameter R by using its MLE, we first need to assess how many samples are needed to construct a $100(1 - \alpha)\%$ confidence interval. We consider two sequential procedures and seek to answer the question of whether the procedures attain a prescribed confidence level or not. Regarding this point, the rest of the article is composed of the following sections. In section 2, Asymptotic and exact confidence intervals based on a fixed sample size are introduced. Section 3 introduces a two-stage sequential methodology where the fixed-width $2d(d > 0)$ confidence interval is considered. In section 4, a purely sequential procedure is defined, and some characteristics of its stopping time are discussed. Moreover, an approximation of the exact distribution of purely sequential procedure, using the law of large numbers is provided. To clarify the theoretical results of the previous sections, we give some simulations in Section 5. In section 6, using real datasets, we illustrate the useful application of our proposed method. Finally section 7, ends with the conclusion.

2 Asymptotic and Exact Confidence Interval Based on a Fixed Sample Size

In this section, the inferential procedure about the reliability parameter $R = P(X < Y)$, where X and Y are independent, and both have exponential distributions with different scale parameters, is represented based on fixed sample sizes. An exponential distribution denoted by $\text{Exp}(\theta)$ has the probability density function (pdf) given as follows:

$$f(x;\theta) = \frac{1}{\theta}\exp(-\frac{x}{\theta}), x > 0, \theta > 0,$$

where θ is the scale parameter. Let $X \sim \text{Exp}(\theta_1)$ be independent of $Y \sim \text{Exp}(\theta_2)$, then the stress-strength reliability parameter can be expressed as follows:

$$R = P(X < Y) = \frac{\theta_2}{\theta_1 + \theta_2}.$$

Assume that the scale parameters θ_1 and θ_2 are both unknown. We want to estimate stress-strength reliability parameter $R = \theta_2/(\theta_1 + \theta_2)$. Let $X_1, ..., X_n$ and $Y_1, ..., Y_n$ be two independent random samples from $\text{Exp}(\theta_1)$ and $\text{Exp}(\theta_2)$, respectively. So, the MLE of R is

$$\widehat{R}_n = \frac{\bar{Y}_n}{\bar{X}_n + \bar{Y}_n},$$

where $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$ and $\bar{Y}_n = n^{-1}\sum_{i=1}^{n} Y_i$. Considering the multivariate central limit theorem (CLT), the asymptotic distribution of $(\bar{X}_n, \bar{Y}_n)^T$ is given by

$$\sqrt{n}\left(\begin{pmatrix}\bar{X}_n\\\bar{Y}_n\end{pmatrix} - \begin{pmatrix}\theta_1\\\theta_2\end{pmatrix}\right) \xrightarrow{D} N_2\left(\mathbf{0}, \begin{bmatrix}\theta_1^2 & 0\\0 & \theta_2^2\end{bmatrix}\right) \quad \text{as } n \longrightarrow \infty,$$

where N_2 denotes bivariate normal distribution and $\xrightarrow{D}$ stands for convergence in distribution. So, the asymptotic distribution of $\hat{R}_n$ is (see Chapter 7 of Ferguson (1996))

$$\sqrt{n}\left(\widehat{R}_n - R\right) \xrightarrow{D} N\left(0, \frac{2\theta_1^2\theta_2^2}{(\theta_1 + \theta_2)^4}\right) \quad \text{as } n \longrightarrow \infty. \tag{2.1}$$

Using (2.1) for a given $d > 0$ and $\alpha \in (0,1)$, a confidence interval for R with length $2d$ based on $\hat{R}_n$ is $I_n = [\hat{R}_n - d, \hat{R}_n + d]$. Lim *et al.* (2004) gave a purely sequential fixed-width confidence interval estimation procedure for a "smooth" function of the two scale parameters θ_1 and θ_2 when $d \longrightarrow 0$. Moreover for d sufficiently small Govindarajulu

(2004) provided a sequential procedure to construct a fixed-width $2d$ confidence interval for R when (X, Y) has a bivariate normal distribution with an unknown mean vector and an unknown covariance matrix. In this case, if one choose d so large, lower and upper limits of confidence interval I_n might be negative or greater than 1, respectively. So, assume that d is sufficiently small ($d \longrightarrow 0$). For the coverage probability (CP) of I_n with confidence at least $1 - \alpha$, we should have

$$P(R \in I_n) = P\left(\left|\widehat{R}_n - R\right| \leq d\right) \geq 1 - \alpha. \tag{2.2}$$

Let $a = Z_{1-\alpha/2} > 0$ be such that $\Phi(a) = 1 - \alpha/2$ where $\Phi(\cdot)$ is standard normal cumulative distribution function. For

$$n^* = \frac{2a^2 \theta_1^2 \theta_2^2}{d^2 (\theta_1 + \theta_2)^4},$$

it follows from (2.1) that once n^* is sufficiently large, for all $n > n^*$,

$$P(R \in I_n) = P\left(\left|\frac{\sqrt{n}(\widehat{R}_n - R)}{\sqrt{\frac{2\theta_1^2\theta_2^2}{(\theta_1+\theta_2)^4}}}\right| \leq \frac{\sqrt{n}d}{\sqrt{\frac{2\theta_1^2\theta_2^2}{(\theta_1+\theta_2)^4}}}\right)$$

$$\geq P\left(\left|\frac{\sqrt{n}(\widehat{R}_n - R)}{\sqrt{\frac{2\theta_1^2\theta_2^2}{(\theta_1+\theta_2)^4}}}\right| \leq a\right) \approx 1 - \alpha,$$

that is, n^* is the optimal (smallest) fixed sample size, which approximately satisfies (2.2). The exact coverage probability of I_n is given in the following theorem.

Theorem 2.1. *Let* $CP(R) = P(R - d \leq \hat{R}_n \leq R + d)$, *then*

$$CP(R) = \begin{cases} F_{\theta_1,\theta_2}\left(\frac{1}{R-d} - 1; n\right) - F_{\theta_1,\theta_2}\left(\frac{1}{R+d} - 1; n\right) & R > d \\ 1 - F_{\theta_1,\theta_2}\left(\frac{1}{R+d} - 1; n\right) & R \leq d, \end{cases} \tag{2.3}$$

where $F_{\theta_1,\theta_2}(.;n)$ *is the cumulative distribution function (CDF) of random variable* $T_n = \bar{X}_n/\bar{Y}_n$.

Proof. The proof involves looking at CDF of random variable T_n. Let $Po(\cdot; \mu)$ denotes the cumulative distribution function of a Poisson distribution with mean μ. Assume

that $X \sim G(h; 1)$ in which $G(h; 1)$ stands for a gamma random variable with the shape parameter h and scale parameter 1. The gamma-Poisson formula (see Kao (1997), p. 51) is given by

$$P(X > x) = Po(h - 1; x), \quad 0 \le x < \infty. \tag{2.4}$$

Let $F_{\theta_1, \theta_2}(t; n) = P(T_n \le t)$, according to the iterative expectation formula, we have

$$F_{\theta_1, \theta_2}(t; n) = P\left(\frac{\sum\limits_{i=1}^{n} X_i}{\sum\limits_{i=1}^{n} Y_i} \le t\right)$$

$$= E\left\{E\left[I\left\{\frac{\sum\limits_{i=1}^{n} X_i}{\theta_1} \le \frac{t\sum\limits_{i=1}^{n} Y_i}{\theta_1}\right\}\Big|\sum\limits_{i=1}^{n} Y_i\right]\right\},$$

where $I\{\}$ is the indicator function. Given that $\theta_1^{-1} \sum_{i=1}^{n} X_i \sim G(n, 1)$, $\bar{X}_n$ and $\bar{Y}_n$ are independent and also using gamma-Poisson formula (2.4), we have

$$E\left[I\left\{\frac{\sum\limits_{i=1}^{n} X_i}{\theta_1} \le \frac{t\sum\limits_{i=1}^{n} Y_i}{\theta_1}\right\}\Big|\sum\limits_{i=1}^{n} Y_i = y\right] = P\left(\frac{\sum\limits_{i=1}^{n} X_i}{\theta_1} \le \frac{ty}{\theta_1}\right)$$

$$= 1 - \sum_{j=0}^{n-1} \frac{(yt)^j \exp\left\{-\frac{yt}{\theta_1}\right\}}{\theta_1^j j!}.$$

Therefore,

$$F_{\theta_1, \theta_2}(t; n) = 1 - E\left[\sum_{j=0}^{n-1} \frac{\left(t\sum\limits_{i=1}^{n} Y_i\right)^j \exp\left\{-\frac{t\sum\limits_{i=1}^{n} Y_i}{\theta_1}\right\}}{\theta_1^j j!}\right]$$

$$= 1 - \sum_{j=0}^{n-1} \frac{1}{\theta_1^j j!} E\left[\left(t\sum\limits_{i=1}^{n} Y_i\right)^j \exp\left\{-\frac{t\sum\limits_{i=1}^{n} Y_i}{\theta_1}\right\}\right].$$

Now let $U = t \sum_{i=1}^{n} Y_i$, obviously $U \sim G(n, 1/(t\theta_2))$, so

$$\begin{aligned}
E\left[U^j \exp\{\frac{-U}{\theta_1}\}\right] &= \int_0^\infty \frac{(t\theta_2)^{-n}}{\Gamma(n)} u^{n-1} \exp\{\frac{-u}{t\theta_2}\} u^j \exp\{\frac{-u}{\theta_1}\} du \\
&= \int_0^\infty \frac{(t\theta_2)^{-n}}{\Gamma(n)} u^{n+j-1} \exp\{-u(\frac{1}{\theta_1} + \frac{1}{t\theta_2})\} du \\
&= \frac{\Gamma(n+j)}{\Gamma(n)} (t\theta_2)^{-n} (\frac{t\theta_1\theta_2}{t\theta_2 + \theta_1})^{n+j} \\
&= \frac{(n+j-1)!}{(n-1)!} (\frac{t\theta_1\theta_2}{t\theta_2 + \theta_1})^j (\frac{\theta_1}{t\theta_2 + \theta_1})^n.
\end{aligned}$$

Hence, for $t > 0$

$$\begin{aligned}
F_{\theta_1,\theta_2}(t;n) &= 1 - \sum_{j=0}^{n-1} \frac{(n+j-1)!}{\theta_1^j j!(n-1)!} (\frac{t\theta_1\theta_2}{t\theta_2 + \theta_1})^j (\frac{\theta_1}{t\theta_2 + \theta_1})^n \\
&= 1 - \sum_{j=0}^{n-1} \binom{n+j-1}{n-1} \left(\frac{t\theta_2}{t\theta_2 + \theta_1}\right)^j \left(\frac{\theta_1}{t\theta_2 + \theta_1}\right)^n,
\end{aligned}$$

equals 0 for $t \leq 0$. Remember that T_n is a positive random variable. On the other hand,

$$\begin{aligned}
CP(R) &= P(R - d \leq \hat{R}_n \leq R + d) \\
&= P(R - d \leq \frac{1}{1 + \frac{\bar{X}_n}{\bar{Y}_n}} \leq R + d) \\
&= \begin{cases} P(T_n \leq \frac{1}{R-d} - 1) - P(T_n \leq \frac{1}{R+d} - 1) & R > d \\ 1 - P(T_n \leq \frac{1}{R+d} - 1) & R \leq d, \end{cases}
\end{aligned}$$

which completes the proof. □

CDF of the random variable T_n and Theorem 2.1 will come into use in the next section and along with defining stopping rule (2.5). The required sample size to obtain an exact coverage probability greater than $1 - \alpha$ is

$$\tilde{n}(R) = \min\{n \geq 1 | CP(R) \geq 1 - \alpha\}. \tag{2.5}$$

Stopping rule (2.5), in which $\hat{R}_n = \bar{Y}_n/(\bar{X}_n + \bar{Y}_n)$ is substituted for R, is too complicated and is asymptotically equivalent to the classical stopping rules based on the asymptotic

coverage probability. We often have the required coverage when $n \geq n^*$ also in small and medium-size samples. As an example, for $\alpha = 0.05$, $\theta_1 = 1$ and $\theta_2 = 2$, by substituting n^* for n in (2.3), the results summarized as in Table 1 are acquired.

Table 1: Exact coverage probability for different values of n^*.

d	n^*	$CP(R)$
0.1377	20	0.9517
0.0616	100	0.9503
0.0436	200	0.9501
0.0275	500	0.9500
0.0195	1000	0.9500

If θ_1 and θ_2 are known, consequently the asymptotically smallest sample size is known, and no sequential methods is needed to be applied. The problem of sequential estimation arises when the parameters are unknown. Therefore, two procedures, namely two-stage sequential and purely sequential procedures, are considered for this problem.

3 Two-Stage Sequential Procedure for Fixed-Width Confidence Interval Estimation of the Stress-Strength Reliability Parameter

Let $X_1, ..., X_m$ and $Y_1, ..., Y_m$ be pilot samples of $\text{Exp}(\theta_1)$ and $\text{Exp}(\theta_2)$, respectively. The statistical and mathematical principles relating to the two-stage procedures have been given by Stein (1945) and are used here to define the stopping rule as follows:

$$N_m = \max\left\{ \left\lfloor \frac{2a^2 \bar{X}_m^2 \bar{Y}_m^2}{d^2(\bar{X}_m + \bar{Y}_m)^4} \right\rfloor + 1, m \right\}, \tag{3.1}$$

where $\lfloor x \rfloor$ denotes the largest integer less than x. If $N_m = m$, we do not take any more observations at the second stage. On the other hand, if $N_m > m$, the initial samples do not suffice and we then sample the difference $(N - m)$ at the second stage. Finally, sampling stops with $X_1, ..., X_{N_m}, Y_1, ..., Y_{N_m}$, and interval $I_{N_m} = [\hat{R}_{N_m} - d, \hat{R}_{N_m} + d]$ based on all the $2N_m$ samples is proposed for R. Our next goal is to determine the distribution of

N_m using CDF of T_n and Theorem 2.1, in which the next result deals with this problem. Throughout this paper, n is an integer number. For each $n < \lfloor a^2/(8d^2) \rfloor + 1$, a_n and b_n are the real-valued roots of the quadratic equation of the form $x^2 - (\sqrt{2a^2/(d^2 l)} - 2)x + 1 = 0$, in which $l = n$, $(a_n < b_n)$.

Theorem 3.1. *we have the following results:*

(i) *The probability mass function (PMF) of N_m, in a two-stage procedure is given by*

$$P(N_m = m) = \begin{cases} 1 - F_{\theta_1,\theta_2}(b_m;m) + F_{\theta_1,\theta_2}(a_m;m) & m < \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1 \\ \\ 1 & m \geq \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1, \end{cases}$$

(ii) *If $m < \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1$, then the PMF becomes*

$$P(N_m = n) = \begin{cases} A_n & m+1 \leq n < \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1 \\ F_{\theta_1,\theta_2}(b_{\left\lfloor \frac{a^2}{8d^2} \right\rfloor};m) - F_{\theta_1,\theta_2}(a_{\left\lfloor \frac{a^2}{8d^2} \right\rfloor};m) & n = \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1 \\ 0 & n > \left\lfloor \frac{a^2}{8d^3} \right\rfloor + 1, \end{cases}$$

where $A_n = F_{\theta_1,\theta_2}(b_{n-1};m) + F_{\theta_1,\theta_2}(a_n;m) - \left(F_{\theta_1,\theta_2}(b_n;m) + F_{\theta_1,\theta_2}(a_{n-1};m) \right)$.

Proof. (i) Using the CDF of T_n and stopping variable (3.1), we have

$$P(N_m = m) = P\left(\frac{2a^2 \bar{X}_m^2 \bar{Y}_m^2}{d^2(\bar{X}_m + \bar{Y}_m)^4} \leq m \right)$$

$$= P\left(T_m^2 - \left(\sqrt{\frac{2a^2}{d^2 m}} - 2 \right) T_m + 1 \geq 0 \right). \tag{3.2}$$

Let Δ be equal to the discriminant of the quadratic equation inside (3.2). If Δ is non-positive, then $P(N_m = m) = 1$. Δ is non-positive when $m \geq a^2/(8d^2)$. It means N_m has a degenerate distribution at m if and only if $m \geq \lfloor a^2/(8d^2) \rfloor + 1$. Δ is positive when $m < \lfloor a^2/(8d^2) \rfloor + 1$, so

$$P(N_m = m) = P\left(T_m^2 - \left(\sqrt{\frac{2a^2}{d^2 m}} - 2 \right) T_m + 1 \geq 0 \right)$$

$$= 1 - P\left(a_m \leq T_m \leq b_m\right)$$

$$= 1 - F_{\theta_1,\theta_2}(b_m; m) + F_{\theta_1,\theta_2}(a_m; m).$$

(*ii*) If $m \geq \lfloor a^2/(8d^2)\rfloor + 1$, then N_m has a degenerate distribution at initial sample size m and consequently $P(N_m = n) = 0$ for each $n \neq m$. For $m < \lfloor a^2/(8d^2)\rfloor + 1$, the proof is as follows:

$$P\left(N_m = n\right) = P\left(n - 1 \leq \frac{2a^2 \bar{X}_m^2 \bar{Y}_m^2}{d^2(\bar{X}_m + \bar{Y}_m)^4} \leq n\right)$$

$$= P\left(\frac{2a^2 \bar{X}_m^2 \bar{Y}_m^2}{d^2(\bar{X}_m + \bar{Y}_m)^4} \leq n\right) - P\left(\frac{2a^2 \bar{X}_m^2 \bar{Y}_m^2}{d^2(\bar{X}_m + \bar{Y}_m)^4} \leq n - 1\right)$$

$$= P\left(T_m^2 - \left(\sqrt{\frac{2a^2}{d^2 n}} - 2\right)T_m + 1 \geq 0\right)$$

$$- P\left(T_m^2 - \left(\sqrt{\frac{2a^2}{d^2(n-1)}} - 2\right)T_m + 1 \geq 0\right). \tag{3.3}$$

The first term on the right hand side of (3.3) is equal to 1 when $n \geq a^2/(8d^2)$ and the second one is equal to 1 when $n \geq a^2/(8d^2) + 1$. So for each $n > \lfloor a^2/(8d^2)\rfloor + 1$, $P(N_m = n) = 0$. Let $n = \lfloor a^2/(8d^2)\rfloor + 1$, then

$$P\left(N_m = n\right) = P\left(T_m^2 - \left(\sqrt{\frac{2a^2}{d^2 n}} - 2\right)T_m + 1 \geq 0\right)$$

$$- P\left(T_m^2 - \left(\sqrt{\frac{2a^2}{d^2(n-1)}} - 2\right)T_m + 1 \geq 0\right)$$

$$= 1 - [1 - P\left(a_{n-1} \leq T_m \leq b_{n-1}\right)]$$

$$= F_{\theta_1,\theta_2}(b_{n-1}; m) - F_{\theta_1,\theta_2}(a_{n-1}; m)$$

$$= F_{\theta_1,\theta_2}(b_{\lfloor a^2/(8d^2)\rfloor}; m) - F_{\theta_1,\theta_2}(a_{\lfloor a^2/(8d^2)\rfloor}).$$

Finally, for each $m + 1 \leq n < \lfloor a^2/(8d^2)\rfloor + 1$,

$$P\left(N_m = n\right) = P\left(T_m^2 - \left(\sqrt{\frac{2a^2}{d^2 n}} - 2\right)T_m + 1 \geq 0\right)$$

$$- P\left(T_m^2 - \left(\sqrt{\frac{2a^2}{d^2(n-1)}} - 2\right)T_m + 1 \geq 0\right)$$

$$= P\left(a_{n-1} \le T_m \le b_{n-1}\right) - P\left(a_n \le T_m \le b_n\right)$$

$$= F_{\theta_1,\theta_2}(b_{n-1}; m) + F_{\theta_1,\theta_2}(a_n; m) - \left(F_{\theta_1,\theta_2}(b_n; m) + F_{\theta_1,\theta_2}(a_{n-1}; m)\right).$$

□

Theorem 3.1 summarizes by saying that, if starting sample size $m \ge \lfloor a^2/(8d^2)\rfloor + 1$, then the random variable N_m has a degenerate distribution at the initial sample size m. N_m is finite and attains values $m, ..., \lfloor a^2/(8d^2)\rfloor + 1$ when $m < \lfloor a^2/(8d^2)\rfloor + 1$. Stopping rule N_m is the estimate of the unknown optimum sample size n^*, so we are interested in computing the expectation of this random variable. The next theorem deals with the expectation of the stopping rule N_m for the case $m < \lfloor a^2/(8d^2)\rfloor + 1$, because if $m \ge \lfloor a^2/(8d^2)\rfloor + 1$, it has been proved in part (*i*) of Theorem 3.1 that N_m has a degenerate distribution at m and hence $E[N_m] = m$. The following Lemma will help us to compute the expectation of random variable N_m.

Lemma 3.1. *The cumulative distribution function of N_m is*

$$P\left(N_m \le k\right) = \begin{cases} 0 & k < m \\[2mm] 1 - F_{\theta_1,\theta_2}(b_{\langle k\rangle}; m) + F_{\theta_1,\theta_2}(a_{\langle k\rangle}; m) & m \le k < \left\lfloor \frac{a^2}{8d^2}\right\rfloor + 1 \\[2mm] 1 & k \ge \left\lfloor \frac{a^2}{8d^2}\right\rfloor + 1. \end{cases} \tag{3.4}$$

Where $\langle x\rangle$ denotes the greatest integer less than or equal to x.

Proof. If $k < m$ or $k \ge \lfloor a^2/(8d^2)\rfloor + 1$, the proof is evident and the details are avoided. For the case in which $m \le k < \lfloor a^2/(8d^2)\rfloor + 1$, the proof is as follows:

$$P\left(N_m \le k\right) = \sum_{i=m}^{\langle k\rangle} P\left(N_m = i\right)$$

$$= P\left(N_m = m\right) + \sum_{i=m+1}^{\langle k\rangle} P\left(N_m = i\right),$$

note that $\sum_{i=m+1}^{\langle k\rangle} P(N_m = i)$, is a telescoping series, so we have

$$\sum_{i=m+1}^{\langle k\rangle} P\left(N_m = i\right) = F_{\theta_1,\theta_2}(b_m; m) - F_{\theta_1,\theta_2}(a_m; m) + F_{\theta_1,\theta_2}(a_{\langle k\rangle}; m) - F_{\theta_1,\theta_2}(b_{\langle k\rangle}; m). \tag{3.5}$$

Therefore,

$$P\left(N_m \leq k\right) = P\left(N_m = m\right) + \sum_{i=m+1}^{\langle k \rangle} P\left(N_m = i\right)$$

$$= 1 - F_{\theta_1,\theta_2}(b_{\langle k \rangle}; m) + F_{\theta_1,\theta_2}(a_{\langle k \rangle}; m).$$

□

Theorem 3.2. *For the case $m < \lfloor a^2/(8d^2) \rfloor + 1$, the expectation of the stopping variable N_m is given as*

$$E\left[N_m\right] = m + \sum_{i=m+1}^{\lfloor a^2/(8d^2) \rfloor +1} \left(F_{\theta_1,\theta_2}(b_{i-1}; m) - F_{\theta_1,\theta_2}(a_{i-1}; m)\right). \tag{3.6}$$

Proof. Since N_m is a discrete random variable with support $m, m + 1, ..., \lfloor a^2/8d^2 \rfloor + 1$, therefore,

$$E\left[N_m\right] = m + \sum_{i=m+1}^{\lfloor a^2/(8d^2) \rfloor +1} P\left(N_m \geq i\right). \tag{3.7}$$

Now using Lemma 3.1, we have

$$P\left(N_m > k\right) = \begin{cases} 1 & k < m \\ F_{\theta_1,\theta_2}\left(b_{\langle k \rangle}; m\right) - F_{\theta_1,\theta_2}\left(a_{\langle k \rangle}; m\right) & m \leq k < \lfloor \frac{a^2}{8d^2} \rfloor + 1 \\ 0 & k \geq \lfloor \frac{a^2}{8d^2} \rfloor + 1. \end{cases} \tag{3.8}$$

Hence,

$$E\left[N_m\right] = m + \sum_{i=m+1}^{\lfloor a^2/(8d^2) \rfloor +1} P\left(N_m \geq i\right)$$

$$= m + P(N_m > m) + P(N_m > m + 1) + \cdots + P(N_m > \lfloor a^2/(8d^2) \rfloor)$$

$$= m + \sum_{i=m+1}^{\lfloor a^2/(8d^2) \rfloor +1} \left(F_{\theta_1,\theta_2}\left(b_{i-1}; m\right) - F_{\theta_1,\theta_2}\left(a_{i-1}; m\right)\right).$$

□

Theorem 3.3. *The first and second moments of $\hat{R}_{N_m}$ are, given respectively by*

$$E\left[\widehat{R}_{N_m}\right] = \sum_{n=m}^{\lfloor a^2/(8d^2)\rfloor+1} \frac{k(n,\rho)}{B(n,n)} P\left(N_m = n\right), \tag{3.9}$$

where

$$k\left(n,\rho\right) = \int_0^1 \frac{t^n(1-t)^{n-1}}{\rho+(1-\rho)t} dt,$$

$\rho = \theta_1/\theta_2$ *and $B(\cdot,\cdot)$ is Euler's integral of the first kind (also known as beta function), and*

$$E\left[\widehat{R}_{N_m}^2\right] = \sum_{n=m}^{\lfloor a^2/8d^2\rfloor+1} \left[\left(\frac{n(1+\rho)}{1-\rho}+1\right)\frac{k(n,\rho)}{B(n,n)} - \frac{n}{1-\rho}\right] P\left(N_m = n\right). \tag{3.10}$$

Proof. According to Sathe and Shah (1981), we have

$$E\left[\widehat{R}_n\right] = \frac{k(n,\rho)}{B(n,n)},$$

and

$$E\left[\widehat{R}_n^2\right] = \left(\frac{n(1+\rho)}{1-\rho}+1\right)\frac{k(n,\rho)}{B(n,n)} - \frac{n}{1-\rho}.$$

So, $E[\hat{R}_{N_m}]$ and $E[\hat{R}_{N_m}^2]$ are given, respectively by

$$\begin{aligned}
E\left[\widehat{R}_{N_m}\right] &= E\left\{E\left[\widehat{R}_{N_m}\Big|N_m\right]\right\} \\
&= E\left[\frac{k(N_m,\rho)}{B(N_m,N_m)}\right] \\
&= \sum_{n=m}^{\lfloor a^2/8d^2\rfloor+1} \frac{k(n,\rho)}{B(n,n)} P\left(N_m = n\right),
\end{aligned}$$

and

$$\begin{aligned}
E\left[\widehat{R}_{N_m}^2\right] &= E\left\{E\left[\widehat{R}_{N_m}^2\Big|N_m\right]\right\} \\
&= E\left[\left(\frac{N_m(1+\rho)}{1-\rho}+1\right)\frac{k(N_m,\rho)}{B(N_m,N_m)} - \frac{N_m}{1-\rho}\right]
\end{aligned}$$

$$= \sum_{n=m}^{\lfloor a^2/(8d^2) \rfloor + 1} \left[\left(\frac{n(1+\rho)}{1-\rho} + 1 \right) \frac{k(n,\rho)}{B(n,n)} - \frac{n}{1-\rho} \right] P(N_m = n).$$

□

Using Theorem 3.3 the MSE of $\hat{R}_{N_m}$ according to R is given by

$$MSE\left[\widehat{R}_{N_m}\right] = E\left[\widehat{R}_{N_m}^2\right] - 2RE\left[\widehat{R}_{N_m}\right] + R^2. \tag{3.11}$$

Theorem 3.4. *The coverage probability* $(\hat{R}_{N_m} - d, \hat{R}_{N_m} + d)$ *is as follows:*

$$CP(R) = \sum_{n=m}^{\lfloor a^2/(8d^2) \rfloor + 1} \left(F_{\theta_1,\theta_2}(\frac{1}{R-d} - 1; n) - F_{\theta_1,\theta_2}(\frac{1}{R+d} - 1; n)P(N_m = n) \right) I\{R > d\}$$

$$+ \sum_{n=m}^{\lfloor a^2/(8d^2) \rfloor + 1} \left(1 - F_{\theta_1,\theta_2}(\frac{1}{R+d} - 1; n)P(N_m = n) \right) I\{R \le d\}. \tag{3.12}$$

Proof. Using Theorem (2.1) and iterative expectation formula, we have

$$CP(R) = P\left(R - d \le \hat{R}_{N_m} \le R + d \right)$$

$$= E\left\{ E\left[I\{R - d \le \hat{R}_{N_m} \le R + d\} \big| N_m \right] \right\}$$

$$= E[\left(F_{\theta_1,\theta_2}(\frac{1}{R-d} - 1; N_m) - F_{\theta_1,\theta_2}(\frac{1}{R+d} - 1; N_m) \right) I\{R > d\}$$

$$+ \left(1 - F_{\theta_1,\theta_2}(\frac{1}{R+d} - 1; N_m) \right) I\{R \le d\}]$$

$$= \sum_{n=m}^{\lfloor a^2/8d^2 \rfloor + 1} \left(F_{\theta_1,\theta_2}(\frac{1}{R-d} - 1; n) - F_{\theta_1,\theta_2}(\frac{1}{R+d} - 1; n)P(N_m = n) \right) I\{R > d\}$$

$$+ \sum_{n=m}^{\lfloor a^2/8d^2 \rfloor + 1} \left(1 - F_{\theta_1,\theta_2}(\frac{1}{R+d} - 1; n)P(N_m = n) \right) I\{R \le d\}.$$

□

4 Purely Sequential-Based for Fixed-Width Confidence Interval Estimation of the Stress-Strength Reliability Parameter

Stein's two-stage procedure (Stein (1945)) is considered one of the most exciting and practical sequential procedure for hypotheses testing and estimation. This procedure

estimates the required sample size, N_m using a small number of observations as, m. However, the procedure tends to oversample and the amount of oversampling increases with the increase in $(N_m - m)$. To reduce oversampling by the two-stage procedure, we have to proceed the estimation of R successively in a sequential manner. We begin with a few observations (say, $m > 2$) and then we continue to take one additional observation at-a-time, but terminate sampling when we have gathered enough observations. As soon as a sample size exceeds the corresponding estimate of n^*, the sampling is terminated right there, and an interval estimator of R is constructed. In this section, a sequential procedure for estimating the stress-strength reliability parameter for exponential distributions is proposed, and using the law of large numbers, the characteristics of its stopping time are depicted. First, note that

$$N_d = \min\left\{ n \geq m \middle| n \geq \frac{2a^2 \bar{X}_n^2 \bar{Y}_n^2}{d^2 \left(\bar{X}_n + \bar{Y}_n\right)^4} \right\}, \tag{4.1}$$

in which $m \geq 2$ is the starting (or pilot) sample size. N_d can be expressed in terms of $T_n = \bar{X}_n/\bar{Y}_n$ as follows:

$$N_d = \min\left\{ n \geq m \middle| \frac{\left(\bar{X}_n + \bar{Y}_n\right)^4}{\bar{X}_n^2 \bar{Y}_n^2} \geq \frac{2a^2}{nd^2} \right\}$$

$$= \min\left\{ n \geq m \middle| T_n^2 - \left(\sqrt{\frac{2a^2}{nd^2}} - 2 \right) T_n + 1 \geq 0 \right\}. \tag{4.2}$$

Note that there is no real solution to the quadratic equation inside (4.2) for $n > a^2/8d^2$.

Theorem 4.1. *Let a_l and b_l be the real-valued roots of the quadratic equation inside (4.2) for $n = l$, such that $a_l < b_l$, then*

(i)

$$P(N_d = m) = \begin{cases} 1 - P\left(a_m \leq T_m \leq b_m\right) & m < \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1 \\ 1 & m \geq \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1, \end{cases}$$

(ii)

$$P(N_d \leq k) = \begin{cases} 0 & k < m \\ 1 - P\left(\bigcap_{i=m}^{\langle k \rangle} \{a_i \leq T_i \leq b_i\} \right) & m \leq k < \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1 \\ 1 & k \geq \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1. \end{cases}$$

Proof. (*i*) The discriminant of the quadratic equation inside (4.2) is non-positive for, $n \geq a^2/(8d^2)$. It means that for each $n \geq a^2/(8d^2)$, $T_n^2 - (\sqrt{2a^2/(nd^2)} - 2)T_n + 1 \geq 0$. So for each $n \geq m \geq \lfloor a^2/(8d^2) \rfloor + 1$, the quadratic equation inside (4.2) is non-negative. Therefore, the minimum $n \geq m$ such that $T_n^2 - (\sqrt{2a^2/(nd^2)} - 2)T_n + 1 \geq 0$ is m itself. Thus if $m \geq \lfloor a^2/(8d^2) \rfloor + 1$, then $P(N_d = m) = 1$ and if $m < \lfloor a^2/(8d^2) \rfloor + 1$,

$$P(N_d = m) = P\left(T_m^2 - (\sqrt{\frac{2a^2}{md^2}} - 2)T_m + 1 \geq 0 \right)$$

$$= 1 - P(a_m \leq T_m \leq b_m).$$

(*ii*) It is apparent that $P(N_d \leq k) = 0$ if $k < m$. For $m \leq k < \lceil a^2/(8d^2) \rceil + 1$,

$$P(N_d > k) = P(N_d \neq m, ..., N_d \neq k)$$

$$= P\left(\bigcap_{i=m}^{\langle k \rangle} \left\{ T_i^2 - (\sqrt{\frac{2a^2}{id^2}} - 2)T_n + 1 < 0 \right\} \right)$$

$$= P\left(\bigcap_{i=m}^{\langle k \rangle} \{ a_i \leq T_i \leq b_i \} \right).$$

Finally, if $k = \lfloor a^2/(8d^2) \rfloor + 1$, in the view of the fact that for each $n \geq a^2/(8d^2)$, $T_n^2 - (\sqrt{2a^2/(nd^2)} - 2)T_n + 1 \geqslant 0$ (see part (*i*)), then we have

$$P(N_d \leq \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1) = 1 - P\left(N_d > \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1 \right)$$

$$= 1 - P(N_d \neq m, ..., N_d \neq \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1)$$

$$= 1 - P\left(\bigcap_{i=m}^{\left(\lfloor \frac{a^2}{8d^2} \rfloor + 1 \right)} \left\{ T_i^2 - (\sqrt{\frac{2a^2}{id^2}} - 2)T_n + 1 < 0 \right\} \right)$$

$$= 1.$$

So, if $k = \lfloor a^2/(8d^2) \rfloor + 1$ then $T_k^2 - (\sqrt{2a^2/(kd^2)} - 2)T_k + 1 \geqslant 0$, the intersection is an empty set, which is the desired result. □

Part (*i*) of Theorem 4.1 shows that if the starting sample size $m \geq \lfloor a^2/(8d^2) \rfloor + 1$, then N_d has a degenerate distribution at m. Part (*ii*) of Theorem 4.1 indicates that N_d can

attain values $m, m + 1, ..., \lfloor a^2/(8d^2) \rfloor + 1$, proving that not only $P(N_d < \infty) = 1$ but also $P(N_d \leq \lfloor a^2/(8d^2) \rfloor + 1) = 1$.

We can approximate $P(N_d > k)$, by using the law of large numbers. Using part (*ii*) of Theorem 4.1, for each $m \leq k < \lfloor a^2/(8d^2) \rfloor + 1$, we have

$$P(N_d > k) = P\left(\bigcap_{i=m}^{\langle k \rangle} \{a_i \leq T_i \leq b_i\}\right)$$

$$= E\left[I\left\{\bigcap_{i=m}^{\langle k \rangle} \{a_i \leq T_i \leq b_i\}\right\}\right].$$

In order to apply the strong law of large numbers, we only need to show that $E[|I\{\bigcap_{i=m}^{\langle k \rangle} \{a_i \leq T_i \leq b_i\}\}|] < \infty$, which in this case is trivial because $|I\{\bigcap_{i=m}^{\langle k \rangle} \{a_i \leq T_i \leq b_i\}\}| \leq 1$. It may be observed that, generating random numbers from the joint distribution of $(T_m, ..., T_{\langle k \rangle})$ is very straightforward by using the definition of $(T_m, ..., T_{\langle k \rangle})$. More precisely, first generate $X_1, ..., X_{\langle k \rangle}$ and $Y_1, ..., Y_{\langle k \rangle}$ from $\text{Exp}(\theta_1)$ and $\text{Exp}(\theta_2)$ distributions, respectively. Then set

$$T_j = \frac{\sum_{i=1}^{j} X_i}{\sum_{i=1}^{j} Y_i}, \quad j = m, ..., \langle k \rangle.$$

Now, let $\mathbf{t_m} = (t_{m1}, ..., t_{mn})$ be n independent observations from T_m. Using Monte Carlo integration, for sufficiently large n ($n \longrightarrow \infty$), we have

$$\widehat{P}(N_d > k) = \frac{1}{n} \sum_{j=1}^{n} I\left\{\bigcap_{i=m}^{k} \{a_i \leq t_{ij} \leq b_i\}\right\} \longrightarrow E\left[I\left\{\bigcap_{i=m}^{k} \{a_i \leq T_i \leq b_i\}\right\}\right] = P(N_d > k).$$

Therefore, using part (*ii*) of Theorem 3.4 and Monte Carlo method, we can approximate the CDF of N_d as follows:

$$\widehat{P}(N_d \leq k) = \begin{cases} 0 & k < m \\ 1 - \widehat{P}(N_d > k) & m \leq k < \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1 \\ 1 & k \geq \left\lfloor \frac{a^2}{8d^2} \right\rfloor + 1, \end{cases}$$

So, $\widehat{P}(N_d = n) = \widehat{P}(N_d \leq n) - \widehat{P}(N_d \leq n - 1)$. Having approximated the exact distribution of N_d, we can estimate the average stopping time, the average estimate of R, MSE of $\widehat{R}_{N_d}$

according to R and the coverage probability of R. Let $\widehat{P}(N_d = n)$ be the approximation of $P(N_d = n)$, so we approximate the average stopping time, $E[N_d]$ by $E[N_d]^*$ as follows:

$$E[N_d]^* = \sum_{i=m}^{\lfloor a^2/8d^2 \rfloor + 1} i\widehat{P}(N_d = i). \tag{4.3}$$

To approximate $E[\hat{R}_{N_d}]$, $E[\hat{R}^2_{N_d}]$, MSE of $\hat{R}_{N_d}$ according to R and CP, we use (3.9)-(3.12), respectively, in which N_d is substituted for N_m and $\widehat{P}(N_d = n)$ is substituted for $P(N_m = n)$. So we have

$$E\left[\widehat{R}_{N_d}\right]^* = \sum_{n=m}^{\lfloor a^2/(8d^2) \rfloor + 1} \frac{k(n,\rho)}{B(n,n)}\widehat{P}(N_d = n), \tag{4.4}$$

$$E\left[\widehat{R}^2_{N_d}\right]^* = \sum_{n=m}^{\lfloor a^2/8d^2 \rfloor + 1} \left[\left(\frac{n(1+\rho)}{1-\rho} + 1\right)\frac{k(n,\rho)}{B(n,n)} - \frac{n}{1-\rho}\right]\widehat{P}(N_d = n), \tag{4.5}$$

and the MSE of $\hat{R}_{N_d}$ according to R is

$$MSE\left(\widehat{R}_{N_d}\right)^* = E\left[\widehat{R}^2_{N_d}\right]^* - 2RE\left[\widehat{R}_{N_d}\right]^* + R^2. \tag{4.6}$$

Finally, the CP is as follows:

$$CP(R)^* = \sum_{n=m}^{\lfloor a^2/(8d^2) \rfloor + 1} \left(F_{\theta_1,\theta_2}(\frac{1}{R-d} - 1; n) - F_{\theta_1,\theta_2}(\frac{1}{R+d} - 1; n)\widehat{P}(N_d = n)\right)I\{R > d\}$$

$$+ \sum_{n=m}^{\lfloor a^2/(8d^2) \rfloor + 1} \left(1 - F_{\theta_1,\theta_2}(\frac{1}{R+d} - 1; n)\widehat{P}(N_d = n)\right)I\{R \leq d\}. \tag{4.7}$$

5   Computation and Simulation Results

We have done detailed computations to justify the results of the previous sections, but for the sake of brevity, we only present a summary here. Purely and two-stage procedures are considered when $\theta_1 = 1$ and $\theta_2 = 2, 3, 5$ and 7, the confidence level is $1 - \alpha = 0.95$, the initial sample size $m = 5, 10, 20$ and d is chosen such that $n^* = 20, 100, 200,$

500 and 1000. For each set of values, we ran $h = 10{,}000$ replications by letting MATLAB ("MATLAB 2018a, The MathWorks, Natick, 2018") draw samples from the preassigned exponential populations. Suppose that in the ith replication there are n_i observations. Based on these data, we estimate R by $\bar{Y}_{n_i}/(\bar{X}_{n_i} + \bar{Y}_{n_i})$. That is, for the two stage sequential procedures; we observed $N_m = n_i$ and set $\hat{R}_{N_m} = \bar{Y}_{n_i}/(\bar{X}_{n_i} + \bar{Y}_{n_i})$. Similarly, for the purely sequential procedures, we observed $N_d = n_i$ and set $\hat{R}_{N_d} = \bar{Y}_{n_i}/(\bar{X}_{n_i} + \bar{Y}_{n_i})$. The distribution of N_d for the purely sequential procedure is approximated using 10,000 replications. Table 2 represents the notations used to summarize our results.

Table 2: Index of notation that is used in Table 3-10.

Sampling scheme	Notation	Formula	Explain
Two-stage (Tables 3-6)	$E[N_m]$	(3.6)	Exact expectation of N_m
	$\widehat{E}[N_m]$	$h^{-1}\sum_{i=1}^{h} n_i$	Simulated value of $E[N_m]$
	$E[\widehat{R}_{N_m}]$	(3.9)	Exact expectation of $\widehat{R}_{N_m}$
	$\widehat{E[\widehat{R}_{N_m}]}$	$h^{-1}\sum_{i=1}^{h} \hat{R}_{n_i}$	Simulated value of expectation of $\widehat{R}_{N_m}$
	$MSE[\widehat{R}_{N_m}]$	(3.11)	Exact value of $MSE[\widehat{R}_{N_m}]$ according to R
	$\widehat{MSE}$	$h^{-1}\sum_{i=1}^{h}(\hat{R}_{n_i} - R)^2$	Simulated value of $MSE[\widehat{R}_{N_m}]$ according to R
	CP	(3.12)	Exact value of coverage probability of R
	$\widehat{CP}$	$h^{-1}\sum_{i=1}^{h} I\left\{\hat{R}_{n_i} - d \le R \le \hat{R}_{n_i} + d\right\}$	Simulated value of coverage probability of R
Purely- sequential (Tables 7-10)	$E[N_d]^*$	(4.3)	Approximation of the average stopping time
	$\widehat{E}[N_d]^*$	$h^{-1}\sum_{i=1}^{h} n_i$	Simulated value of $E[N_d]^*$
	$E[\widehat{R}_{N_d}]^*$	(4.4)	Approximation of the expectation of $\widehat{R}_{N_d}$
	$\widehat{E[\widehat{R}_{N_d}]}^*$	$h^{-1}\sum_{i=1}^{h} \hat{R}_{n_i}$	Simulated value of $E[\widehat{R}_{N_d}]^*$
	$MSE[\widehat{R}_{N_d}]^*$	(4.6)	Approximation of $MSE[\hat{R}_{N_d}]$ according to R
	$\widehat{MSE}^*$	$h^{-1}\sum_{i=1}^{h}(\hat{R}_{n_i} - R)^2$	Simulated value of $MSE[\hat{R}_{N_d}]^*$ according to R
	CP^*	(4.7)	Approximation of coverage probability of R
	$\widehat{CP}^*$	$h^{-1}\sum_{i=1}^{h} I\left\{\hat{R}_{n_i} - d \le R \le \hat{R}_{n_i} + d\right\}$	Simulated value of CP^*

As Tables 3-10 show, the accuracy, precision, and coverage probability increase with m. The coverage probability is below the target for some cases, so we need to find the initial sample size m in such a way that the coverage probability gets at least $1 - \alpha$. We considered it only for the two-stage sequential procedure because we have found an exact distribution only for this case. Table 11, indicates the characteristic of the two-stage procedure and the smallest m such that the coverage probability be at least $1 - \alpha$. In Table 11, oversampling by the two-stage procedure is obvious. Some of the main results of the simulations are listed as follows:

1. The results derived from both of the proposed sequential procedures are almost

the same.

2. For both proposed sequential procedures, the results of simulations and exact computations (approximations) are extremely coincide with each other, confirming the theorem's accuracy mentioned in the previous sections.

3. Given a fixed value of m, as d decreases, $E[N_m]$ and $E[\hat{R}_{N_m}]$ increase with $E[\hat{R}_{N_m}]$ gets closer to R. Similarly, $E[N_d]^*$ and $E[\hat{R}_{N_d}]^*$ increase, with $E[\hat{R}_{N_d}]^*$ nearing to R.

4. Given either a fixed or decreasing value of d, as the initial sample size, m increases, CP (CP^*) increases, and nears its nominal value to $1 - \alpha$.

5. Comparing the results in Tables 3-6 (two-stage procedure) and Tables 7-10 (purely sequential procedure) we realize that, in the cases examined, the expected sample size in the two-stage procedure is generally larger than n^*, whereas in the sequential procedure it is generally smaller than n^*. The bias of the estimators of R is negative in both procedures and is almost the same magnitude. Similar results is found about the MSE of the estimators. The coverage probabilities in both procedures are generally below the prescribed $1 - \alpha$ value. These values approach the prescribed values, as m getting larger. The coverage probability in the sequential procedure is more than the coverage probability in the two-stage procedure. In all cases, the purely sequential procedure is more efficient.

Although we approximate the exact distribution of N_d, one may notice that purely sequential strategy outperforms the two-stage sequential estimation strategy. Indeed, both procedures are fully expected to perform very well, but the two-stage sequential procedure is logistically more straightforward to be implemented than a purely sequential estimation one. Based on the simulation results, we can conclude that the proposed sequential procedures are working quite well, and they can be used quite effectively for data analysis purposes.

Table 3: Characteristics and simulated values of two-stage procedure with $\alpha = 0.05$, $\theta_1 = 1$, $\theta_2 = 2$, $m = 5, 10, 20$ and $R = 0.6666$.

m	n^*	20	100	200	500	1000
	d	0.1377	0.0616	0.0436	0.0275	0.0195
	$E[N_m]$	18.6806	91.0953	181.3078	454.9795	904.3905
	$\widehat{E}[N_m]$	18.6808	91.0525	181.2331	455.003	904.0718
	$E[\widehat{R}_{N_m}]$	0.6620	0.6656	0.6661	0.6665	0.6666
5	$\widehat{E}[\widehat{R}_{N_m}]$	0.6622	0.6656	0.6661	0.6664	0.6665
	$MSE[\widehat{R}_{N_m}]$	6.300E-03	1.400E-03	7.043E-04	2.820E-04	1.421E-04
	$\widehat{MSE}$	6.400E-03	1.400E-03	7.064E-04	2.819E-04	1.441E-04
	CP	0.9215	0.9130	0.9122	0.9117	0.9116
	$\widehat{CP}$	0.9213	0.9128	0.9119	0.9114	0.9118
	$E[N_m]$	19.6836	95.8195	190.7647	478.7536	951.6705
	$\widehat{E}[N_m]$	19.6755	95.8336	190.7202	478.8408	951.5641
	$E[\widehat{R}_{N_m}]$	0.6627	0.6658	0.6662	0.6665	0.6666
10	$\widehat{E}[\widehat{R}_{N_m}]$	0.6628	0.6658	0.6662	0.6665	0.6666
	$MSE[\hat{R}_{N_m}]$	5.400E-03	1.100E-03	5.748E-04	2.292E-04	1.153E-04
	$\widehat{MSE}$	5.400E-03	1.100E-03	5.700E-04	2.289E-04	1.149E-04
	CP	0.9401	0.9327	0.9320	0.9316	0.9315
	$\widehat{CP}$	0.9402	0.9324	0.9323	0.9314	0.9314
	$E[N_m]$	21.5975	98.193	195.5053	490.6745	975.3743
	$\widehat{E}[N_m]$	21.6000	98.1677	195.5746	490.7444	975.5926
	$E[\widehat{R}_{N_m}]$	0.6633	0.6659	0.6663	0.6665	0.6666
20	$\widehat{E}[\widehat{R}_{N_m}]$	0.6633	0.6659	0.6663	0.6665	0.6666
	$MSE[\widehat{R}_{N_m}]$	4.600E-03	1.100E-03	5.299E-04	2.112E-04	1.062E-04
	$\widehat{MSE}$	4.600E-03	1.100E-03	5.316E-04	2.096E-04	1.063E-04
	CP	0.9586	0.9420	0.9415	0.9412	0.9410
	$\widehat{CP}$	0.9587	0.9422	0.9417	0.9415	0.9408

Table 4: Characteristics and simulated values of two-stage procedure with $\alpha = 0.05$, $\theta_1 = 1$, $\theta_2 = 3$, $m = 5, 10, 20$ and $R = 0.7500$.

m	n^*	20	100	200	500	100
	d	0.1162	0.0520	0.0367	0.0232	0.0164
	$E[N_m]$	20.6674	100.9704	201.4414	502.8482	1005.1994
	$\widehat{E}[N_m]$	20.6543	101.1541	201.8539	502.0154	1006.0814
	$E[\widehat{R}_{N_m}]$	0.7438	0.7486	0.7493	0.7497	0.7499
5	$\widehat{E}[\widehat{R}_{N_m}]$	0.7438	0.7486	0.7492	0.7496	0.7498
	$MSE[\widehat{R}_{N_m}]$	4.92E-03	1.10E-03	5.58E-04	2.24E-04	1.12E-04
	$\widehat{MSE}$	4.94E-03	1.08E-03	5.53E-04	2.26E-04	1.13E-04
	CP	0.9150	0.9051	0.9044	0.9040	0.9038
	$\widehat{CP}$	0.9157	0.9062	0.9051	0.9036	0.9047
	$E[N_m]$	20.9463	101.6958	202.8921	506.4787	1012.4584
	$\widehat{E}[N_m]$	20.9310	101.5722	202.8118	506.4290	1013.5033
	$E[\widehat{R}_{N_m}]$	0.7449	0.7489	0.7494	0.7498	0.7499
10	$\widehat{E}[\widehat{R}_{N_m}]$	0.7451	0.7488	0.7494	0.7497	0.7498
	$MSE[\widehat{R}_{N_m}]$	4.01E-03	8.59E-04	4.30E-04	1.72E-04	8.59E-05
	$\widehat{MSE}$	4.04E-03	8.64E-04	4.34E-04	1.71E-04	8.60E-05
	CP	0.9361	0.9268	0.9264	0.9261	0.9260
	$\widehat{CP}$	0.9354	0.9269	0.9248	0.9263	0.9261
	$E[N_m]$	22.7565	101.4015	202.3025	505.0060	1009.5122
	$\widehat{E}[N_m]$	22.7634	101.4896	202.2260	505.0895	1008.7894
	$E[\widehat{R}_{N_m}]$	0.7458	0.7490	0.7495	0.7498	0.7499
20	$\widehat{E}[\widehat{R}_{N_m}]$	0.7457	0.7490	0.7495	0.7498	0.7499
	$MSE[\widehat{R}_{N_m}]$	3.25E-03	7.73E-04	3.86E-04	1.54E-04	7.72E-05
	$\widehat{MSE}$	3.24E-03	7.75E-04	3.87E-04	1.54E-04	7.77E-05
	CP	0.9587	0.9384	0.9381	0.9379	0.9378
	$\widehat{CP}$	0.9594	0.9378	0.9374	0.9390	0.9365

Table 5: Characteristics and simulated values of two-stage procedure with $\alpha = 0.05$, $\theta_1 = 1$, $\theta_2 = 5$, $m = 5, 10, 20$ and $R = 0.8333$.

m	n^*	20	100	200	500	1000
	d	0.0861	0.0385	0.0272	0.0172	0.0122
	$E[N_m]$	24.2817	118.8283	237.1490	592.1213	1183.7429
	$\widehat{E}[N_m]$	24.2631	118.9932	236.8460	591.1848	1176.5900
	$E[\widehat{R}_{N_m}]$	0.8269	0.8317	0.8325	0.8330	0.8332
5	$\widehat{E}[\widehat{R}_{N_m}]$	0.8268	0.8317	0.8326	0.8330	0.8332
	$MSE[\widehat{R}_{N_m}]$	3.09E-03	7.08E-04	3.56E-04	1.42E-04	7.08E-05
	$\widehat{MSE}$	3.10E-03	7.10E-04	3.58E-04	1.43E-04	7.27E-05
	CP	0.9074	0.8945	0.8936	0.8932	0.8930
	$\widehat{CP}$	0.9069	0.8947	0.8936	0.8937	0.8917
	$E[N_m]$	22.9999	111.2373	221.9707	554.1766	1107.8533
	$\widehat{E}[N_m]$	22.9813	111.3915	221.7133	553.5728	1107.2376
	$E[\widehat{R}_{N_m}]$	0.8282	0.8321	0.8327	0.8331	0.8332
10	$\widehat{E}[\widehat{R}_{N_m}]$	0.8281	0.8322	0.8328	0.8331	0.8333
	$MSE[\widehat{R}_{N_m}]$	2.34E-03	5.10E-04	2.53E-04	1.01E-04	5.02E-05
	$\widehat{MSE}$	2.35E-03	5.02E-04	2.53E-04	1.01E-04	5.00E-05
	CP	0.9317	0.9194	0.9190	0.9188	0.9187
	$\widehat{CP}$	0.9317	0.9207	0.9190	0.9186	0.9186
	$E[N_m]$	24.1967	106.2557	212.0079	529.2696	1058.0393
	$\widehat{E}[N_m]$	24.2183	106.2909	212.1401	530.3853	1059.8462
	$E[\widehat{R}_{N_m}]$	0.8293	0.8323	0.8328	0.8331	0.8332
20	$\widehat{E}[\widehat{R}_{N_m}]$	0.8294	0.8323	0.8328	0.8331	0.8332
	$MSE[\widehat{R}_{N_m}]$	1.79E-03	4.42E-04	2.20E-04	8.75E-05	4.37E-05
	$\widehat{MSE}$	1.79E-03	4.41E-04	2.19E-04	8.71E-05	4.40E-05
	CP	0.9572	0.9337	0.9335	0.9335	0.9335
	$\widehat{CP}$	0.9572	0.9331	0.9336	0.9335	0.9322

Table 6: Characteristics and simulated values of two-stage procedure with $\alpha = 0.05$, $\theta_1 = 1$, $\theta_2 = 7$, $m =$5, 10, 20 and $R = 0.875$.

m	n^*	20	100	200	500	1000
	d	0.0678	0.0303	0.0214	0.0136	0.0096
	$E[N_m]$	27.1543	133.0503	265.5876	663.2155	1325.9306
	$\widehat{E}[N_m]$	27.1849	132.9368	266.0099	663.1108	1329.4560
	$E[\widehat{R}_{N_m}]$	0.8690	0.8735	0.874217	0.8747	0.8748
5	$\widehat{E}[\widehat{R}_{N_m}]$	0.8689	0.8734	0.874237	0.8747	0.8748
	$MSE[\widehat{R}_{N_m}]$	2.10E-03	4.84E-04	2.42E-04	9.60E-05	4.76E-05
	$\widehat{MSE}$	2.09E-03	4.84E-04	2.40E-04	9.56E-05	4.84E-05
	CP	0.9038	0.8887	0.887705	0.8872	0.8871
	$\widehat{CP}$	0.9042	0.8882	0.88737	0.8869	0.8865
	$E[N_m]$	24.4308	117.9825	235.4582	587.8949	1175.2899
	$\widehat{E}[N_m]$	24.3310	118.0597	235.364	588.6521	1177.0300
	$E[\widehat{R}_{N_m}]$	0.8704	0.8739	0.874449	0.8748	0.8749
10	$\widehat{E}[\widehat{R}_{N_m}]$	0.8705	0.8738	0.874447	0.8748	0.8749
	$MSE[\widehat{R}_{N_m}]$	1.51E-03	3.32E-04	1.64E-04	6.50E-05	3.24E-05
	$\widehat{MSE}$	1.49E-03	3.34E-04	1.65E-04	6.57E-05	3.27E-05
	CP	0.9296	0.9153	0.914875	0.9147	0.9146
	$\widehat{CP}$	0.9300	0.9146	0.91502	0.9131	0.9148
	$E[N_m]$	25.0496	109.4479	218.3886	545.2214	1089.9429
	$\widehat{E}[N_m]$	25.0325	109.3863	218.7623	546.4983	1093.5911
	$E[\widehat{R}_{N_m}]$	0.8714	0.8741	0.874528	0.8748	0.8749
20	$\widehat{E}[\widehat{R}_{N_m}]$	0.8715	0.8740	0.874464	0.8748	0.8749
	$MSE[\widehat{R}_{N_m}]$	1.12E-03	2.81E-04	1.39E-04	5.54E-05	2.76E-05
	$\widehat{MSE}$	1.12E-03	2.83E-04	1.39E-04	5.49E-05	2.73E-05
	CP	0.9561	0.9310	0.930961	0.9310	0.9310
	$\widehat{CP}$	0.9574	0.9308	0.93061	0.9324	0.9319

Table 7: Characteristics and simulated values of purely sequential procedure with $\alpha = 0.05$, $\theta_1 = 1$, $\theta_2 = 2$, $m = 5, 10, 20$ and $R = 0.6666$.

m	n^*	20	100	200	500	1000
	d	0.1377	0.0616	0.0436	0.0275	0.0195
	$E[N_d]^*$	18.9280	99.2708	199.0051	501.1108	997.0738
	$\widehat{E}[N_d]^*$	18.9148	99.2436	199.1469	501.6570	997.9636
	$E[\widehat{R}_{N_d}]^*$	0.6623	0.6659	0.6663	0.6665	0.6666
5	$\widehat{E}[\widehat{R}_{N_d}]^*$	0.6622	0.6659	0.6663	0.6665	0.6666
	$MSE[\widehat{R}_{N_d}]^*$	6.000E-03	1.000E-03	5.033E-04	1.975E-04	9.912E-05
	$\widehat{MSE}^*$	6.100E-03	1.000E-03	5.035E-04	1.949E-04	9.733E-05
	CP^*	0.9281	0.9470	0.9493	0.9496	0.9499
	$\widehat{CP}^*$	0.9286	0.9474	0.9491	0.9498	0.9498
	$E[N_{N_d}]^*$	19.4449	99.3574	199.2510	500.8986	997.0674
	$\widehat{E}[N_{N_d}]^*$	19.4396	99.2483	199.1443	501.1976	997.2442
	$E[\widehat{R}_{N_d}]^*$	0.6627	0.6659	0.6663	0.6665	0.6666
10	$\widehat{E}[\widehat{R}_{N_d}]^*$	0.6627	0.6659	0.6663	0.6665	0.6666
	$MSE[\widehat{R}_{N_d}]^*$	5.400E-03	1.000E-03	4.987E-04	1.975E-04	9.907E-05
	$\widehat{MSE}^*$	5.400E-03	1.000E-03	5.006E-04	1.957E-04	9.982E-05
	CP^*	0.9408	0.9480	0.9493	0.9497	0.9499
	$\widehat{CP}^*$	0.9400	0.9477	0.9492	0.9496	0.9498
	$E[N_{N_d}]^*$	21.5103	99.3548	199.2791	501.4209	997.3726
	$\widehat{E}[N_{N_d}]^*$	21.5255	99.2791	199.1347	501.3926	997.3315
	$E[\widehat{R}_{N_d}]^*$	0.6633	0.6659	0.6663	0.6665	0.6666
20	$\widehat{E}[\widehat{R}_{N_d}]^*$	0.6633	0.6659	0.6663	0.6665	0.6666
	$MSE[\widehat{R}_{N_d}]^*$	4.600E-03	1.000E-03	4.986E-04	1.974E-04	9.912E-05
	$\widehat{MSE}^*$	4.700E-03	1.000E-03	4.872E-04	1.934E-04	9.568E-05
	CP^*	0.9588	0.9481	0.9493	0.9499	0.9499
	$\widehat{CP}^*$	0.9583	0.9481	0.9493	0.9498	0.9499

Table 8: Characteristics and simulated values of purely sequential procedure with $\alpha = 0.05$, $\theta_1 = 1$, $\theta_2 = 3$, $m =$ 5, 10, 20 and $R = 0.7500$.

m	n^*	20	100	200	500	1000
	d	0.1162	0.0520	0.0367	0.0232	0.0164
	$E[N_d]^*$	18.6937	98.9637	198.9288	498.8600	998.2622
	$\widehat{E}[N_d]^*$	18.8218	98.9522	199.0617	499.5763	999.2656
	$E[\widehat{R}_{N_d}]^*$	0.7437	0.7490	0.7495	0.7498	0.749906
5	$\widehat{E}[\widehat{R}_{N_d}]^*$	0.7446	0.7492	0.7495	0.7495	0.749953
	$MSE[\widehat{R}_{N_d}]^*$	5.00E-03	7.66E-04	3.63E-04	1.42E-04	7.06E-05
	$\widehat{MSE}^*$	4.81E-03	7.53E-04	3.62E-04	1.42E-04	6.82E-05
	CP^*	0.9135	0.9439	0.9475	0.9491	0.949495
	$\widehat{CP}^*$	0.9183	0.9434	0.9457	0.9496	0.9534
	$E[N_d]^*$	19.6192	99.1603	199.3606	499.1454	999.3403
	$\widehat{E}[N_d]^*$	19.5277	99.0741	199.5256	499.0761	999.4349
	$E[\widehat{R}_{N_d}]^*$	0.7447	0.7490	0.7495	0.7498	0.749906
10	$\widehat{E}[\widehat{R}_{N_d}]^*$	0.7447	0.7491	0.7494	0.7498	0.74976
	$MSE[\widehat{R}_{N_d}]^*$	4.13E-03	7.37E-04	3.58E-04	1.42E-04	7.05E-05
	$\widehat{MSE}^*$	4.14E-03	7.51E-04	3.58E-04	1.40E-04	7.08E-05
	CP^*	0.9331	0.9450	0.9480	0.9492	0.949624
	$\widehat{CP}^*$	0.9358	0.9430	0.9489	0.9513	0.9487
	$E[N_d]^*$	22.3833	99.1211	199.1688	499.3151	999.6274
	$\widehat{E}[N_d]^*$	22.3847	99.3918	199.3486	499.2030	999.8737
	$E[\widehat{R}_{N_d}]^*$	0.7458	0.7490	0.7495	0.7498	0.749906
20	$\widehat{E}[\widehat{R}_{N_d}]^*$	0.7465	0.7491	0.7495	0.7497	0.749952
	$MSE[\widehat{R}_{N_d}]^*$	3.28E-03	7.34E-04	3.58E-04	1.42E-04	7.05E-05
	$\widehat{MSE}^*$	3.24E-03	7.53E-04	3.56E-04	1.41E-04	7.14E-05
	CP^*	0.9577	0.9452	0.9479	0.9492	0.9496
	$\widehat{CP}^*$	0.9607	0.9421	0.9501	0.9509	0.9477

Table 9: Characteristics and simulated values of purely sequential procedure with $\alpha = 0.05$, $\theta_1 = 1$, $\theta_2 = 5$, $m =$ 5, 10, 20 and $R = 0.8333$.

m	n^*	20	100	200	500	1000
	d	0.0861	0.0385	0.0272	0.0172	0.0122
5	$E[N_d]^*$	18.6361	98.0376	198.2179	498.7504	999.1778
	$\widehat{E}[N_d]^*$	18.5870	98.2458	198.3566	498.7384	999.5550
	$E[\widehat{R}_{N_d}]^*$	0.8265	0.8322	0.8328	0.8331	0.8332
	$\widehat{E}[\widehat{R}_{N_d}]^*$	0.8268	0.8319	0.8328	0.8332	0.8333
	$MSE[\widehat{R}_{N_d}]^*$	3.26E-03	4.91E-04	2.15E-04	7.91E-05	3.88E-05
	$\widehat{MSE}^*$	3.29E-03	4.67E-04	2.03E-04	7.78E-05	3.84E-05
	CP^*	0.9009	0.9360	0.9447	0.9484	0.9493
	$\widehat{CP}^*$	0.9034	0.9350	0.9457	0.9480	0.9500
10	$E[N_d]^*$	19.6835	98.2149	198.9431	499.5781	1000.3711
	$\widehat{E}[N_d]^*$	19.9368	98.4462	199.0357	498.5999	999.3145
	$E[\widehat{R}_{N_d}]^*$	0.8278	0.8323	0.8329	0.8331	0.8332
	$\widehat{E}[\widehat{R}_{N_d}]^*$	0.8279	0.8322	0.8330	0.8331	0.8332
	$MSE[\widehat{R}_{N_d}]^*$	2.51E-03	4.28E-04	2.00E-04	7.80E-05	3.88E-05
	$\widehat{MSE}^*$	2.49E-03	4.21E-04	1.99E-04	7.77E-05	3.96E-05
	CP^*	0.9246	0.9395	0.9461	0.9487	0.9494
	$\widehat{CP}^*$	0.9255	0.9438	0.9468	0.9455	0.9458
20	$E[N_d]^*$	23.1967	98.7593	198.9704	498.2473	998.8203
	$\widehat{E}[N_d]^*$	23.1966	98.6641	198.5786	499.0432	1000.2725
	$E[\widehat{R}_{N_d}]^*$	0.8292	0.8323	0.8329	0.8331	0.8332
	$\widehat{E}[\widehat{R}_{N_d}]^*$	0.8285	0.8322	0.8327	0.8331	0.8332
	$MSE[\widehat{R}_{N_d}]^*$	1.83E-03	4.17E-04	1.99E-04	7.83E-05	3.88E-05
	$\widehat{MSE}^*$	1.87E-03	4.28E-04	2.02E-04	7.74E-05	3.88E-05
	CP^*	0.9555	0.9414	0.9462	0.9484	0.9493
	$\widehat{CP}^*$	0.9523	0.9392	0.9447	0.9519	0.9484

Table 10: Characteristics and simulated values of purely sequential procedure with $\alpha = 0.05$, $\theta_1 = 1$, $\theta_2 = 7$, $m = 5, 10, 20$ and $R = 0.875$.

m	n^*	20	100	200	500	1000
	d	0.0678	0.0303	0.0214	0.0136	0.0096
	$E[N_d]^*$	18.5717	97.6680	198.3381	498.8221	999.3101
	$\widehat{E}[N_d]^*$	18.5204	96.8969	198.2913	499.5355	998.3575
	$E[\widehat{R}_{N_d}]^*$	0.8685	0.8739	0.8745	0.8748	0.8749
5	$\widehat{E}[\widehat{R}_{N_d}]^*$	0.8688	0.8743	0.8746	0.8749	0.8749
	$MSE[\widehat{R}_{N_d}]^*$	2.27E-03	3.29E-04	1.38E-04	5.03E-05	2.54E-05
	$\widehat{MSE}^*$	2.23E-03	3.59E-04	1.37E-04	4.93E-05	2.42E-05
	CP^*	0.8937	0.9320	0.9435	0.9479	0.9490
	$\widehat{CP}^*$	0.8954	0.9305	0.9405	0.9469	0.9451
	$E[N_d]^*$	19.7899	98.2148	198.4795	498.5058	998.1902
	$\widehat{E}[N_d]^*$	20.1036	98.1968	198.2880	498.6699	999.7197
	$E[\widehat{R}_{N_d}]^*$	0.8700	0.8741	0.8746	0.8748	0.8749
10	$\widehat{E}[\widehat{R}_{N_d}]^*$	0.8707	0.8740	0.8745	0.8748	0.8749
	$MSE[\widehat{R}_{N_d}]^*$	1.64E-03	2.77E-04	1.25E-04	4.87E-05	2.42E-05
	$\widehat{MSE}^*$	1.53E-03	2.80E-04	1.25E-04	4.82E-05	2.43E-05
	CP^*	0.9208	0.9365	0.9450	0.9481	0.9490
	$\widehat{CP}^*$	0.9275	0.9360	0.9434	0.9489	0.9487
	$E[N_d]^*$	23.6220	97.9493	199.1339	498.9835	998.5301
	$\widehat{E}[N_d]^*$	23.5805	98.3716	198.3858	498.8461	999.3820
	$E[\widehat{R}_{N_d}]^*$	0.8713	0.8741	0.8746	0.8748	0.8749
20	$\widehat{E}[\widehat{R}_{N_d}]^*$	0.8715	0.8741	0.8746	0.8748	0.8750
	$MSE[\widehat{R}_{N_d}]^*$	1.15E-03	2.69E-04	1.25E-04	4.86E-05	2.42E-05
	$\widehat{MSE}^*$	1.18E-03	2.71E-04	1.27E-04	5.04E-05	2.37E-05
	CP^*	0.9539	0.9376	0.9451	0.9482	0.9491
	$\widehat{CP}^*$	0.9524	0.9354	0.9450	0.9421	0.9513

Table 11: Characteristic of the two-stage procedure and the smallest m such that coverage probability be at least 95%.

	n^*	20	100	200	500	1000
$\theta_1 = 1, \theta_2 = 2$	d	0.1377	0.0616	0.0616	0.0275	0.0195
	m	16	88	181	464	939
	$E[N_m]$	20.4368	100.6199	200.6407	500.5397	1000.3440
	$\widehat{E}[N_m]$	20.4368	100.6246	200.6659	500.6219	1000.4066
	$E[\widehat{R}_{N_m}]$	0.6630	0.6659	0.6663	0.6665	0.6666
	$\widehat{E}[\widehat{R}_{N_m}]$	0.6631	0.6660	0.6663	0.6665	0.6666
	$MSE[\widehat{R}_{N_m}]$	5.00E-03	9.89E-04	5.00E-04	2.00E-04	9.88E-05
	$\widehat{MSE}$	4.90E-03	9.91E-04	5.00E-04	2.00E-04	9.90E-05
	CP	0.9505	0.9500	0.9500	0.9500	0.9500
	$\widehat{CP}$	0.9507	0.9503	0.9493	0.9496	0.9495
$\theta_1 = 1, \theta_2 = 3$	d	0.1162	0.0520	0.0367	0.0232	0.0164
	m	17	87	178	458	931
	$E[N_m]$	21.7933	102.1451	202.1986	502.2104	1002.0449
	$\widehat{E}[N_m]$	21.8044	102.1311	202.2000	502.1621	1001.9552
	$E[\widehat{R}_{N_m}]$	0.7455	0.7491	0.7495	0.7498	0.7499
	$\widehat{E}[\widehat{R}_{N_m}]$	0.7455	0.7490	0.7495	0.7498	0.7499
	$MSE[\widehat{R}_{N_m}]$	3.48E-03	7.03E-04	3.52E-04	1.41E-04	7.03E-05
	$\widehat{MSE}$	3.48E-03	7.03E-04	3.53E-04	1.40E-04	7.06E-05
	CP	0.9518	0.9501	0.9500	0.9500	0.9500
	$\widehat{CP}$	0.9516	0.9491	0.9495	0.9501	0.9498
$\theta_1 = 1, \theta_2 = 5$	d	0.0861	0.0385	0.0272	0.0172	0.0122
	m	17	86	176	452	922
	$E[N_m]$	23.3406	104.1099	204.3535	504.4301	1004.4449
	$\widehat{E}[N_m]$	23.2954	104.1361	204.3072	504.3212	1004.7583
	$E[\widehat{R}_{N_m}]$	0.8290	0.8324	0.8329	0.8331	0.8332
	$\widehat{E}[\widehat{R}_{N_m}]$	0.8290	0.8325	0.8329	0.8331	0.8332
	$MSE[\widehat{R}_{N_m}]$	1.93E-03	3.86E-04	1.93E-04	7.72E-05	3.86E-05
	$\widehat{MSE}$	1.93E-03	3.89E-04	1.93E-04	7.66E-05	3.86E-05
	CP	0.9501	0.9500	0.9500	0.9500	0.9500
	$\widehat{CP}$	0.9503	0.9496	0.9503	0.9508	0.9497
$\theta_1 = 1, \theta_2 = 7$	d	0.0678	0.0303	0.0214	0.0136	0.0096
	m	17	86	175	450	917
	$E[N_m]$	24.4921	104.1099	205.5915	505.8347	1005.8066
	$\widehat{E}[N_m]$	24.5044	104.1459	205.5118	506.0394	1005.5856
	$E[\widehat{R}_{N_m}]$	0.8713	0.8324	0.8746	0.8748	0.8749
	$\widehat{E}[\widehat{R}_{N_m}]$	0.8713	0.8324	0.8746	0.8748	0.8749
	$MSE[\widehat{R}_{N_m}]$	1.18E-03	3.86E-04	1.20E-04	4.79E-05	2.40E-05
	$\widehat{MSE}$	1.17E-03	3.84E-04	1.21E-04	4.78E-05	2.39E-05
	CP	0.9513	0.9500	0.9500	0.9500	0.9500
	$\widehat{CP}$	0.9517	0.9498	0.9490	0.9501	0.9501

6   Real Data

In this section, the analysis of a pair of real datasets is presented for illustrative purposes. Breaking strengths of jute fiber into two different gauge lengths are shown in Table 12-13. These two datasets were used by Xia *et al*. (2009) and presented earlier in Mirjalili *et al*. (2016). According to the latter, the data in Tables 12-13 have exponential distribution. Let random variables, *X* and *Y* be breaking strengths of jute fiber of gauge length 10mm and 20mm, respectively. Define reliability parameter under the stress-strength model, as the probability that jute fiber of gauge length 20 mm endures more than that of gauge length 10 mm.

Table 12: Dataset 1 (Breaking strength of jute fiber of gauge length 10 mm).

693.73	704.66	323.83	778.17	123.06	637.66	383.43	151.48	108.94	50.16
671.49	183.16	257.44	727.23	291.27	101.15	376.42	163.40	141.38	700.74
262.90	353.24	422.11	43.93	590.48	212.13	303.90	506.60	530.55	177.25

Table 13: Dataset 2 (Breaking strength of jute fiber of gauge length 20 mm).

71.46	419.02	284.64	585.57	456.60	113.85	187.85	688.16	662.66	45.58
578.62	756.70	594.29	166.49	99.72	707.36	765.14	187.13	145.96	350.70
547.44	116.99	375.81	581.60	119.86	48.01	200.16	36.75	244.53	83.55

Treating these two datasets as universal, we implemented both two-stage and purely sequential procedures, drawing observations (X, Y) from the full set of data as needed. To obtain the fixed-width $2d$ asymptotic confidence interval for parameter R, we carried out a single run under both procedures. Tables 14-15 provide the results derived from implementing the stopping rules from (3.1) and (4.1) respectively, when the initial sample size m =5, 10, $\alpha = 0.05$ and $d = 0.3, 0.2, 0.15$ are chosen arbitrarily. Under both methodologies, the final estimators $\hat{R}_{N_m}$ and $\hat{R}_{N_d}$ tended to get closer to their exact value $\hat{R} = 0.482$ obtained from the full data as the initial sample size m increases. Both estimators $\hat{R}_{N_m}$ and $\hat{R}_{N_d}$ are less than 0.5. It means that jute fibers of gauge length of 20 mm do not endure more than jute fibers of gauge length of 10 mm.

Table 14: An illustration with breaking strength of jute fiber of gauge length 10 mm, 20 mm, with d =0.3, 0.2, 0.15, m =5, 10 and α =0.05 using the two-stage procedure.

d	m =5	m =10
0.3	Pilot Data: X: 50.16, 108.94, 163.40, 376.42, 637.66 Y: 594.29, 113.85, 756.70, 36.75, 48.01, $\bar{X}_5$=418.976, $\bar{Y}_5$=490.518 $\longrightarrow N_m = 6$ Second Stage Data Size: X :530.55 Y :662.66 $\bar{X}_{N_m}$=418.976, $\bar{Y}_{N_m}$=490.518 $\longrightarrow \widehat{R}_{N_m} = 0.5390$ $(\widehat{R}_{N_m} - d, \widehat{R}_{N_m} + d) = (0.239, 0.839).$	Pilot Data: X : 727.23, 163.4, 637.66, 123.06, 50.16, 671.49, 353.24, 700.74, 212.13, 257.44 Y : 200.16, 145.96, 350.70, 578.62, 187.13, 756.70, 36.75, 375.81, 113.85, 166.49 $\bar{X}_{10}$=389.655, $\bar{Y}_{10}$=291.217 $\longrightarrow N_m = 10$ Second Stage Data Size: Pilot Data is enough. $\bar{X}_{N_m}$=389.655, $\bar{Y}_{N_m}$=291.217 $\longrightarrow \widehat{R}_{N_m} = 0.4277$ $(\widehat{R}_{N_m} - d, \widehat{R}_{N_m} + d) = (0.1277, 0.7277).$
0.2	Pilot Data: X : 303.90, 212.13, 291.27, 693.73, 383.43 Y : 116.99, 45.58, 581.60, 707.36, 119.86 $\bar{X}_5$=376.892, $\bar{Y}_5$=314.278 $\longrightarrow N_m = 12$ Second Stage Data Size: X : 151.48, 727.23, 637.66, 353.24, 530.55, 177.25, 101.15 Y : 662.66, 375.81, 688.16, 145.96, 48.01, 284.64, 113.85 $\bar{X}_{N_m}$=380.2517, $\bar{Y}_{N_m}$=324.2067 $\longrightarrow \widehat{R}_{N_m} = 0.4602$ $(\widehat{R}_{N_m} - d, \widehat{R}_{N_m} + d) = (0.2602, 0.6602).$	Pilot Data: X : 212.13, 704.66, 727.23, 303.9, 530.55, 323.83, 123.06, 693.73, 141.38, 506.6 Y : 48.01, 581.60, 765.14, 244.53, 284.64, 116.99, 350.70, 662.66, 756.70, 113.85 $\bar{X}_{10}$=426.707, $\bar{Y}_{10}$=392.482 $\longrightarrow N_m = 12$ Second Stage Data Size: X : 506.60, 212.13 Y : 350.70, 116.99 $\bar{X}_{N_m}$=415.4833, $\bar{Y}_{N_m}$=366.0425 $\longrightarrow \widehat{R}_{N_m} = 0.4684$ $(\widehat{R}_{N_m} - d, \widehat{R}_{N_m} + d) = (0.2684, 0.6684).$
0.15	Pilot Data: X : 257.44, 323.83, 303.9, 506.6, 123.06 Y 578.62, 166.49, 244.53, 36.75, 113.85, $\bar{X}_5$=302.966 , $\bar{Y}_5$=228.048 $\longrightarrow N_m = 21$ Second Stage Data Size: X : 530.55, 422.11, 323.83, 700.74, 123.06, 108.94, 183.16, 163.40, 590.48, 177.25, 778.17, 727.23, 101.15, 212.13, 50.16, 262.9 Y : 594.29, 113.85, 36.75, 145.96, 375.81, 187.85, 765.14, 547.44, 116.99, 419.02, 71.46, 662.66, 99.72, 581.60, 244.53, 48.01 $\bar{X}_{N_m}$=331.909, $\bar{Y}_{N_m}$=292.92 $\longrightarrow \widehat{R}_{N_m} = 0.4688$ $(\widehat{R}_{N_m} - d, \widehat{R}_{N_m} + d) = (0.3188, 0.6188).$	Pilot Data: X : 141.38, 530.55, 376.42, 353.24, 693.73, 637.66, 43.93, 291.27, 212.13, 383.43 Y 119.86, 765.14, 350.70, 200.16, 375.81, 116.99, 662.66, 244.53, 707.36, 166.49 $\bar{X}_{10}$=366.374, $\bar{Y}_{10}$=370.97 $\longrightarrow N_m = 22$ Second Stage Data Size: X : 506.60, 177.25, 43.93, 108.94, 183.16, 671.49, 353.24, 141.38, 727.23, 50.16, 778.17, 123.06 Y : 200.16, 45.58, 113.85, 594.29, 585.57, 71.46, 662.66, 419.02, 116.99, 166.49, 284.64, 48.01 $\bar{X}_{N_m}$=342.1977, $\bar{Y}_{N_m}$=319.0191 $\longrightarrow \widehat{R}_{N_m} = 0.4825$ $(\widehat{R}_{N_m} - d, \widehat{R}_{N_m} + d) = (0.3325, 0.6325).$

Table 15: An illustration with breaking strength of jute fiber of gauge length 10 mm, 20 mm with d =0.3, 0.2, 0.15, m =5, 10 and α =0.05 using the purely sequential procedure.

d	m =5	m =10
0.3	Pilot Data: X: 141.38, 101.15, 671.49, 163.40, 151.48 Y: 187.85, 688.16, 48.01, 284.64, 547.44 Samples X: 257.44 Y: 83.55 $\longrightarrow N_d$=6 $\bar{X}_{N_d}$= 247.723, $\bar{Y}_{N_d}$ = 306.608, $\widehat{R}_{N_d}$ =0.553 $(\widehat{R}_{N_d} - d, \widehat{R}_{N_d} + d)$ = (0.253, 0.853)	Pilot Data: X: 671.49, 323.83, 506.60, 123.06, 151.48, 183.16, 177.25, 163.40, 637.66, 376.42 Y: 187.85, 113.85, 585.57, 707.36, 284.64, 99.72, 145.96, 166.49, 419.02, 375.81 Samples Initial Sample is sufficient. $\longrightarrow N_d$=10 $\bar{X}_{N_d}$ =331.435, $\bar{Y}_{N_d}$ = 308.627, $\widehat{R}_{N_d}$ =0.482 $(\widehat{R}_{N_d} - d, \widehat{R}_{N_d} + d)$ = (0.182, 0.782)
0.2	Pilot Data: X: 700.74, 257.44, 177.25, 212.13, 778.17 Y: 166.49, 594.29, 547.44, 200.16, 45.58 Samples X: 671.49, 376.42, 637.66, 123.06, 123.06 Y: 119.86, 187.85, 166.49, 99.72, 36.75 $\longrightarrow N_d$=10, $\bar{X}_{N_d}$=405.742, $\bar{Y}_{N_d}$ =216.463, $\widehat{R}_{N_d}$ =0.347 $(\widehat{R}_{N_d} - d, \widehat{R}_{N_d} + d)$ = (0.147, 0.547)	Pilot Data: X: 778.17, 693.73, 141.38, 727.23, 212.13, 177.25, 108.94, 123.06, 671.49, 506.60 Y: 36.75, 48.01, 581.60, 200.16, 350.70, 707.36, 662.66, 594.29, 71.46, 419.02 Samples X: 177.25, 671.49 Y: 187.85, 200.16 $\longrightarrow N_d$=12, $\bar{X}_{N_d}$=415.726, $\bar{Y}_{N_d}$ = 338.335, $\widehat{R}_{N_d}$ =0.448 $(\widehat{R}_{N_d} - d, \widehat{R}_{N_d} + d)$ = (0.248, 0.648)
0.15	Pilot Data: X: 43.93, 212.13, 291.27, 422.11, 163.40 Y: 48.01, 578.62, 99.72, 200.16, 594.29 Samples X : 303.90, 262.90, 108.94, 671.49, 123.06, 141.38, 177.25, 177.25, 262.90, 212.13, 530.55, 257.44, 704.66, 704.66, 101.15, 506.60, 262.90 Y : 688.16, 707.36, 45.58, 688.16, 350.70, 166.49, 456.60, 113.85, 45.58, 36.75, 116.99, 662.66, 244.53, 145.96, 244.53, 284.64, 578.62 $\longrightarrow N_d$=22, $\bar{X}_{N_d}$= 301.909, $\bar{Y}_{N_d}$ =322.634, $\widehat{R}_{N_d}$ =0.516 $(\widehat{R}_{N_d} - d, \widehat{R}_{N_d} + d)$ = (0.366, 0.666)	Pilot Data: X: 383.43, 353.24, 530.55, 151.48, 163.40, 700.74, 212.13, 376.42, 262.90, 257.44 Y: 456.60, 48.01, 350.70, 578.62, 113.85, 707.36, 547.44, 244.53, 45.58, 116.99 Samples X : 257.44, 257.44, 693.73, 50.16, 671.49, 50.16, 163.40, 376.42, 704.66, 183.16, 506.60, 671.49 Y : 585.57, 187.13, 578.62, 113.85, 284.64, 187.85, 578.62, 83.55, 585.57, 71.46, 375.81, 99.72 $\longrightarrow N_d$=22, $\bar{X}_{N_d}$=362.630, $\bar{Y}_{N_d}$ =315.548, $\widehat{R}_{N_d}$ =0.465 $(\widehat{R}_{N_d} - d, \widehat{R}_{N_d} + d)$ = (0.315, 0.615)

To show that the coverage frequency is correspondingly higher than 0.95, we carried out 100 runs under both procedures for each case of Tables 14-15. We found, $\hat{R}_{N_d}$ for purely sequential and $\hat{R}_{N_m}$ for two-stage sequential procedures, then constructed $2d$ fixed-width confidence interval $(\hat{R}_{N_d}^{(i)} - d, \hat{R}_{N_d}^{(i)} + d)$ and $(\hat{R}_{N_m}^{(i)} - d, \hat{R}_{N_m}^{(i)} + d)$ for purely and

two-stage sequential procedures, respectively, for the ith run. Define $Z^{(i)}_{N_d}$ such that

$$Z^{(i)}_{N_d} = \begin{cases} 1 & \hat{R} \in (\hat{R}^{(i)}_{N_d} - d, \hat{R}^{(i)}_{N_d} + d) \\ 0 & O.W. \end{cases}$$

Similarly define $Z^{(i)}_{N_m}$ in which N_m is substituted for N_d. The coverage probability (frequency) is $\sum_{i=1}^{100} z^{(i)}_{N_d}/100$ for purely sequential or $\sum_{i=1}^{100} z^{(i)}_{N_m}/100$ for two-stage sequential procedures. The results of coverage frequency for purely and two-stage sequential procedures are summarized in Table 16. It can be seen that for all the cases, the coverage frequencies are higher than 0.95.

Table 16: Coverage frequency for breaking strength of jute fiber of gauge length 10 mm, 20 mm with 100 runs.

procedures	m	$d = 0.3$	$d = 0.2$	d=0.15
purely sequential	5	0.99	0.98	0.98
	10	1.00	0.99	0.98
two-stage sequential	5	0.99	0.97	0.96
	10	1.00	0.98	0.98

7 Conclusion

Two-stage and purely sequential procedures have been investigated to construct the fixed-width $2d$ confidence interval for reliability parameter under the stress-strength model in the exponential two-sample problem. Approximation of exact distribution of purely sequential procedure, and the exact distribution of two-stage sequential procedure, and their characteristics have also been discussed for the different pilot sample sizes. The accuracy of our method to approximate the distribution of purely sequential procedure has been assessed by using simulation results. Both of these procedures have different routes, merits, and demerits. A two-stage technique is often easy to implement, whereas a purely sequential methodology is proved to be more accurate. One should, however, balance any logistical concerns, such as feasibility and cost.

In the current research, it has been assumed that the samples taken from two

populations are of the same size. However, it might not be the case in reality. So, the MLE of R can be written as,

$$\widehat{R}_{n_1,n_2} = \frac{\bar{Y}_{n_2}}{\bar{Y}_{n_2} + \bar{X}_{n_1}},\tag{7.1}$$

where n_1 is the size of the sample taken from the stress (X) population, and n_2 is the size of the sample from the strength (Y) population. Therefore, defining a stopping rule to find optimal n_1^*, n_2^* for constructing a fixed-width $2d$ confidence interval with confidence at least $1 - \alpha$ for R based on $\hat{R}_{n_1,n_2}$, is yet to be studied and deserves a comprehensive research. Moreover, the exact distribution and asymptotic second-order properties for the proposed purely sequential procedure in this research, are yet to be addressed.

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