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Admissibility in a One-parameter Non-regular Family with Squared-log Error Loss Function

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Abstract. Consider an estimation problem in a one-parameter non-regular distribution when both endpoints of the support depend on a single parameter. In this paper, we give sufficient conditions for a generalized Bayes estimator of a parametric function to be admissible. Some examples are given.

Keywords. Admissibility, Generalized Bayes estimator, Non-regular distribution, Squared-log error loss function.

MSC: 62C15; 62F15.

1 Introduction

Consider a random variable X whose probability density function depends on an unknown parameter. Under a quadratic loss function, the admissibility of estimators was discussed by many authors. For a one-parameter exponential family of distributions, Karlin (1958) gave sufficient conditions for linear estimators of the form aX to be admissible in estimating the mean of X. The result of Karlin was generalized by Ghosh & Meeden (1977), Ralescu & Ralescu (1981), Hoffmann (1985), Pulskamp & Ralescu (1991) and Tanaka (2010). Moreover, for the non-regular case when the dimension of

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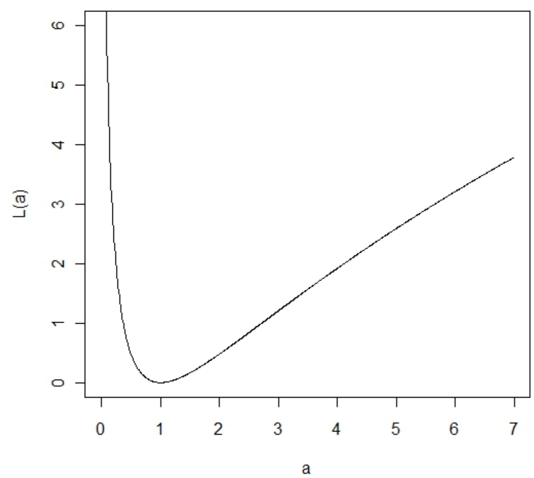

Figure 1: Plot of the squared-log error loss function.

the minimal sufficient statistic is one, sufficient conditions for admissibility are given by Karlin (1958), Sinha & Gupta (1984) and Kim (1994) under quadratic loss function, by Sanjari Farsipour (2003) under entopy loss function, by Tanaka (2011) under LINEX loss function and by Zakerzadeh & Moradi Zahraie (2015) under squared-log error loss function. In addition Kim & Meeden (1994), Sanjari Farsipour (2007) and Tanaka (2012) considered the admissibility of generalized Bayes estimators when the dimension of the minimal sufficient statistic is two.

Consider the asymmetric squared-log error loss of the form

$$L(\delta, h(\theta)) = (\ln(\nabla))^2, \tag{1.1}$$

where $\nabla := \delta/h(\theta)$ and both $h(\theta)$ and δ are positive. This loss function was introduced by Brown (1968); see also Pal and Ling (1996).

The Bayes estimator for $h(\theta)$ under the latter loss function may be given as

$$\delta_\pi(x) = \exp\left\{E_\theta\left[\ln\left(h(\theta)\right)|X = x\right]\right\},$$

where $E_\theta(.|X = x)$ denotes the posterior expectation.

From Figure 1, we see that:

(i) It is asymmetric;

(ii) It is convex when $\nabla \leq e$ and concave otherwise;

(iii) It has a unique minimum at $\nabla = 1$;

(iv) It is a balanced loss function in the sense that $L(\theta, \delta) \to \infty$ as $\delta \to 0$ or $\delta \to \infty$. A balanced loss function takes both error of estimation and goodness of fit into account but the unbalanced loss function only considers error of estimation;

(v) On the basis of this loss, under-estimation/under-prediction is penalized more heavily, per unit distance, than over-estimation/over-prediction.

For estimation under the loss (1.1), see Sanjari Farsipour & Zakerzadeh (2006), Mahmoudi & Zakerzadeh (2011), Kiapour & Nematollahi (2011), Nematollahi & Jafari Tabrizi (2012) and Zakerzadeh & Moradi Zahraie (2015).

Now, suppose that $Y_1, ..., Y_n$ are independent and identically distributed random variables according to the probability distribution function

$$p(y;\theta) = \begin{cases} q^*(\theta) r^*(y) & \text{if } \alpha(\theta) < y < \beta(\theta) \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

for some functions $\alpha(\theta)$ and $\beta(\theta)$, where $\theta \in \Theta =: (\underline{\theta}, \bar{\theta})$ and Θ is a nondegenerate interval (possibly infinite) on the real line. Also $r^*(y)$ is a positive measurable function of y and

$$q^{*-1}(\theta) = \int_{\alpha(\theta)}^{\beta(\theta)} r^*(y) dy < \infty \quad \text{for} \quad \theta \in \Theta.$$

The family of distributions with the density (1.2) is referred to as a non-regular family of distributions.

Before expressing the purpose of this paper, we need the following notations:

Let $X_1 := \min_{1 \leq i \leq n} Y_i$, $X_n := \max_{1 \leq i \leq n} Y_i$, $\mathbf{X} := (X_1, X_n)$ and $\mathbf{x} := (x_1, x_n)$ be an observed value of $\mathbf{X}$. Suppose $I_A(\mathbf{x})$ be the indicator function of set A and, for $\theta \in \Theta$, define

$$\mathbf{x} + \theta := (x_1 + \theta, x_n + \theta),$$

$$\theta\mathbf{x} := (\theta x_1, \theta x_n),$$

$$\chi_\theta := \{\mathbf{x} \in \mathbf{R}^2 | \alpha(\theta) < x_1 < x_n < \beta(\theta)\},$$

and

$$\Theta_\mathbf{x} := \{\theta \in \Theta | \beta^{-1}(x_n) < \theta < \alpha^{-1}(x_1)\}.$$

In (1.2), if both $\alpha(\theta)$ and $\beta(\theta)$ are strictly increasing, then $\mathbf{X}$ is a sufficient statistic for θ and its probability density function is given by

$$f(\mathbf{x};\theta) = \begin{cases} q(\theta)r(\mathbf{x}) & \text{if } \mathbf{x} \in \chi_\theta \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

where $q(\theta)$ and $r(\mathbf{x})$ are positive. Let $\pi(\theta)$ be an improper prior density over Θ which is positive. The generalized Bayes estimator of $h(\theta)$ with respect to $\pi(\theta)$ is given by $\delta_\pi(\mathbf{X})$, where

$$\delta_\pi(\mathbf{x}) = \exp\left\{ \frac{\int_\Theta \{\ln(h(\theta))\} q(\theta)\pi(\theta) I_{\Theta_\mathbf{x}}(\theta) d\theta}{\int_\Theta q(\theta)\pi(\theta) I_{\Theta_\mathbf{x}}(\theta) d\theta} \right\}, \tag{1.4}$$

provided that the integrals exist and are finite.

Since any admissible estimator should be a generalized Bayes estimator, it is enough to focus on generalized Bayes estimator with respect to improper priors. Further, the generalized Bayes estimator based on $Y_1, ..., Y_n$ depends only on the sufficient statistic. Therefore, we can start from the probability density function (1.3).

The rest of this paper is organized as follows. In Section 2, using Karlin's technique, we derive sufficient conditions for admissibility of generalized Bayes estimators in the one parameter non-regular family of distributions with the density of the form (1.3) under the loss (1.1). In Section 3, we treat the estimation of a bounded parametric function and then we consider an estimation problem with a special class of prior densities. Some examples are given.

2 Main Results

In this section, we restrict estimators to the class

$$\Delta := \{\delta | \mathbf{C1} \text{ and } \mathbf{C2} \text{ are satisfied}\},$$

where

C1: $E_\theta[\{\ln(\delta(\mathbf{X}))\}^2] < \infty$ for all $\theta \in \Theta$,

C2: $\int_a^b E_\theta[\{\ln(\frac{\delta(\mathbf{X})}{h(\theta)})\}^2]\pi(\theta)d\theta < \infty$ for all $a < b$ $(a, b \in \Theta)$.

Our purpose is providing sufficient conditions for admissibility of the generalized Bayes estimator (1.4). The next lemma is the main tool for us.

Lemma 2.1. *Let $S(\theta)$ be a continuous and non-negative function over Θ. Suppose that there exists a positive function $R(\theta)$ such that*

$$\int_a^b S(\theta)d\theta \le \sqrt{S(b)}\sqrt{R(b)} + \sqrt{S(a)}\sqrt{R(a)},$$

for all a and b, when $\underline{\theta} < a < b < \bar{\theta}$. If there exists $v \in \Theta$ such that

$$\lim_{u\to\bar{\theta}} \int_v^u \frac{d\theta}{R(\theta)} = \lim_{u\to\underline{\theta}} \int_u^v \frac{d\theta}{R(\theta)} = \infty,$$

then $S(\theta) = 0$ for almost all $\theta \in \Theta$.

The proof is discussed by Karlin (1958).

We need the following proposition for the proof of our main theorem.

Proposition 2.1. *Suppose $[a, b] \subset \Theta$ be a finite interval. If we define*

$$K(\mathbf{x}, \theta) := \int_{\underline{\theta}}^{\theta} \left\{\ln\left(\frac{\delta_\pi(\mathbf{x})}{h(t)}\right)\right\} q(t)\pi(t)I_{\Theta_{\mathbf{x}}}(t)dt, \tag{2.1}$$

then $K(\mathbf{x}, b) = K(\mathbf{x}, b)I_{\mathcal{X}_b}(\mathbf{x})$.

Proof. If $\beta^{-1}(x_n) > b$, then from definition of $I_{\Theta_{\mathbf{x}}}(b)$ and this fact that $I_{\chi_\theta}(\mathbf{x}) = I_{\Theta_{\mathbf{x}}}(\theta)$, we have $I_{\chi_b}(\mathbf{x}) = 0$. If $\alpha^{-1}(x_1) < b$, then from definition of $I_{\chi_b}(\mathbf{x})$, we have $I_{\chi_b}(\mathbf{x}) = 0$. Since in these cases $K(\mathbf{x}, b) = 0$, $K(\mathbf{x}, b) = K(\mathbf{x}, b)I_{\chi_b}(\mathbf{x})$. □

The following theorem provides sufficient conditions for Δ-admissibility of (1.4) under the loss (1.1).

Theorem 2.1. *Suppose that* $\delta_\pi \in \Delta$ *and put*

$$\gamma(\theta) := \frac{1}{q(\theta)\pi(\theta)} \int_{\mathbf{R}^2} K^2(\mathbf{x}, \theta) r(\mathbf{x}) I_{\chi_\theta}(\mathbf{x}) d\mathbf{x}.$$

If $\gamma(\theta) < \infty$ *for all* $\theta \in \Theta$ *and there exists* $v \in \Theta$ *such that*

$$\lim_{u \to \bar{\theta}} \int_v^u \frac{d\theta}{\gamma(\theta)} = \lim_{u \to \underline{\theta}} \int_u^v \frac{d\theta}{\gamma(\theta)} = \infty, \tag{2.2}$$

then $\delta_\pi(\mathbf{X})$ *is* Δ*-admissible for* $h(\theta)$ *under the loss (1.1).*

Proof. According to the definition of admissibility, if δ_π is not Δ-admissible, then there exists an estimator $\delta \in \Delta$ such that

$$E_\theta[L(\delta(\mathbf{X}), h(\theta))] \le E_\theta[L(\delta_\pi(\mathbf{X}), h(\theta))] \tag{2.3}$$

for all $\theta \in \Theta$ with strict inequality for at least one θ. From Condition **C1**, we see that (2.3) is equivalent to

$$E_\theta\left[\left\{\ln\left(\frac{\delta(\mathbf{X})}{\delta_\pi(\mathbf{X})}\right)\right\}^2\right] \le 2E_\theta\left[\left\{\ln\left(\frac{\delta_\pi(\mathbf{X})}{h(\theta)}\right)\right\}\left\{\ln\left(\frac{\delta_\pi(\mathbf{X})}{\delta(\mathbf{X})}\right)\right\}\right], \tag{2.4}$$

for all $\theta \in \Theta$. Multiplying both sides of (2.4) by $\pi(\theta)$, and integrating with respect to θ over the finite interval $[a, b] \subset \Theta$, we obtain

$$\int_a^b E_\theta\left[\left\{\ln\left(\frac{\delta(\mathbf{X})}{\delta_\pi(\mathbf{X})}\right)\right\}^2\right]\pi(\theta)d\theta \le 2\int_a^b E_\theta\left[\left\{\ln\left(\frac{\delta_\pi(\mathbf{X})}{h(\theta)}\right)\right\}\left\{\ln\left(\frac{\delta_\pi(\mathbf{X})}{\delta(\mathbf{X})}\right)\right\}\right]\pi(\theta)d\theta.$$

An application of the Fubini's theorem gives

$$\begin{aligned}
&\int_a^b \int_{\mathbf{R}^2} \left\{\ln\left(\frac{\delta(\mathbf{x})}{\delta_\pi(\mathbf{x})}\right)\right\}^2 r(\mathbf{x}) I_{\chi_\theta}(\mathbf{x}) d\mathbf{x} q(\theta)\pi(\theta)d\theta \qquad (2.5)\\
\le\ & 2\int_{\mathbf{R}^2} r(\mathbf{x}) \int_a^b \left[\left\{\ln\left(\frac{\delta_\pi(\mathbf{x})}{h(\theta)}\right)\right\}\left\{\ln\left(\frac{\delta_\pi(\mathbf{x})}{\delta(\mathbf{x})}\right)\right\}\right] q(\theta)\pi(\theta) I_{\Theta_{\mathbf{x}}}(\theta) d\theta d\mathbf{x},
\end{aligned}$$

which is guaranteed by Condition **C2**. From Proposition 2.1, right hand side of (2.5) is rewritten as

$$\begin{aligned} &2\int_{\mathbf{R}^2} r(\mathbf{x})\left\{\ln\left(\frac{\delta(\mathbf{x})}{\delta_\pi(\mathbf{x})}\right)\right\}K(\mathbf{x},b)I_{\chi_b}(\mathbf{x})d\mathbf{x} \\ &-2\int_{\mathbf{R}^2} r(\mathbf{x})\left\{\ln\left(\frac{\delta(\mathbf{x})}{\delta_\pi(\mathbf{x})}\right)\right\}K(\mathbf{x},a)I_{\chi_a}(\mathbf{x})d\mathbf{x}. \end{aligned} \tag{2.6}$$

Using the Cauchy-Schwartz inequality, (2.6) is less than

$$2\left\{\int_{\mathbf{R}^2}\left\{\ln\left(\frac{\delta(\mathbf{x})}{\delta_\pi(\mathbf{x})}\right)\right\}^2 r(\mathbf{x})I_{\chi_b}(\mathbf{x})d\mathbf{x}\right\}^{1/2}\left\{\int_{\mathbf{R}^2}K^2(\mathbf{x},b)r(\mathbf{x})d\mathbf{x}\right\}^{1/2}. \tag{2.7}$$

Hence, if we define

$$T(\theta) := \int_{\mathbf{R}^2}\left\{\ln\left(\frac{\delta(\mathbf{x})}{\delta_\pi(\mathbf{x})}\right)\right\}^2 r(\mathbf{x})I_{\chi_\theta}(\mathbf{x})d\mathbf{x},$$

then, combining (2.5) and (2.7), we have

$$\begin{aligned} \int_a^b T(\theta)\pi(\theta)q(\theta)d\theta \quad &\leq \quad 2\left\{T(b)\pi(b)q(b)\right\}^{1/2}\gamma^{1/2}(b) \\ &+ \quad 2\{T(a)\pi(a)q(a)\}^{1/2}\gamma^{1/2}(a). \end{aligned}$$

Now from (2.2) and Lemma 2.1, we obtain $T(\theta) = 0$ for almost all $\theta \in \Theta$. From Condition **C1**, $\delta(\mathbf{x}) = \delta_\pi(\mathbf{x})$ for almost all $\mathbf{x} \in \chi_\theta$ and the proof is completed. □

By putting (1.4) in (2.1), we get

$$K(\mathbf{x},\theta) = \frac{\int_\theta^{\bar\theta}\int_{\underline\theta}^{\theta}\left\{\ln\left(\frac{h(s)}{h(t)}\right)\right\}q(s)\pi(s)I_{\Theta_\mathbf{x}}(s)q(t)\pi(t)I_{\Theta_\mathbf{x}}(t)dtds}{\int_{\underline\theta}^{\bar\theta}q(u)\pi(u)I_{\Theta_\mathbf{x}}(u)du},$$

which is easier to handle than (2.1). Note that the above result is obtained by using this fact that $\int_{\underline\theta}^{\bar\theta}\int_{\underline\theta}^{\theta}$ can be written as $\int_{\underline\theta}^{\theta}\int_{\underline\theta}^{\theta} + \int_{\theta}^{\bar\theta}\int_{\underline\theta}^{\theta}$, and since $\int_{\underline\theta}^{\theta}\int_{\underline\theta}^{\theta}$ is equal to zero, $\int_{\underline\theta}^{\bar\theta}\int_{\underline\theta}^{\theta} = \int_{\theta}^{\bar\theta}\int_{\underline\theta}^{\theta}$.

In the next example, we apply the result of Theorem 2.1.

Examples 2.1. Suppose that $Y_1, ..., Y_n$ $(n > 2)$ are independent and identically distributed random variables according to the probability density function

$$p(y;\theta) = \begin{cases} 1 & \text{if } \theta < y < \theta + 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta(\in \mathbf{R})$ is unknown. Then, the probability density function of $\mathbf{X}$ is given by (1.3) when $q(\theta) = 1, r(\mathbf{x}) = n(n-1)(x_n - x_1)^{n-2}$ and $\chi_\theta = \{\mathbf{x} \in \mathbf{R}^2 | \theta < x_1 < x_n < \theta + 1\}$. We want to estimate $h(\theta) = e^{\eta\theta}$ under the loss (1.1), when $\eta \neq 0$ is a real number. The generalized Bayes estimator with respect to $\pi(\theta) = 1$ is given by $\delta_\pi(\mathbf{x}) = exp(\eta(x_1 + x_n - 1)/2)$. We can obtain $r(\mathbf{x} + \theta) = r(\mathbf{x})$ and $K(\mathbf{x} + \theta, \theta) = K(\mathbf{x}, 0)$. Hence

$$\gamma(\theta) = \int_{\mathbf{R}^2} K^2(\mathbf{x}, 0) r(\mathbf{x}) I_{\chi_0}(\mathbf{x}) d\mathbf{x} < \infty,$$

and consequently, $\delta_\pi(\mathbf{X})$ is Δ-admissible from Theorem 2.1.

In the above example, the moment generating function of Y_1, $\{(e^\eta - 1)/\eta\} e^{\eta\theta}$, is a multiple of $h(\theta)$.

3 Two Special Cases

In this section, we consider two special cases. In the first case, we suppose that $h(\theta)$ is bounded and, in the second case, we consider a special class of priors densities.

3.1 $h(\theta)$ is Bounded

It is difficult to express $\gamma(\theta)$ explicitly and it can have a complicated form. So, in order to apply Theorem 2.1, we have to seek a suitable upper bound of $\gamma(\theta)$. For the case when $h(\theta)$ is bounded, we can get the following corollary.

Corollary 3.1. Suppose that $h(\theta)$ is bounded and $\delta_\pi \in \Delta$. For $\theta \in \Theta$ and $\mathbf{x} \in \chi_\theta$, put

$$\tilde{K}(\mathbf{x}, \theta) := \frac{\int_\theta^{\bar{\theta}} q(s)\pi(s) I_{\Theta_\mathbf{x}}(s) ds \int_{\underline{\theta}}^{\theta} q(t)\pi(t) I_{\Theta_\mathbf{x}}(t) dt}{\int_{\underline{\theta}}^{\bar{\theta}} q(u)\pi(u) I_{\Theta_\mathbf{x}}(u) du},$$

and

$$\tilde{\gamma}(\theta) := \frac{1}{q(\theta)\pi(\theta)} \int_{\mathbf{R}^2} \tilde{K}^2(\mathbf{x}, \theta) r(\mathbf{x}) I_{\chi_\theta}(\mathbf{x}) d\mathbf{x}.$$

If $\tilde{\gamma}(\theta) < \infty$ for all $\theta \in \Theta$ and there exists $v \in \Theta$ such that

$$\lim_{u \to \bar{\theta}} \int_v^u \frac{d\theta}{\tilde{\gamma}(\theta)} = \lim_{u \to \underline{\theta}} \int_u^v \frac{d\theta}{\tilde{\gamma}(\theta)} = \infty,$$

then $\delta_\pi(\mathbf{X})$ is Δ-admissible for $h(\theta)$ under the loss (1.1).

Proof. It can be easily shown that there exists a constant C such that $|K(\mathbf{x},\theta)| \le C\tilde{K}(\mathbf{x},\theta)$, for all $(\mathbf{x},\theta) \in \{(\mathbf{x},\theta)|\mathbf{x} \in \chi_\theta, \theta \in \Theta\}$. This completes the proof by Theorem 2.1. □

We now give an application of the above corollary.

Examples 3.1. In Example 2.1, consider the estimation problem of

$$h(\theta) = P_\theta(Y_1 \le 1) = (1-\theta)I_{\{0<\theta<1\}}(\theta) + I_{\{\theta<0\}}(\theta),$$

which is clearly bounded. The generalized Bayes estimator of $h(\theta)$ with respect to $\pi(\theta) = 1$ is given by

$$\delta_\pi(\mathbf{x}) = \begin{cases} 1 & \text{if } x_1 < 0 \\ \exp\left\{\frac{(1-x_1)(\ln(1-x_1)-1)+1}{x_1+x_n-1}\right\} & \text{if } x_n - 1 < 0 < x_1 < 1 \\ \exp\left\{\frac{(1-x_1)(\ln(1-x_1)-1)-(2-x_n)(\ln(2-x_n)-1)}{x_1+x_n-1}\right\} & \text{if } 0 < x_n - 1 < x_1 < 1 \\ \exp\left\{\frac{-(2-x_n)(\ln(2-x_n)-1)}{x_1+x_n-1}\right\} & \text{if } 0 < x_n - 1 < 1 < x_1 \\ 0 & \text{if } 1 < x_n - 1 < x_1 < 2. \end{cases}$$

Since $\tilde{K}(\mathbf{x}+\theta,\theta) = \tilde{K}(\mathbf{x},0)$ and hence

$$\tilde{\gamma}(\theta) = \int_{\mathbf{R}^2} \tilde{K}^2(\mathbf{x},0)r(\mathbf{x})I_{\chi_0}(\mathbf{x})d\mathbf{x} < \infty,$$

$\delta_\pi(\mathbf{X})$ is Δ-admissible under the loss (1.1) from Corollary 3.1.

3.2 $\pi(\theta)$ Has a Special Form

Let the random variable $\mathbf{X}$ have a density of the form (1.3) and suppose that $h(\theta)$ is strictly increasing and differentiable. Consider the class of prior density functions treated by Sinha & Gupta (1984) and Kim & Meeden (1994) in the form

$$\pi_g(\theta) := \frac{h'(\theta)}{q(\theta)} g(h(\theta)), \qquad \theta \in \Theta, \tag{3.1}$$

for a positive function $g(.)$ defined on the range of h. In this case, the generalized Bayes estimator of $h(\theta)$ with respect to $\pi_g(\theta)$ is given by $\delta_{\pi_g}(\mathbf{X})$, where

$$\delta_{\pi_g}(\mathbf{x}) = \exp\left\{\frac{\int_{h(\Theta)} \{\ln(z)\}\, g(z) I_{h(\Theta_{\mathbf{x}})}(z)dz}{\int_{h(\Theta)} g(z) I_{h(\Theta_{\mathbf{x}})}(z)dz}\right\}. \tag{3.2}$$

From (3.2) certain integrability conditions would have to be imposed on g for δ_{π_g} to be well defined.

The following Corollary gives sufficient conditions for (3.2) to be Δ-admissible.

Corollary 3.2. Suppose that $h(\theta)$ is strictly increasing and differentiable, and $\delta_{\pi_g} \in \Delta$. For $\theta \in \Theta$ and $\mathbf{x} \in \chi_\theta$, put

$$K_g(\mathbf{x}, \theta) := \int_{h(\underline{\theta})}^{h(\theta)} \left\{\ln\left(\frac{\delta_{\pi_g}(\mathbf{x})}{z}\right)\right\} g(z) I_{h(\Theta_{\mathbf{x}})}(z)dz,$$

and

$$\gamma_g(\theta) := \frac{1}{h'(\theta) g(h(\theta))} \int_{\mathbf{R}^2} K_g^2(\mathbf{x}, \theta) r(\mathbf{x}) I_{\chi_\theta}(\mathbf{x}) d\mathbf{x}.$$

If $\gamma_g(\theta) < \infty$ for all $\theta \in \Theta$ and there exists $v \in \Theta$ such that

$$\lim_{u \to \bar{\theta}} \int_v^u \frac{d\theta}{\gamma_g(\theta)} = \lim_{u \to \underline{\theta}} \int_u^v \frac{d\theta}{\gamma_g(\theta)} = \infty,$$

then $\delta_{\pi_g}(\mathbf{X})$ is Δ-admissible for $h(\theta)$ under the loss (1.1).

The proof is omitted because it is an immediate consequence of Theorem 2.1.

We now close this section with an applications of Corollary 3.2 in which we work with the choice of $g(u) = u^{-\alpha} (\alpha > 1)$.

Examples 3.2. Suppose that $Y_1, ..., Y_n$ $(n > 2)$ are independent and identically distributed random variables according to the probability density function

$$p(y; \theta) = \begin{cases} \frac{1}{\theta} & \text{if } \xi\theta < y < (\xi + 1)\theta \\ 0 & \text{otherwise,} \end{cases}$$

where $\xi \in (0, \infty)$ is known and $\theta \in (0, \infty)$ is unknown. In this case, the probability density function of $\mathbf{X}$ is given by (1.3), when $q(\theta) = \theta^{-n}$, $r(\mathbf{x}) = n(n-1)(x_n - x_1)^{n-2}$ and $\chi_\theta = \{\mathbf{x} \in \mathbf{R}^2 | \xi\theta < x_1 < x_n < (\xi + 1)\theta\}$. Consider the estimation of $h(\theta) = \theta^\beta$ where $\beta > 0$. Let the prior density of θ be $\pi_\alpha(\theta) = \beta\theta^{n-\beta\alpha-1}$, where $\alpha > 0$, which is (3.1) with $g(u) = u^{-\alpha}$. The generalized Bayes estimator of $h(\theta)$ is given by

$$\delta_{\pi_\alpha}(\mathbf{x}) = e^{\frac{1}{\alpha}} \exp\left\{\beta\Big(\frac{\ln\left(\frac{x_n}{\xi+1}\right)}{\left(\frac{x_n}{\xi+1}\right)^{\beta\alpha}} - \frac{\ln\left(\frac{x_1}{\xi}\right)}{\left(\frac{x_1}{\xi}\right)^{\beta\alpha}}\Big)\Big/\Big(\frac{1}{\left(\frac{x_n}{\xi+1}\right)^{\beta\alpha}} - \frac{1}{\left(\frac{x_1}{\xi}\right)^{\beta\alpha}}\Big)\right\}.$$

We can obtain $r(\theta\mathbf{x}) = \theta^{n-2} r(\mathbf{x})$ and $K_g(\theta\mathbf{x}, \theta) = \theta^{-\beta\alpha} K_g(\mathbf{x}, 1)$. These imply that

$$\gamma_g(\theta) = \frac{\theta^{n-\beta(\alpha+1)-1}}{\beta} \int_{\mathbf{R}^2} K_g^2(\mathbf{x}, 1) r(\mathbf{x}) I_{\chi_1}(\mathbf{x}) d\mathbf{x}.$$

Therefore, from Corollary 3.2, for $\alpha = \frac{n-1}{\beta} - 1$ where $0 < \beta < n-1$, $\delta_{\pi_\alpha}(\mathbf{X})$ is Δ-admissible under the loss (1.1).

In the above example, all moments are multiples of $h(\theta) = \theta^\beta$ for appropriate β's. For instance, the mean $(= (\xi + 1/2)\theta)$ and the variance $(= (1/12)\theta^2)$ are multiples of θ and θ^2, respectively. Furthermore, the quantiles $q_p = (\xi + p)\theta$, where $0 < p < 1$ are multiples of θ.

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