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A Goodness of Fit Test For Normality Based on Balakrishnan-Sanghvi Information

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Abstract. We introduce a new goodness of fit test for normality based on Balakrishnan-Sanghvi divergence measure. In order to estimate the divergence measure, we use a method similar to Vasicek's for estimating the Shannon entropy. Also, the test statistic based on kernel density estimation is investigated. Critical values and the power of tests are computed by Monte Carlo simulation. It is shown that the tests are consistent. Further, by comparing the power of proposed tests with other normality tests, we suggest the new entropic based test according to Balakrishnan-Sanghvi divergence measure using Vasicek method.

Keywords. Divergence measure, Entropy, Normality test, Vasicek sample entropy.

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1 Introduction

The goodness of fit test for normality has been widely discussed in different literature. Normal distribution plays a predominant role in various sciences like management, engineering and reliability. So in many applications, we need to test whether the underlying data follow a normal distribution or not. Since the introduction of the chi-squared goodness of fit test for normality by Pearson in 1900, considerable attention has been given to the problem of testing normality and various tests, such as the moment-based test, regression-based test and entropy-based test.

Let f denote the density of a given population, then the null and alternative hypotheses of normality are stated formally as $H_0 : f(x) = f_0(x; \mu, \sigma)$ and $H_1 : f(x) \neq f_0(x; \mu, \sigma)$, where μ and σ are unknown parameters such that

$$f_0(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2\right\}.$$

Cramer Von Mises and Kolmogorov-Smirnov proposed their tests in 1931 and 1933 respectively. Some other popular tests for normality founded by Shapiro-Wilk (1965), Anderson-Darling (1954), Kuiper (1960) and Watson (1961). For the first time, Vasicek (1976) introduced a normality test based on entropy. The entropy of X with distribution function $F(x)$ and a continuous density function $f(x)$ is defined by Shannon (1948) as

$$H(f) = - \int_{-\infty}^{\infty} f(x) \log f(x) dx. \quad (1)$$

The problem of estimating Shannon entropy has been considered by many authors, including Ahmad and Lin (1976), Vasicek (1976), Dudewicz and Van der Meulen (1981), Joe (1989), Van Es (1992), Ebrahimi *et al.* (1994), Corea (1995), Wiczorkowski and Grzegorzewski (1999), Yousefzadeh and Arghami (2008) and Alizadeh (2010). Among these various entropy estimators, Vasicek's sample entropy has been the most widely used in developing entropy-based statistical procedures. With the transformation $F(x) = p$, Vasicek (1976) rewrote equation (1) as

$$H(f) = \int_0^1 \log\left\{\frac{d}{dp}F^{-1}(p)\right\}dp. \quad (2)$$

By replacing the distribution function F with its empirical distribution function F_n and using the well-known difference operator instead of the used differential operator, the

estimator of $H(f)$ is given as

$$H_{n,m} = \frac{1}{n} \sum_{i=1}^n \log \left\{ \frac{n}{2m} (X_{(i+m)} - X_{(i-m)}) \right\}, \tag{3}$$

where $X_{(1)} \leq \dots \leq X_{(n)}$ are the order statistics and m is a positive integer, $m \leq n/2$ and $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$.

The sample entropy was considered in establishing a goodness of fit test statistic for normal distribution by Vasicek (1976) and Arizono and Ohta (1989), also Dudewicz and Van der Meulen (1981) used it for uniform distribution, and Ebrahimi *et al.* (1992) applied it for the exponential distribution.

Vasicek (1976) introduced the test statistic based on the property of maximum entropy of the normal distribution as

$$K_{n,m} = \frac{\exp(H_{n,m})}{\hat{\sigma}} = \frac{n}{2m\hat{\sigma}} \left(\prod_{i=1}^n [X_{(i+m)} - X_{(i-m)}] \right)^{1/n}, \tag{4}$$

where

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}.$$

Park (1999) provided a test of normality based on the sample entropy of order statistics. Choi (2008) improved Vasicek's entropy test as

$$K_{n,m}^c = \frac{R_{n,m}}{\hat{\sigma}_{n-1}} \left(\prod_{i=1}^n [X_{(i+m)} - X_{(i-m)}] \right)^{1/n}, \tag{5}$$

where

$$R_{n,m} = \exp \left\{ - (1 - 2m/n)\psi(2m) + \psi(n + 1) - 2/n \sum_{i=1}^n \psi(k + m - 1) \right\},$$

and ψ is the digamma function such that

$$\psi(t) = \frac{d \ln \Gamma(t)}{dt} = \frac{d \ln \int_0^\infty x^{t-1} t^x dx}{dt},$$

and $\hat{\sigma}_{n-1} = \{n/(n-1)\}^{1/2}\hat{\sigma}$.

Esteban *et al.* (2001) compared four tests of normality using four statistics based on different entropy estimators namely, Vasicek (1976), Van Es (1992), Corea (1995) and Wiczorkowski and Grzegorzewski (1999). Similar to our work, Alizadeh (2010) compared the four mentioned tests with a test based on a new estimator of entropy as

$$TA_{mn} = \frac{\exp\{H_a(m, n)\}}{\hat{\sigma}}, \quad (6)$$

where $\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}$ and $H_a(m, n)$ is the new estimator of entropy as

$$H_a(m, n) = -\frac{1}{n} \sum_{i=1}^n \log\left\{\frac{\hat{f}(X_{(i+m)}) + \hat{f}(X_{(i-m)})}{2}\right\},$$

where $\hat{f}(X) = \frac{1}{nh} \sum_{i=1}^n k\left(\frac{X-X_{(i)}}{h}\right)$, and the kernel function k is chosen to be the standard normal density function and the bandwidth h is chosen to be the normal optimal smoothing formula with $h = 1.06sn^{-\frac{1}{5}}$ where s is the sample standard deviation.

In other works, Alizadeh and Arghami (2011a) compared seven different tests of normality such as the Kolmogorov-Smirnov (1933), Anderson-Darling (1954), Kuiper (1960), Jarque-Bera (1987), Cramer-Von Mises (1931), Shapiro-Wilk (1965) and Vasicek (1976).

This paper is organized as follows. In Section 2, we introduce two test statistics based on Balakrishnan-Sanghvi divergence measure. These statistics are estimated in the manner of Vasicek for estimating the entropy and also estimated using kernel density function. Other methods of testing normality like Corea (1995) and Van Es (1992) are investigated, but their results are not given because of their weak performances. Also in this section, we show the tests are asymptotically consistent. In Section 3, using Monte Carlo simulation, it is shown that the proposed test based on Vasicek method has greater power than its competitors in most of the times.

2 Test statistics

Balakrishnan-Sanghvi divergence measure (Balakrishnan and Sanghvi (1968)) for two density function $f(x)$ and $g(x)$ is defined as

$$D_{BS}(f, g) = \int_{-\infty}^{\infty} \left(\frac{f(x) - g(x)}{f(x) + g(x)} \right)^2 f(x) dx. \tag{7}$$

It is obvious that $D_{BS}(f, g) \geq 0$ and the equality holds if and only if $f(x) = g(x)$. So it motivates us to use Balakrishnan-Sanghvi information as a test statistic for normality. Plugging

$$g(x) = f_0(x; \mu, \sigma) = (1/\sqrt{2\pi\sigma^2}) \exp\{-(x - \mu)^2/2\sigma^2\},$$

in (7), we get

$$D_{BS}(f, g) = \int_{-\infty}^{\infty} f(x) \left(\frac{f(x) - (1/\sqrt{2\pi\sigma^2}) \exp\{-(x - \mu)^2/2\sigma^2\}}{f(x) + (1/\sqrt{2\pi\sigma^2}) \exp\{-(x - \mu)^2/2\sigma^2\}} \right)^2 dx. \tag{8}$$

Now, similar to Vasicek's method for estimating entropy, using $F(x) = p$, equation (8) is expressed as

$$D_{BS}(f, g) = \int_0^1 \left(\frac{\left(\frac{dF^{-1}(p)}{dp}\right)^{-1} - (1/\sqrt{2\pi\sigma^2}) \exp\{-(F^{-1}(p) - \mu)^2/2\sigma^2\}}{\left(\frac{dF^{-1}(p)}{dp}\right)^{-1} + (1/\sqrt{2\pi\sigma^2}) \exp\{-(F^{-1}(p) - \mu)^2/2\sigma^2\}} \right)^2 dp. \tag{9}$$

By replacing F by F_n and using the well-known the difference operator in place of differential operator, we get an estimator B_V of $D_{BS}(f, g)$ as follow

$$B_V = \frac{1}{n} \sum_{i=1}^n \left(\frac{\frac{2m}{n(X_{(i+m)} - X_{(i-m)})} - \left(\frac{1}{\sqrt{2\pi\hat{\sigma}^2}}\right) \exp\left\{\frac{-(X_{(i)} - \hat{\mu})^2}{2\hat{\sigma}^2}\right\}}{\frac{2m}{n(X_{(i+m)} - X_{(i-m)})} + \left(\frac{1}{\sqrt{2\pi\hat{\sigma}^2}}\right) \exp\left\{\frac{-(X_{(i)} - \hat{\mu})^2}{2\hat{\sigma}^2}\right\}} \right)^2, \tag{10}$$

where $X_{(i)} = X_{(1)}$ for $i < 1$, $X_{(i)} = X_{(n)}$ for $i > n$. Also $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2$ are the MLEs of μ and σ^2 under the null hypothesis.

It is obvious that B_V is invariant with respect to scale and location transformations. For the consistency of the test, similar to the proof of Alizadeh (2010) , we prove B_V is

consistent. That is the power of the test under the alternative hypothesis increases to one as $n \rightarrow \infty$. The following statement shows this.

Theorem 2.1. *Let F be an unknown continuous distribution on the real line and let F_0 be distributed as a normal distribution with unknown parameters. Then the test based on B_V is consistent.*

Proof. As $n, m \rightarrow \infty$ and $m/n \rightarrow 0$, we have

$$\begin{aligned} \frac{2m}{n} &= F_n(X_{(i+m)}) - F_n(X_{(i-m)}) \simeq F(X_{(i+m)}) - F(X_{(i-m)}) \\ &\simeq \frac{f(X_{(i+m)}) + f(X_{(i-m)})}{2} (X_{(i+m)} - X_{(i-m)}), \end{aligned}$$

where $F_n(a) = \#\{x_i \leq a\}/n = (1/n) \sum I_{(-\infty, X_i]}(a)$ and I is the indicator function. Therefore noting that $\hat{\mu}$ and $\hat{\sigma}^2$ are consistent estimators, we have

$$\begin{aligned} B_V &= \frac{1}{n} \sum_{i=1}^n \left(\frac{\frac{2m}{n(X_{(i+m)} - X_{(i-m)})} - \left(\frac{1}{\sqrt{2\pi\hat{\sigma}^2}}\right) \exp\left\{\frac{-(X_{(i)} - \hat{\mu})^2}{2\hat{\sigma}^2}\right\}}{\frac{2m}{n(X_{(i+m)} - X_{(i-m)})} + \left(\frac{1}{\sqrt{2\pi\hat{\sigma}^2}}\right) \exp\left\{\frac{-(X_{(i)} - \hat{\mu})^2}{2\hat{\sigma}^2}\right\}} \right)^2 \\ &\simeq \frac{1}{n} \sum_{i=1}^n \left(\frac{\frac{F(X_{(i+m)}) - F(X_{(i-m)})}{X_{(i+m)} - X_{(i-m)}} - \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) \exp\left\{\frac{-(X_{(i)} - \mu)^2}{2\sigma^2}\right\}}{\frac{F(X_{(i+m)}) - F(X_{(i-m)})}{X_{(i+m)} - X_{(i-m)}} + \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) \exp\left\{\frac{-(X_{(i)} - \mu)^2}{2\sigma^2}\right\}} \right)^2 \\ &\rightarrow E \left(\frac{\frac{F(X_{(i+m)}) - F(X_{(i-m)})}{X_{(i+m)} - X_{(i-m)}} - \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) \exp\left\{\frac{-(X - \mu)^2}{2\sigma^2}\right\}}{\frac{F(X_{(i+m)}) - F(X_{(i-m)})}{X_{(i+m)} - X_{(i-m)}} + \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) \exp\left\{\frac{-(X - \mu)^2}{2\sigma^2}\right\}} \right)^2. \end{aligned}$$

where the last limit holds by the law of large numbers. We have $X_{(i-m)}$ and $X_{(i+m)}$ belong to an interval in which $f(x)$ is positive and continuous, then there exists a value $X'_i \in (X_{(i-m)}, X_{(i+m)})$ such that

$$\frac{F(X_{(i+m)}) - F(X_{(i-m)})}{X_{(i+m)} - X_{(i-m)}} = f(X'_i).$$

Therefore $B_V \rightarrow D_{BS}(f, f_0)$. So, the test based on B_V is consistent.

We have estimated Balakrishnan-Sanghvi divergence measure based on spacing method. Also, one can estimate this divergence using methods like kernel-based approaches.

The Balakrishnan-Sanghvi test statistic based on kernel density function is

$$B_K = \frac{1}{n} \sum_{i=1}^n \left(\frac{\hat{f}(x_i) - g(x_i)}{\hat{f}(x_i) + g(x_i)} \right),$$

in which f can be replaced by the kernel density estimator

$$\hat{f}(x_i) = \frac{1}{nh} \sum_{j=1}^n k\left(\frac{x_i - x_j}{h}\right),$$

where the kernel function k is the standard normal density function and the bandwidth h is chosen to be the normal optimal smoothing formula with $h = 1.06sn^{-\frac{1}{5}}$, also g is a normal density function. Similar to Vasicek-based test, we can prove the test based on B_K is consistent.

3 Simulation Study

A simulation study is performed to analyze the efficiency of the proposed test statistics. The critical values are determined using Monte Carlo simulation with 10,000 replicates at the significance level 0.05. Namely, for the test statistic like B_V , we have generated n samples from the standard normal distribution and calculated its sample value ($B_{V,1}$). We did this work for 10,000 simulated random samples and calculated ($B_{V,1}, \dots, B_{V,10,000}$). Since large values of B_V reject the null hypothesis, we specify the 9500th order statistic [$B_{V,(9500)}$] as the critical value of B_V (because $\alpha = 0.05$).

Based on (10) different values of m are given in Table 1. Table 2 gives the critical values of B_V , B_K and the competitor tests for various sample sizes. As can be seen in Table 2, the critical values of B_V and B_K decrease as the sample size n increase unlike other competitors. The reason for this is that in the proposed test large values of the test statistics reject the null hypothesis (in other competitor tests, small values reject the null hypothesis). So, the power of the tests ($P(\text{reject } H_0 | H_1 \text{ is True})$) tends to one when n increases and we expect this matter because the test is consistent. That is, the test statistic estimates $D_{BS}(f, g)$ which is zero under the null hypothesis. So, we expect that the estimator B_V (and therefore critical values) tend to zero under the null hypothesis and as n increases.

Table 1: The values of m for test B_V .

n	m
5 – 9	2
10 – 19	3
20 – 29	5
30 – 49	9
50 – 79	15
80 – 100	24

Table 2: Critical values of B_V and B_K and other competitor tests.

n	W	A^2	$K_{n,m}$	$K_{n,m}^c$	B_V	B_K
5	0.775		1.509	2.679	0.259	0.018
6	0.792		1.632	2.683	0.255	0.024
7	0.809		1.741	2.736	0.246	0.028
8	0.823	0.667	1.815	2.790	0.237	0.030
9	0.834	0.678	1.940	2.833	0.228	0.031
10	0.845	0.686	2.038	3.020	0.194	0.031
11	0.855	0.693	2.112	3.052	0.189	0.031
12	0.861	0.699	2.212	3.091	0.183	0.032
13	0.869	0.703	2.266	3.149	0.174	0.031
14	0.875	0.707	2.334	3.178	0.170	0.030
15	0.882	0.711	2.396	3.235	0.161	0.031
20	0.904	0.722	2.707	3.403	0.135	0.029
25	0.920	0.729	2.875	3.512	0.117	0.026
30	0.930	0.733	2.996	3.606	0.118	0.024
40	0.945	0.738	3.159	3.725	0.097	0.021
50	0.954	0.741	3.318	3.801	0.102	0.019
60	0.961	0.743	3.390	3.870	0.088	0.017
70	0.965	0.745	3.450	3.887	0.080	0.015
80	0.969	0.746	3.443	3.948	0.092	0.014
90	0.972	0.747	3.515	3.978	0.084	0.013
100	0.975	0.747	3.571	4.029	0.078	0.013

In order to compute the powers of B_V and B_K , a battery of Monte Carlo simulation was

Table 3: Critical values of $A_{n,m}(m)$ test.

n	m									
	1	2	3	4	5	6	7	8	9	10
5	3.198	3.309								
6	3.209	3.302	3.386							
7	3.243	3.293	3.363							
8	3.273	3.314	3.381	3.441						
9	3.302	3.333	3.385	3.441						
10	3.336	3.350	3.391	3.440	3.481					
11	3.373	3.383	3.417	3.458	3.498					
12	3.385	3.386	3.417	3.459	3.495	3.527				
13	3.429	3.421	3.442	3.477	3.516	3.547				
14	3.443	3.434	3.452	3.482	3.512	3.545	3.573			
15	3.457	3.448	3.463	3.490	3.521	3.550	3.576			
20	3.547	3.521	3.518	3.531	3.550	3.573	3.595	3.616	3.635	3.653
25	3.621	3.598	3.586	3.588	3.599	3.615	3.634	3.651	3.666	3.681
30	3.671	3.647	3.633	3.629	3.631	3.639	3.651	3.664	3.678	3.691
40	3.748	3.724	3.708	3.700	3.695	3.696	3.699	3.705	3.713	3.722
50	3.794	3.773	3.758	3.748	3.742	3.738	3.737	3.738	3.742	3.747
60	3.831	3.812	3.796	3.785	3.777	3.771	3.768	3.766	3.767	3.769
70	3.866	3.847	3.832	3.822	3.813	3.807	3.802	3.799	3.798	3.798
80	3.892	3.872	3.858	3.847	3.839	3.832	3.828	3.824	3.822	3.821
90	3.904	3.887	3.874	3.864	3.855	3.848	3.842	3.838	3.835	3.833
100	3.922	3.906	3.893	3.882	3.873	3.865	3.859	3.855	3.850	3.848

also applied. We regard the test statistics of normality discussed in Choi (2008) as the competitor tests. These tests are the Vasicek test ($K_{n,m}$), the Anderson-Darling test (A^2), the Shapiro-Wilk test (W) (these three tests have the greatest power among the seven tests of Alizadeh and Arghami (2011a)) and the Choi test ($K_{n,m}^c$). Moreover, the proposed test with the Alizadeh (2010) test $A_{m,n}$ was compared. To compare powers, we consider 7 alternatives used by Choi (2008) which are Uniform (0,1), Exponential (1), Weibull (1.2,2), Gamma (1.5,2), Lognormal (0,1), Chi-squared (2) and Inverse Gaussian (1,1). For each alternative, we generated 10,000 samples of size 10, 20, 30, 50 and 100. For each sample the statistics (A^2 , W , $K_{n,m}$, $K_{n,m}^c$, $A_{n,m}(m)$, B_K and B_V) were evaluated and the power of the corresponding test was estimated by the frequency of the event "the statistic is in the critical region".

Table 4: Estimated powers of the B_V and the B_K tests against the non-normal distribution at the significance level 0.05. (The best values are bolded)

Distribution	Sample size	Test statistic					B_V	B_K
		A^2	W	$K_{n,m}$	$K_{n,m}^c$	$A_{m,n}(m)$		
Uniform(0,1)	10	0.084	0.087	0.174	0.153	0.152(5)	0.105	0.015
	20	0.169	0.198	0.432	0.443	0.451(10)	0.357	0.028
	30	0.300	0.388	0.649	0.704	0.734(15)	0.604	0.186
	50	0.582	0.755	0.928	0.969	0.974(25)	0.915	0.662
	100	0.948	0.997	1	1	1(45)	1	0.989
Exponential(1)	10	0.406	0.438	0.423	0.455	0.457(4)	0.494	0.354
	20	0.771	0.832	0.847	0.854	0.859(9)	0.882	0.774
	30	0.934	0.967	0.970	0.970	0.979(13)	0.982	0.934
	50	0.996	1	0.999	0.999	1(25)	1	0.998
	100	1	1	1	1	1(1)	1	1
Weibull(1.2,2)	10	0.283	0.310	0.276	0.302	0.324(4)	0.337	0.253
	20	0.584	0.659	0.645	0.665	0.692(9)	0.882	0.607
	30	0.791	0.873	0.849	0.879	0.899(13)	0.907	0.806
	50	0.961	0.989	0.986	0.986	0.986(22)	0.990	0.979
	100	1	1	1	0.992	1(1)	1	1
Gamma(1.5,2)	10	0.285	0.310	0.266	0.295	0.326(4)	0.331	0.272
	20	0.589	0.661	0.625	0.644	0.687(8)	0.698	0.607
	30	0.795	0.870	0.829	0.855	0.893(13)	0.895	0.818
	50	0.964	0.987	0.979	0.981	0.993(23)	0.987	0.976
	100	1	1	1	0.986	1(1)	1	1
Lognormal(0,1)	10	0.576	0.607	0.569	0.602	0.622(4)	0.642	0.521
	20	0.906	0.931	0.926	0.932	0.943(9)	0.947	0.908
	30	0.984	0.992	0.988	0.991	0.995(14)	0.996	0.982
	50	0.999	1	1	1	1(20)	1	0.999
	100	1	1	1	1	1(8)	1	1
Chi-squared(2)	10	0.411	0.445	0.430	0.452	0.459(4)	0.496	0.349
	20	0.779	0.839	0.852	0.859	0.863(9)	0.890	0.773
	30	0.936	0.970	0.971	0.974	0.980(14)	0.984	0.935
	50	0.997	0.999	0.999	1	1(20)	1	0.998
	100	1	1	1	1	1(1)	1	1
Inverse Gaussian(1,1)	10	0.515	0.542	0.554	0.529	0.536(4)	0.574	0.466
	20	0.877	0.907	0.894	0.900	0.921(10)	0.925	0.880
	30	0.973	0.986	0.978	0.983	0.990(14)	0.990	0.977
	50	1	1	1	1	1(21)	1	1
	100	1	1	1	1	1(35)	1	1

The goodness of fit test based on entropy involves choosing the best integer parameter m . Unfortunately, there are no choices out of the criterion of m , and in general it depends on the alternative. Vasicek (1976) selected the values of m , which maximize the powers of the test. It is shown by Alizadeh (2010) that, there is no m being appropriate for all alternatives. The values of m that attain maximum powers are also suggested and presented in Table 1 for the B_V test. Table 4 shows the estimated power of the tests B_V and B_K and those of the competing tests, at the significance level $\alpha = 0.05$ based on 10,000 iterations with sample sizes 10, 20, 30, 50 and 100.

According to Table 4, for a symmetric alternative means uniform distribution, the greatest power is related to $K_{n,m}$ test for $n = 10$, and to $A_{m,n}$ test for $n = 20$, $n = 30$ and $n = 50$. However, for the asymmetric alternatives, the power of the B_V test is better than the other tests for all alternatives and for all sample sizes (except Gamma when $n=50$). In addition, the B_V test, has the advantage of owning a fixed m in comparison with $A_{m,n}$ test. So, we can suggest the B_V test as a test of fit for normality for asymmetric alternatives.

Concerning B_K , it can be understood that its powers are weaker in comparison with the other tests (except A^2 for large sample sizes in most of the times). But B_K , just has the advantage of being independent on fixed m .

4 Conclusion

In the present paper, a goodness of fit test for normality based on Balakrishnan-Sanghvi divergence was introduced. To construct the test statistic, two methods were investigated. The first method was similar to Vasicek's method for estimating the Shannon entropy and the second one, was based on kernel density function. By a simulation study, the powers of the proposed tests were computed under several alternatives and different sample sizes. It is shown that, the B_V test, compares favorably with the leading competitors for every sample size especially for the asymmetric alternatives.

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