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A Kotz-Riesz-type Distribution

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Abstract. This article derives the distribution of the random matrix $\mathbf{X}$ associated with the transformation $\mathbf{Y} = \mathbf{X}^*\mathbf{X}$, such that $\mathbf{Y}$ has a Riesz distribution for real normed division algebras. Two versions of this distributions are proposed and some of their properties are studied.

Keywords. Generalised power, Kotz distribution, Kotz-Riesz distribution, Riesz distribution, Real normed division algebras, Wishart distribution.

MSC: Primary 60E05, 62E15; Secondary 15A52.

1 Introduction

Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be real independent $\mathcal{N}_m(\boldsymbol{\mu}, \mathbf{\Sigma})$, that is, $\mathbf{X}_i$ has an *m-dimensional normal distribution* with expected value $\boldsymbol{\mu} \in \Re^m$ and $m \times m$ positive definite covariance matrix $\mathbf{\Sigma} > \mathbf{0}$. Let $\mathbf{X}$ be the $n \times m$ $(n \geq m)$ random matrix

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}'_1 \\ \vdots \\ \mathbf{X}'_n \end{bmatrix}$$

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and

$$\mathrm{E}(\mathbf{X}) = \begin{bmatrix} \boldsymbol{\mu}' \\ \vdots \\ \boldsymbol{\mu}' \end{bmatrix} = \mathbf{1}\boldsymbol{\mu}', \quad \text{where } \mathbf{1} = (1, \dots, 1)' \in \Re^n$$

and $\mathrm{Cov}(\mathrm{vec}\,\mathbf{X}') = \mathbf{I}_n \otimes \boldsymbol{\Sigma}$, that is, $\mathbf{X}$ has a $n \times m$ *matrix variate normal distribution*, denoted this fact as $\mathbf{X} \sim \mathcal{N}_{n\times m}(\mathbf{1}\boldsymbol{\mu}', \mathbf{I}_n, \boldsymbol{\Sigma})$, see Muirhead (1982, pp. 79-80), among others. If $\mathbf{Y} = \mathbf{X}'\mathbf{X}$ where the $n \times m$ random matrix $\mathbf{X} \sim \mathcal{N}_{n\times m}(\mathbf{0}, \mathbf{I}_n, \boldsymbol{\Sigma})$, then $\mathbf{Y}$ is said to have the Wishart distribution with n degrees of freedom and scale parameter $\boldsymbol{\Sigma}$.

For a long time, statistical theory based on real numbers was unconnected from the corresponding complex case. It forced that both approaches were studied separately. The real case was usually considered at first and the complex version appeared as a translation of the preceding one without any relation. Later, some concepts and results derived from abstract algebra allowed a unified way of constructing a general theory considering not only real and complex cases but also quaternion and octonion fields. Part of this approach has been used in random matrix theory, see Dumitriu (2002) and Edelman and Rao (2005) and, in the context of statistics, see Kabe (1984) and Bhavsar (2002) among many others.

Based on the theory of Jordan algebras, a family of distributions on symmetric cones, termed the *Riesz distributions*, were first introduced by Hassairi and Lajmi (2001) under the name of Riesz natural exponential family (Riesz NEF). They were based on a special case of the so-called *Riesz measure* from (Faraut and Korányi, 1994, p.137), going back to Riesz (1949). This Riesz distribution *generalises the matrix multivariate gamma and Wishart distributions*, containing them as particular cases. Recently, Díaz-García (2015) proposed two versions of the Riesz distribution. A diverse range of their properties were studied for real normed division algebras.

The main purpose of this article is to introduce the distribution of the random matrix $\mathbf{X}$, such that $\mathbf{Y} = \mathbf{X}^*\mathbf{X}$ has one of the two versions of the Riesz distributions. We explore some of their basic properties for real normed division algebras. These distributions shall be termed *Kotz-Riesz distributions*. It is a simple matter to check that the Kotz-Riesz distribution belongs to the *matrix multivariate elliptical-spherical distributions*. Moreover, the Kotz-Riesz distribution contains the matrix multivariate Kotz distribution as a particular case, a property that has been taken into account to name it as Kotz-Riesz distribution, see Fang and Zhang (1990, pp. 102-013) and Fang and Li (2013, Example 4.1).

This article studies two versions of Kotz-Riesz distributions for real normed division algebras. Section 2 reviews some definitions and notation on real normed division algebras, and introduces other mathematical tools as two definitions for the generalised gamma function on symmetric cones. Several integration results for real normed division algebras are found in Section 3. Section 4 introduces Kotz-Riesz distributions for real normed division algebras, and also studies the relationship between the Riesz distributions and the Kotz-Riesz distributions.

2 Preliminary results

A detailed discussion of real normed division algebras may be found in Baez (2002) and Ebbinghaus *et al.* (1990). For your convenience, we shall introduce some notation, though in general we adhere to standard notation forms.

For our purposes, let $\mathbb{F}$ be a field. An *algebra* $\mathfrak{A}$ over $\mathbb{F}$ is a pair, $(\mathfrak{A}; m)$, where $\mathfrak{A}$ is a *finite-dimensional vector space* over $\mathbb{F}$ and *multiplication* $m : \mathfrak{A} \times \mathfrak{A} \to A$ is an $\mathbb{F}$-bilinear map; that is, for all $\lambda \in \mathbb{F}$, $x, y, z \in \mathfrak{A}$,

$$\begin{aligned} m(x, \lambda y + z) &= \lambda m(x; y) + m(x; z) \\ m(\lambda x + y; z) &= \lambda m(x; z) + m(y; z). \end{aligned}$$

Two algebras $(\mathfrak{A}; m)$ and $(\mathfrak{E}; n)$ over $\mathbb{F}$ are said to be *isomorphic* if there is an invertible map $\phi : \mathfrak{A} \to \mathfrak{E}$, such that for all $x, y \in \mathfrak{A}$,

$$\phi(m(x, y)) = n(\phi(x), \phi(y)).$$

By simplicity, we write $m(x; y) = xy$ for all $x, y \in \mathfrak{A}$. Let $\mathfrak{A}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{A}$ is said to be

1. *alternative* if $x(xy) = (xx)y$ and $x(yy) = (xy)y$ for all $x, y \in \mathfrak{A}$,
2. *associative* if $x(yz) = (xy)z$ for all $x, y, z \in \mathfrak{A}$,
3. *commutative* if $xy = yx$ for all $x, y \in \mathfrak{A}$, and
4. *unital* if there is a $1 \in \mathfrak{A}$ such that $x1 = x = 1x$ for all $x \in \mathfrak{A}$.

If $\mathfrak{A}$ is unital, then the identity 1 is uniquely determined.

An algebra $\mathfrak{A}$ over $\mathbb{F}$ is said to be a *division algebra* if $\mathfrak{A}$ is nonzero and $xy = 0_{\mathfrak{A}} \Rightarrow x = 0_{\mathfrak{A}}$ or $y = 0_{\mathfrak{A}}$ for all $x, y \in \mathfrak{A}$.

The term "division algebra", comes from the following proposition, which shows that, in such an algebra, left and right division can be unambiguously performed.

Let $\mathfrak{A}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{A}$ is a division algebra if, and only if, $\mathfrak{A}$ is nonzero and for all $a, b \in \mathfrak{A}$, with $b \neq 0_{\mathfrak{A}}$, the equations $bx = a$ and $yb = a$ have unique solutions $x, y \in \mathfrak{A}$.

In the sequel, we assume that $\mathbb{F} = \Re$ and consider classes of division algebras over $\Re$ or "*real division algebras*" for short.

We introduce the algebras of *real numbers* $\Re$, *complex numbers* $\mathfrak{C}$, *quaternions* $\mathfrak{H}$ and *octonions* $\mathfrak{O}$. Then, if $\mathfrak{A}$ is an alternative real division algebra, $\mathfrak{A}$ is isomorphic to $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$.

Let $\mathfrak{A}$ be a real division algebra with identity 1. Then $\mathfrak{A}$ is said to be *normed* if there is an inner product $(\cdot, \cdot)$ on $\mathfrak{A}$ such that

$$(xy, xy) = (x, x)(y, y) \qquad \text{for all } x, y \in \mathfrak{A}.$$

If $\mathfrak{A}$ is a *real normed division algebra*, then $\mathfrak{A}$ is isomorphic $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$.

There are exactly four normed division algebras: real numbers ($\Re$), complex numbers ($\mathfrak{C}$), quaternions ($\mathfrak{H}$) and octonions ($\mathfrak{O}$), see Baez (2002). We take into account that $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ and $\mathfrak{O}$ are the only normed division algebras. Furthermore, they are the only alternative division algebras.

Let $\mathfrak{A}$ be a division algebra over the real numbers. Then $\mathfrak{A}$ has dimension β, either 1, 2, 4 or 8. Other branches of mathematics used the parameters $\alpha = 2/\beta$ and $t = \beta/4$, see Edelman and Rao (2005) and Kabe (1984), respectively.

Finally, we observe that

- $\Re$ is a real commutative associative normed division algebra.
- $\mathfrak{C}$ is a commutative associative normed division algebra.
- $\mathfrak{H}$ is an associative normed division algebra.
- $\mathfrak{O}$ is an alternative normed division algebra.

Let $\mathfrak{L}^{\beta}_{n,m}$ be the set of all $n \times m$ matrices of rank $m \leq n$ over $\mathfrak{A}$ with m distinct positive singular values, where $\mathfrak{A}$ denotes a *real finite-dimensional normed division algebra*. Let $\mathfrak{A}^{n\times m}$ be the set of all $n \times m$ matrices over $\mathfrak{A}$. The dimension of $\mathfrak{A}^{m\times n}$ over $\Re$ is βmn. Let $\mathbf{A} \in \mathfrak{A}^{n\times m}$. Then $\mathbf{A}^* = \bar{\mathbf{A}}^T$ denotes the usual conjugate transpose. Table 1 sets out the equivalence between the same concepts in the four normed division algebras.

It is denoted by $\mathfrak{S}^{\beta}_m$ the real vector space of all $\mathbf{S} \in \mathfrak{A}^{m\times m}$, such that $\mathbf{S} = \mathbf{S}^*$. In addition, let $\mathfrak{P}^{\beta}_m$ be the *cone of positive definite matrices* $\mathbf{S} \in \mathfrak{A}^{m\times m}$. Thus, $\mathfrak{P}^{\beta}_m$ consists of all matrices $\mathbf{S} = \mathbf{X}^*\mathbf{X}$ with $\mathbf{X} \in \mathfrak{L}^{\beta}_{m,n}$. Then $\mathfrak{P}^{\beta}_m$ is an open subset of $\mathfrak{S}^{\beta}_m$.

Table 1: Notation

Real	Complex	Quaternion	Octonion	Generic notation
Semi-orthogonal	Semi-unitary	Semi-symplectic	Semi-exceptional type	$\mathcal{V}_{m,n}^{\beta}$
Orthogonal	Unitary	Symplectic	Exceptional type	$\mathfrak{U}^{\beta}(m)$
Symmetric	Hermitian	Quaternion hermitian	Octonion hermitian	$\mathfrak{S}_m^{\beta}$

Let $\mathfrak{D}_m^{\beta}$ be the *diagonal subgroup* of $\mathcal{L}_{m,m}^{\beta}$ consisting of all $\mathbf{D} \in \mathfrak{A}^{m\times m}$, $\mathbf{D} = \operatorname{diag}(d_1,\dots,d_m)$. Let $\mathfrak{T}_U^{\beta}(m)$ be the subgroup of all *upper triangular* matrices $\mathbf{T} \in \mathfrak{A}^{m\times m}$, such that $t_{ij} = 0$ for $1 \le j < i \le m$. The set of matrices $\mathbf{H}_1 \in \mathfrak{F}^{n\times m}$, where $\mathbf{H}_1^*\mathbf{H}_1 = \mathbf{I}_m$ is a manifold denoted $\mathcal{V}_{m,n}^{\beta}$, is termed the *Stiefel manifold* ($\mathbf{H}_1$ is also known as *semi-orthogonal*).

For any matrix $\mathbf{X} \in \mathfrak{A}^{n\times m}$, $d\mathbf{X}$ denotes the *matrix of differentials* (dx_{ij}). Finally, we define the *measure* or volume element $(d\mathbf{X})$ when $\mathbf{X} \in \mathfrak{A}^{n\times m}, \mathfrak{S}_m^{\beta}$, $\mathfrak{D}_m^{\beta}$ or $\mathcal{V}_{m,n}^{\beta}$, see Díaz-García and Gutiérrez-Jáimez (2011).

If $\mathbf{X} \in \mathfrak{A}^{n\times m}$, then $(d\mathbf{X})$ (the Lebesgue measure in $\mathfrak{A}^{n\times m}$) denotes the exterior product of the βmn functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^{m}\bigwedge_{j=1}^{n} dx_{ij} \quad \text{where} \quad dx_{ij} = \bigwedge_{k=1}^{\beta} dx_{ij}^{(k)}.$$

If $\mathbf{S} \in \mathfrak{S}_m^{\beta}$ (or $\mathbf{S} \in \mathfrak{T}_U^{\beta}(m)$ with $t_{ii} > 0$, $i = 1,\dots,m$), then $(d\mathbf{S})$ (the Lebesgue measure in $\mathfrak{S}_m^{\beta}$ or in $\mathfrak{T}_U^{\beta}(m)$) denotes the exterior product of the $m(m-1)\beta/2 + m$ functionally independent variables, that is,

$$(d\mathbf{S}) = \bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i<j}^{m}\bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}.$$

For the Lebesgue measure $(d\mathbf{S})$ defined, it is required that $\mathbf{S} \in \mathfrak{P}_m^{\beta}$, that is, $\mathbf{S}$ must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If $\mathbf{\Lambda} \in \mathfrak{D}_m^{\beta}$, then $(d\mathbf{\Lambda})$ (the Legesgue measure in $\mathfrak{D}_m^{\beta}$) denotes the exterior product of the βm functionally independent variables, that is,

$$(d\mathbf{\Lambda}) = \bigwedge_{i=1}^{n}\bigwedge_{k=1}^{\beta} d\lambda_i^{(k)}.$$

If $\mathbf{H}_1 \in \mathcal{V}_{m,n}^{\beta}$ is such that $\mathbf{H}_1 = (\mathbf{h}_1, \dots, \mathbf{h}_m)$, where $\mathbf{h}_j$, $j = 1, \dots, m$, are their columns, then

$$(\mathbf{H}_1^* d\mathbf{H}_1) = \bigwedge_{i=1}^{m} \bigwedge_{j=i+1}^{n} \mathbf{h}_j^* d\mathbf{h}_i, \tag{2.1}$$

where the partitioned matrix $\mathbf{H} = (\mathbf{H}_1|\mathbf{H}_2) = (\mathbf{h}_1, \dots, \mathbf{h}_m|\mathbf{h}_{m+1}, \dots, \mathbf{h}_n)$ $\in \mathfrak{U}^{\beta}(n)$ with $\mathbf{H}_2 = (\mathbf{h}_{m+1}, \dots, \mathbf{h}_n)$. It can be proved that this differential form does not depend on the choice of the $\mathbf{H}_2$ matrix. When $n = 1$; $\mathcal{V}_{m,1}^{\beta}$ defines the unit sphere in $\mathfrak{A}^m$. This is, of course, an $n\beta - 1$- dimensional surface in $\mathfrak{A}^m$.

The surface area or volume of the Stiefel manifold $\mathcal{V}_{m,n}^{\beta}$ is

$$\mathrm{Vol}(\mathcal{V}_{m,n}^{\beta}) = \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^{\beta}} (\mathbf{H}_1 d\mathbf{H}_1^*) = \frac{2^m \pi^{mn\beta/2}}{\Gamma_m^{\beta}[n\beta/2]}, \tag{2.2}$$

where $\Gamma_m^{\beta}[a]$ denotes the multivariate *Gamma function* for the space $\mathfrak{S}_m^{\beta}$. This can be obtained as a particular case of the *generalised gamma function of weight* κ for the space $\mathfrak{S}_m^{\beta}$ with $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$, taking $\kappa = (0, 0, \dots, 0) \in \Re^m$ and which for $\mathrm{Re}(a) \geq (m-1)\beta/2 - k_m$ is defined by (see Gross and Richards (1987))

$$\begin{aligned} \Gamma_m^{\beta}[a, \kappa] &= \int_{\mathbf{A} \in \mathfrak{P}_m^{\beta}} \mathrm{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{A}) (d\mathbf{A}) \qquad (2.3) \\ &= \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a + k_i - (i-1)\beta/2] \\ &= [a]_\kappa^{\beta} \Gamma_m^{\beta}[a], \qquad (2.4) \end{aligned}$$

where $\mathrm{etr}(\cdot) = \exp(\mathrm{tr}(\cdot))$, $|\cdot|$ denotes the determinant, and, for $\mathbf{A} \in \mathfrak{S}_m^{\beta}$,

$$q_\kappa(\mathbf{A}) = |\mathbf{A}_m|^{k_m} \prod_{i=1}^{m-1} |\mathbf{A}_i|^{k_i - k_{i+1}} \tag{2.5}$$

with $\mathbf{A}_p = (a_{rs})$, $r, s = 1, 2, \dots, p$, $p = 1, 2, \dots, m$ is termed the *highest weight vector*, see Gross and Richards (1987). In addition,

$$\begin{aligned} \Gamma_m^{\beta}[a] &= \int_{\mathbf{A} \in \mathfrak{P}_m^{\beta}} \mathrm{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} (d\mathbf{A}) \\ &= \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a - (i-1)\beta/2], \end{aligned}$$

and $\text{Re}(a) > (m-1)\beta/2$.

In other branches of mathematics, the *highest weight vector* $q_\kappa(\mathbf{A})$ is also termed the *generalised power* of $\mathbf{A}$ and is denoted as $\Delta_\kappa(\mathbf{A})$, see Faraut and Korányi (1994) and Hassairi and Lajmi (2001).

Additional properties of $q_\kappa(\mathbf{A})$, which are immediate consequences of the definition of $q_\kappa(\mathbf{A})$ and the following property 1, are as follwos.

1. Let $\mathbf{A} = \mathbf{L}^*\mathbf{D}\mathbf{L}$ be the L'DL decomposition of $\mathbf{A} \in \mathfrak{P}_m^\beta$, where $\mathbf{L} \in \mathfrak{T}_U^\beta(m)$ with $l_{ii} = 1$, $i = 1, 2, \dots, m$ and $\mathbf{D} = \text{diag}(\lambda_1, \dots, \lambda_m)$, $\lambda_i \geq 0$, $i = 1, 2, \dots, m$. Then
$$q_\kappa(\mathbf{A}) = \prod_{i=1}^m \lambda_i^{k_i}. \tag{2.6}$$

2. We have
$$q_\kappa(\mathbf{A}^{-1}) = q^*_{-\kappa^*}(\mathbf{A}), \tag{2.7}$$
where $\kappa^* = (k_m, k_{m-1}, \dots, k_1)$, $-\kappa^* = (-k_m, -k_{m-1}, \dots, -k_1)$,
$$q^*_\kappa(\mathbf{A}) = |\mathbf{A}_m|^{k_m} \prod_{i=1}^{m-1} |\mathbf{A}_i|^{k_i - k_{i+1}}, \tag{2.8}$$
and
$$q^*_\kappa(\mathbf{A}) = \prod_{i=1}^m \lambda_i^{k_{m-i+1}}, \tag{2.9}$$
see Faraut and Korányi (1994, pp. 126-127 and Proposition VII.1.5).

   Alternatively, let $\mathbf{A} = \mathbf{T}^*\mathbf{T}$ the Cholesky's decomposition of matrix $\mathbf{A} \in \mathfrak{P}_m^\beta$ with $\mathbf{T} = (t_{ij}) \in \mathfrak{T}_U^\beta(m)$. Then $\lambda_i = t_{ii}^2$, $t_{ii} \geq 0$, $i = 1, 2, \dots, m$, see Hassairi and Lajmi (2001, p. 931, first paragraph), Hassairi *et al.* (2005, p. 390, lines 11-16) and Kołodziejek (2014, p.5, lines 1-6).

3. If $\kappa = (p, \dots, p)$, then
$$q_\kappa(\mathbf{A}) = |\mathbf{A}|^p. \tag{2.10}$$
In particular, if $p = 0$, then $q_\kappa(\mathbf{A}) = 1$.

4. if $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, then
$$q_{\kappa+\tau}(\mathbf{A}) = q_\kappa(\mathbf{A}) q_\tau(\mathbf{A}). \tag{2.11}$$
In particular, if $\tau = (p, p, \dots, p)$, then
$$q_{\kappa+\tau}(\mathbf{A}) \equiv q_{\kappa+p}(\mathbf{A}) = |\mathbf{A}|^p q_\kappa(\mathbf{A}). \tag{2.12}$$

5. Finally, for $\mathbf{B} \in \mathfrak{T}_U^\beta(m)$ in such a manner that $\mathbf{C} = \mathbf{B}^*\mathbf{B} \in \mathfrak{S}_m^\beta$,

$$q_\kappa(\mathbf{B}^*\mathbf{A}\mathbf{B}) = q_\kappa(\mathbf{C})q_\kappa(\mathbf{A}) \tag{2.13}$$

and

$$q_\kappa(\mathbf{B}^{*-1}\mathbf{A}\mathbf{B}^{-1}) = (q_\kappa(\mathbf{C}))^{-1}q_\kappa(\mathbf{A}) = q_{-\kappa}(\mathbf{C})q_\kappa(\mathbf{A}), \tag{2.14}$$

see Hassairi *et al.* (2008, p. 776, eq. (2.1)).

Remark 1. Let $\mathcal{P}(\mathfrak{S}_m^\beta)$ denote the algebra of all polynomial functions on $\mathfrak{S}_m^\beta$, and $\mathcal{P}_k(\mathfrak{S}_m^\beta)$ the subspace of homogeneous polynomials of degree k and let $\mathcal{P}^\kappa(\mathfrak{S}_m^\beta)$ be an irreducible subspace of $\mathcal{P}(\mathfrak{S}_m^\beta)$, such that

$$\mathcal{P}_k(\mathfrak{S}_m^\beta) = \sum_\kappa \bigoplus \mathcal{P}^\kappa(\mathfrak{S}_m^\beta).$$

Note that q_κ is a homogeneous polynomial of degree k. Moreover, $q_\kappa \in \mathcal{P}^\kappa(\mathfrak{S}_m^\beta)$, see Gross and Richards (1987). □

In (2.4), $[a]_\kappa^\beta$ denotes the generalised Pochhammer symbol of weight κ which is defined as

$$\begin{aligned} [a]_\kappa^\beta &= \prod_{i=1}^m (a-(i-1)\beta/2)_{k_i} \\ &= \frac{\pi^{m(m-1)\beta/4} \displaystyle\prod_{i=1}^m \Gamma[a+k_i-(i-1)\beta/2]}{\Gamma_m^\beta[a]} \\ &= \frac{\Gamma_m^\beta[a,\kappa]}{\Gamma_m^\beta[a]}, \end{aligned}$$

where $\operatorname{Re}(a) > (m-1)\beta/2 - k_m$, and

$$(a)_i = a(a+1)\cdots(a+i-1)$$

is the standard Pochhammer symbol.

An alternative definition of the generalised gamma function of weight κ is proposed by Khatri (1966) which is defined as

$$\Gamma_m^\beta[a,-\kappa] = \int_{\mathbf{A}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{A}^{-1})(d\mathbf{A}) \tag{2.15}$$

$$= \pi^{m(m-1)\beta/4}\prod_{i=1}^{m}\Gamma[a-k_i-(m-i)\beta/2]$$

$$= \frac{(-1)^k\Gamma_m^\beta[a]}{[-a+(m-1)\beta/2+1]_\kappa^\beta}, \tag{2.16}$$

where $\text{Re}(a) > (m-1)\beta/2 + k_1$. In addition, by (2.1) and (2.2),

$$(d\mathbf{H}_1) = \frac{1}{\text{Vol}\left(\mathcal{V}_{m,n}^\beta\right)}(\mathbf{H}_1^* d\mathbf{H}_1) = \frac{\Gamma_m^\beta[n\beta/2]}{2^m\pi^{mn\beta/2}}(\mathbf{H}_1^* d\mathbf{H}_1)$$

is the *normalised invariant measure on* $\mathcal{V}_{m,n}^\beta$ and $(d\mathbf{H})$, i.e., with $(m = n)$, it defines the *normalised Haar measure* on $\mathfrak{U}^\beta(m)$.

Finally, the ${}_pF_q^\beta$ is the generalised hypergeometric function with matrix argument defined as

$${}_pF_q^\beta(a_1,\dots,a_p;b_1,\dots,b_q;\mathbf{X}) = \sum_{t=0}^{\infty}\sum_{\tau}\frac{[a_1]_\tau^\beta\cdots[a_p]_\tau^\beta}{[b_1]_\tau^\beta\cdots[b_q]_\tau^\beta}\frac{C_\tau^\beta(\mathbf{X})}{t!},$$

where $\sum_\tau$ denotes summation over all partition $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots t_m \geq 0$, of t, where $\sum_{i=1}^m t_i = t$ and $t_1, t_2, \dots, t_m$ are nonnegative integers, and $C_\tau^\beta(\mathbf{X})$ is the Jack polynomial of $\mathbf{X} \in \mathfrak{S}_m^\beta$ corresponding to t, see Gross and Richards (1987) and Díaz-García (2014).

Jack polynomials for real normed division algebras are also termed spherical functions of symmetric cones in the abstract algebra context, see Sawyer (1997). In addition, in the statistical literature, they are termed real, complex, quaternion and octonion zonal polynomials, or, generically, *general zonal polynomials*, see James (1964), Chapter 7 in Muirhead (1982), Kabe (1984) and Li and Xue (2009). This section is completed, remembering the following result: It is obtained from Gross and Richards (1987, Equation 4.8(2) and Definition 5.3) that

$$C_\kappa^\beta(\mathbf{X}) = C_\kappa^\beta(\mathbf{I}_m)\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} q_\kappa(\mathbf{H}^*\mathbf{X}\mathbf{H})(d\mathbf{H}), \tag{2.17}$$

for all $\mathbf{X} \in \mathfrak{S}_m^\beta$, where $(d\mathbf{H})$ is the normalised Haar measure on $\mathfrak{U}^\beta(m)$, see also Díaz-García (2014).

3 Integration

First, we consider the following result proposed by Zhang and Fang (1990) for the real case and extended to real normed division algebras by Díaz-García (2014).

Lemma 3.1. *The characteristic function of* $\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta$*, the normalised invariant measure on* $\mathcal{V}_{m,n}^\beta$*, is*

$$\begin{aligned}\phi_{\mathbf{H}_1}(\mathbf{T}) &= \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta} \operatorname{etr}\{i\mathbf{H}_1\mathbf{T}^*\}(d\mathbf{H}_1) \\ &= {}_0F_1^\beta(\beta n/2, -\mathbf{T}\mathbf{T}^*/4) \\ &= \sum_{t=0}^\infty \sum_\tau \frac{1}{[\beta n/2]_\tau^\beta} \frac{C_\tau^\beta(-\mathbf{T}\mathbf{T}^*/4)}{t!}, \end{aligned} \tag{3.1}$$

where $\sum_\tau$ *denotes summation over all partition* $\tau = (t_1, t_2, \dots, t_m)$*,* $t_1 \geq t_2 \geq \cdots t_m \geq 0$*, of* t*, where* $\sum_{i=1}^m t_i = t$ *and* $t_1, t_2, \dots, t_m$ *are nonnegative integers.*

The original version of the following result was obtained for the real case in Xu and Fang (1990). Next, we set the version of this result for real normed division algebras.

First, we consider the following concept: let us use the complexification $\mathfrak{S}_m^{\beta,\mathfrak{C}} = \mathfrak{S}_m^\beta + i\mathfrak{S}_m^\beta$ of $\mathfrak{S}_m^\beta$, that is, $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consists of all matrices $\mathbf{Z} \in (\mathfrak{F}^\mathfrak{C})^{m\times m}$ of the form $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$ with $\mathbf{X}, \mathbf{Y} \in \mathfrak{S}_m^\beta$. It comes to $\mathbf{X} = \operatorname{Re}(\mathbf{Z})$ and $\mathbf{Y} = \operatorname{Im}(\mathbf{Z})$ as the *real and imaginary parts* of $\mathbf{Z}$, respectively. The *generalised right half-plane* $\mathbf{\Phi}_m^\beta = \mathfrak{P}_m^\beta + i\mathfrak{S}_m^\beta$ in $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consists of all $\mathbf{Z} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$, such that $\operatorname{Re}(\mathbf{Z}) \in \mathfrak{P}_m^\beta$, see (Gross and Richards, 1987, p. 801).

Lemma 3.2. *Let* $\mathbf{Z} \in \mathbf{\Phi}_m^\beta$ *and* $\mathbf{U} \in \mathfrak{S}_m^\beta$*. Then*

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} f(\mathbf{Z}^{1/2}\mathbf{X}\mathbf{Z}^{1/2})|\mathbf{X}|^{a-(m-1)\beta/2-1} C_\tau^\beta(\mathbf{X}\mathbf{U})(d\mathbf{X})$$

$$= \frac{J(\mathbf{I}_m)}{C_\tau^\beta(\mathbf{I}_m)} |\mathbf{Z}|^{-a} C_\tau^\beta(\mathbf{U}\mathbf{Z}^{-1}), \tag{3.2}$$

where $\mathbf{Z}^{1/2}$ *is the positive definite square root of* $\mathbf{Z}$*, i.e.* $\mathbf{Z}^{1/2}\mathbf{Z}^{1/2} = \mathbf{Z}$*,* $\mathrm{Re}(a) > (m-1)\beta/2 - t_m$*,* $\tau = (t_1, t_2, \dots, t_m)$*,* $t_1 \geq t_2 \geq \cdots t_m \geq 0$*, where* $\sum_{i=1}^{m} t_i = t$ *and* $t_1, t_2, \dots, t_m$ *are nonnegative integers, and*

$$J(\mathbf{I}_m) = \int_{\mathbf{X}\in\mathfrak{P}_m^\beta} f(\mathbf{X})|\mathbf{X}|^{a-(m-1)\beta/2-1} C_\tau^\beta(\mathbf{X})\,(d\mathbf{X}).$$

Proof. The proof is a verbatim copy of given by Xu and Fang (1990), only notice that

1. If $\mathbf{Y} = \mathbf{H}^*\mathbf{X}\mathbf{H}$ for $\mathbf{H} \in \mathfrak{U}^\beta(m)$, then $(d\mathbf{Y}) = (d\mathbf{X})$, and
2. if $\mathbf{B} = \mathbf{Z}^{1/2}\mathbf{X}\mathbf{Z}^{1/2}$ with $\mathbf{B}, \mathbf{X} \in \mathfrak{P}_m^\beta$, then, by Díaz-García and Gutiérrez-Jáimez (2013, Proposition 2), $(d\mathbf{B}) = |\mathbf{Z}|^{(m-1)\beta/2+1}(d\mathbf{X})$, where $\mathbf{Z}^{1/2}$ is the positive definite square root of $\mathbf{Z}$, such that $\mathbf{Z}^{1/2}\mathbf{Z}^{1/2} = \mathbf{Z}$.

□

The following result shows the extension of the Wishart's integral for real normed division algebras stated in Díaz-García (2013).

Lemma 3.3. *Let* $\mathbf{Y} \in \mathfrak{L}_{n,m}^\beta$*.*

$$\int_{\mathbf{Y}^*\mathbf{Y}=\mathbf{R}} f(\mathbf{Y}^*\mathbf{Y})d(\mathbf{Y}) = \frac{\pi^{\beta mn/2}}{\Gamma_m^\beta[\beta n/2]}|\mathbf{R}|^{\beta(n-m+1)/2-1} f(\mathbf{R}),$$

where $\mathrm{Re}(\beta n/2) > (m-1)\beta/2$*.*

Finally, Theorem 1 in Li (1993) is generalised for real normed division algebras, but prior considers the following definition, see Díaz-García and Gutiérrez-Jáimez (2013).

Definition 3.1. Let $\mathbf{X} \in \mathfrak{L}_{n,m}^\beta$ be a random matrix. Then, if $\mathbf{X} \stackrel{d}{=} \mathbf{\Xi}\mathbf{X}$ for every $\mathbf{\Xi} \in \mathfrak{U}^\beta(n)$, $\mathbf{X}$ is termed left-spherical ($\stackrel{d}{=}$ signifies that the two sides have the same distributions).

In addition, note that, if $\mathbf{X}$ has a density with respect to the Lebesgue measure, it is of the form $f(\beta\mathbf{X}^*\mathbf{X})$ and $\mathbf{X}$ can be factorised as

$$\mathbf{X} = \mathbf{H}_1\mathbf{A}, \tag{3.3}$$

where $\mathbf{A}_{m\times m}$ is not unique, it is independent of $\mathbf{H}_1$, and $\mathbf{H}_1$ has a normalised invariant measure on $\mathcal{V}_{m,n}^\beta$.

Theorem 3.1. *Suppose that* $\mathbf{X}$ *is as in Definition 3.1. Then, the characteristic function of* $\mathbf{X}$ *can be expressed as*

$$\phi_{\mathbf{X}}(\mathbf{T}) = \sum_{t=0}^{\infty}\sum_{\tau} \frac{C_{\tau}^{\beta}(-\mathbf{T}\mathbf{T}^{*}/4)}{[\beta n/2]_{\tau}^{\beta}\ C_{\tau}^{\beta}(\mathbf{I}_m)\ t!} \operatorname{E}\left(C_{\tau}^{\beta}(\mathbf{R})\right),$$

where $\mathbf{R} = \mathbf{X}^{*}\mathbf{X}$*,* $\operatorname{Re}(\beta n/2) > (m-1)\beta/2 - t_m$*,* $\tau = (t_1, t_2, \dots, t_m)$*,* $t_1 \geq t_2 \geq \cdots t_m \geq 0$*, wher* $\sum_{i=1}^{m} t_i = t$ *and* $t_1, t_2, \dots, t_m$ *are nonnegative integers.*

Proof. By (3.3) and Lemma 3.1,

$$\begin{aligned}
\phi_{\mathbf{X}}(\mathbf{T}) &= \operatorname{E}_{\mathbf{X}}(\operatorname{etr}\{i\mathbf{X}\mathbf{T}^{*}\}) = \operatorname{E}_{\mathbf{H}_1,\mathbf{A}}(\operatorname{etr}\{i\mathbf{H}_1\mathbf{A}\mathbf{T}^{*}\}) \\
&= \operatorname{E}_{\mathbf{H}_1,\mathbf{A}}(\operatorname{etr}\{i\mathbf{H}_1(\mathbf{T}\mathbf{A}^{*})^{*}\}) = \operatorname{E}_{\mathbf{A}}\left(\phi_{\mathbf{H}_1}(\mathbf{T}\mathbf{A}^{*})\right) \\
&\quad \text{because } \mathbf{H}_1 \text{ and } \mathbf{A} \text{ are independent,} \\
&= \operatorname{E}_{\mathbf{A}}\left({}_0F_1^{\beta}(\beta n/2, -\mathbf{T}\mathbf{A}^{*}(\mathbf{T}\mathbf{A}^{*})^{*}/4)\right) \\
&= \operatorname{E}_{\mathbf{A}}\left({}_0F_1^{\beta}(\beta n/2, -\mathbf{T}^{*}\mathbf{T}\mathbf{A}^{*}\mathbf{A}/4)\right) \\
&= \operatorname{E}_{\mathbf{R}}\left({}_0F_1^{\beta}(\beta n/2, -\mathbf{T}^{*}\mathbf{T}\mathbf{R}/4)\right) \quad \text{with } \mathbf{R} = \mathbf{A}^{*}\mathbf{A} \\
&\quad \text{thus, by Lemma 3.1 and Lemma 3.3,} \\
&= \sum_{t=0}^{\infty}\sum_{\tau} \frac{\pi^{\beta mn/2}}{[\beta n/2]_{\tau}^{\beta}\Gamma_m^{\beta}[\beta n/2]t!} \\
&\qquad \times \int_{\mathbf{R}\in\mathfrak{P}_m^{\beta}} |\mathbf{R}|^{\beta(n-m+1)/2-1} f(\mathbf{R}) C_{\tau}^{\beta}(-\mathbf{T}^{*}\mathbf{T}\mathbf{R}/4)(d\mathbf{R}) \\
&\quad \text{hence, by Lemma 3.2,} \\
&= \sum_{t=0}^{\infty}\sum_{\tau} \frac{C_{\tau}^{\beta}(-\mathbf{T}^{*}\mathbf{T}/4)}{[\beta n/2]_{\tau}^{\beta}C_{\tau}^{\beta}(\mathbf{I}_m)t!} \\
&\qquad \times \int_{\mathbf{R}\in\mathfrak{P}_m^{\beta}} \frac{\pi^{\beta mn/2}}{\Gamma_m^{\beta}[\beta n/2]} |\mathbf{R}|^{\beta(n-m+1)/2-1} f(\mathbf{R}) C_{\tau}^{\beta}(\mathbf{R})(d\mathbf{R}) \\
&= \sum_{t=0}^{\infty}\sum_{\tau} \frac{C_{\tau}^{\beta}(-\mathbf{T}^{*}\mathbf{T}/4)}{[\beta n/2]_{\tau}^{\beta}\ C_{\tau}^{\beta}(\mathbf{I}_m)\ t!} \operatorname{E}_{\mathbf{R}}\left(C_{\tau}^{\beta}(\mathbf{R})\right).
\end{aligned}$$

□

4 Kotz-Riesz distributions

This section introduces two versions of Kotz-Riesz distributions. Keeping this purpose in mind, first we define the spherical Kotz-Riesz distributions.

Definition 4.1. Let $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$.

1. It is said that $\mathbf{Z}$ has a spherical Kotz-Riesz distribution of type I if its density function is of the form

$$\frac{\beta^{mn\beta/2}\Gamma_m^\beta[n\beta/2]}{\pi^{mn\beta/2}\Gamma_m^\beta[n\beta/2,\kappa]} \operatorname{etr}\{-\beta \operatorname{tr} \mathbf{Z}^*\mathbf{Z}\}\, q_\kappa\left[\beta\mathbf{Z}^*\mathbf{Z}\right](d\mathbf{Z}), \tag{4.1}$$

   for $\mathbf{Z} \in \mathfrak{L}_{n,m}^\beta$, $\operatorname{Re}(n\beta/2) > (m-1)\beta/2 - k_m$, denoting this fact as

$$\mathbf{Z} \sim \mathcal{K}\Re_{n\times m}^{\beta,I}(\kappa, \mathbf{0}, \mathbf{I}_n, \mathbf{I}_m).$$

2. It is said that $\mathbf{Z}$ has a Kotz-Riesz distribution of type II if its density function is of the form

$$\frac{\beta^{mn\beta/2}\Gamma_m^\beta[n\beta/2]}{\pi^{mn\beta/2}\Gamma_m^\beta[n\beta/2,-\kappa]} \operatorname{etr}\{-\beta \operatorname{tr} \mathbf{Z}^*\mathbf{Z}\}\, q_\kappa\left[(\beta\mathbf{Z}^*\mathbf{Z})^{-1}\right](d\mathbf{Z}), \tag{4.2}$$

   for $\mathbf{Z} \in \mathfrak{L}_{n,m}^\beta$, $\operatorname{Re}(n\beta/2) > (m-1)\beta/2 + k_1$, denoting this fact as

$$\mathbf{Z} \sim \mathcal{K}\Re_{n\times m}^{\beta,II}(\kappa, \mathbf{0}, \mathbf{I}_n, \mathbf{I}_m).$$

Now, the *elliptical Kotz-Riesz distribution* or simply termed *Kotz-Riesz distribution* is obtained.

Theorem 4.1. *Let* $\mathbf{\Sigma} \in \mathbf{\Phi}_m^\beta$, $\mathbf{\Theta} \in \mathbf{\Phi}_n^\beta$, $\boldsymbol{\mu} \in \mathfrak{L}_{n,m}^\beta$, $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$, *and* $\mathbf{Z} \sim \mathcal{K}\Re_{n\times m}^{\beta,I}(\kappa, \mathbf{0}, \mathbf{I}_n, \mathbf{I}_m)$. *We define* $\mathbf{X} = \boldsymbol{\mu} + \mathcal{U}(\mathbf{\Theta})^*\mathbf{Z}\,\mathcal{U}(\mathbf{\Sigma})$, *where* $\mathcal{U}(\mathbf{B}) \in \mathfrak{T}_U^\beta(m)$ *is such that* $\mathbf{B} = \mathcal{U}(\mathbf{B})^*\mathcal{U}(\mathbf{B})$ *is the Cholesky decomposition of* $\mathbf{B}$. *Then,*

1. $\mathbf{X}$ *is said to have a Kotz-Riesz distribution of type I and its density function is of the form*

$$\frac{\beta^{mn\beta/2+\sum_{i=1}^m k_i}\Gamma_m^\beta[n\beta/2]}{\pi^{mn\beta/2}\Gamma_m^\beta[n\beta/2,\kappa]|\mathbf{\Sigma}|^{n\beta/2}|\mathbf{\Theta}|^{m\beta/2}}$$

$$\times \operatorname{etr}\left\{-\beta \operatorname{tr}\left[\mathbf{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})^*\mathbf{\Theta}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right]\right\}$$
$$\times q_\kappa\left[\mathcal{U}(\mathbf{\Sigma})^{*-1}(\mathbf{X}-\boldsymbol{\mu})^*\mathbf{\Theta}^{-1}(\mathbf{X}-\boldsymbol{\mu})\,\mathcal{U}(\mathbf{\Sigma})^{-1}\right](d\mathbf{X}), \tag{4.3}$$

for $\mathbf{X} \in \mathfrak{L}^\beta_{n,m}$, $\operatorname{Re}(n\beta/2) > (m-1)\beta/2 - k_m$, *denoting this fact as*

$$\mathbf{X} \sim \mathcal{K}\mathfrak{R}^{\beta,I}_{n\times m}(\kappa, \boldsymbol{\mu}, \mathbf{\Theta}, \mathbf{\Sigma}).$$

2. $\mathbf{X}$ *is said to have a Kotz-Riesz distribution of type II and its density function is of the form*

$$\frac{\beta^{mn\beta/2-\sum_{i=1}^m k_i}\Gamma^\beta_m[n\beta/2]}{\pi^{mn\beta/2}\Gamma^\beta_m[n\beta/2,-\kappa]|\mathbf{\Sigma}|^{n\beta/2}|\mathbf{\Theta}|^{m\beta/2}}$$
$$\times \operatorname{etr}\left\{-\beta \operatorname{tr}\left[\mathbf{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})^*\mathbf{\Theta}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right]\right\}$$
$$\times q_\kappa\left[\left(\mathcal{U}(\mathbf{\Sigma})^{*-1}(\mathbf{X}-\boldsymbol{\mu})^*\mathbf{\Theta}^{-1}(\mathbf{X}-\boldsymbol{\mu})\,\mathcal{U}(\mathbf{\Sigma})^{-1}\right)^{-1}\right](d\mathbf{X}) \tag{4.4}$$

for $\mathbf{X} \in \mathfrak{L}^\beta_{n,m}$, $\operatorname{Re}(n\beta/2) > (m-1)\beta/2 + k_1$, *denoting this fact as*

$$\mathbf{X} \sim \mathcal{K}\mathfrak{R}^{\beta,II}_{n\times m}(\kappa, \boldsymbol{\mu}, \mathbf{\Theta}, \mathbf{\Sigma}).$$

Proof. This is an immediate consequence of Theorem 3 in Díaz-García and Gutiérrez-Jáimez (2013) and considering the fact that, for an scalar a,

$$q_\kappa(a\mathbf{A}) = q_\kappa(a\mathbf{I}_m)q_\kappa(\mathbf{A}) = a^{\sum_{i=1}^m k_i}q_\kappa(\mathbf{A}).$$

□

If $\kappa = (0, 0, \dots, 0)$ and $\mathbf{\Sigma} = 2\mathbf{\Sigma}$ in two densities in Theorem 4.1, the matrix multivariate normal distribution for real normed division algebras is obtained, see Díaz-García and Gutiérrez-Jáimez (2011). Moreover, when $\kappa = (l, l, \dots, l)$, $l = a - (m-1)\beta/2 - 1$ and $\mathbf{\Sigma} = \mathbf{\Sigma}/r$, $r > 1$, the original Kotz type distributions are obtained, i.e., for $s = 1$ with the notation of Li (1993).

The main consequence of this article is achieved by finding the two versions of the Riesz distributions in terms of Kotz-Riesz distributions.

Theorem 4.2. 1. *Assume that* $\mathbf{X} \sim \mathcal{K}\mathfrak{R}^{\beta,I}_{n\times m}(\kappa, \mathbf{0}, \mathbf{\Theta}, \mathbf{\Sigma})$, *and define* $\mathbf{Y} = \mathbf{X}^*\mathbf{\Theta}^{-1}\mathbf{X}$. *Then, its density function is of the form*

$$\frac{\beta^{am+\sum_{i=1}^m k_i}}{\Gamma^\beta_m[a,\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma})}\operatorname{etr}\{-\beta\mathbf{\Sigma}^{-1}\mathbf{Y}\}|\mathbf{Y}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{Y})(d\mathbf{Y}), \tag{4.5}$$

for $\mathbf{Y} \in \mathfrak{P}^\beta_m$ *and* $\operatorname{Re}(a) > (m-1)\beta/2 - k_m$, *denoting this fact as* $\mathbf{Y} \sim \mathfrak{R}^{\beta,I}_m(a, \kappa, \mathbf{\Sigma})$.

2. *Suppose that* $\mathbf{X} \sim \mathcal{K}\Re^{\beta,II}_{n\times m}(\kappa,\mathbf{0},\mathbf{\Theta},\mathbf{\Sigma})$*, and define* $\mathbf{Y} = \mathbf{X}^*\mathbf{X}$*. Then, its density function is of the form*

$$\frac{\beta^{am-\sum_{i=1}^m k_i}}{\Gamma^\beta_m[a,-\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma}^{-1})} \operatorname{etr}\{-\beta\mathbf{\Sigma}^{-1}\mathbf{Y}\}|\mathbf{Y}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{Y}^{-1})(d\mathbf{Y}), \qquad (4.6)$$

for $\mathbf{Y} \in \mathfrak{P}^\beta_m$ *and* $\operatorname{Re}(a) > (m-1)\beta/2+k_1$*, denoting this fact as* $\mathbf{Y} \sim \Re^{\beta,II}_m(a,\kappa,\mathbf{\Sigma})$*, where* $\kappa = (k_1,k_2,\dots,k_m) \in \Re^m$*.*

Proof. 1. By applying (2.13), the density of $\mathbf{X}$ is of the form

$$\frac{\beta^{mn\beta/2+\sum_{i=1}^m k_i}\Gamma^\beta_m[n\beta/2]}{\pi^{mn\beta/2}\Gamma^\beta_m[n\beta/2,\kappa]|\mathbf{\Sigma}|^{n\beta/2}q_\kappa(\mathbf{\Sigma})|\mathbf{\Theta}|^{m\beta/2}}$$
$$\times \operatorname{etr}\left\{-\beta \operatorname{tr}\mathbf{\Sigma}^{-1}\mathbf{X}^*\mathbf{\Theta}^{-1}\mathbf{X}\right\} q_\kappa\left[\mathbf{X}^*\mathbf{\Theta}^{-1}\mathbf{X}\right](d\mathbf{X}).$$

Now, let $\mathbf{\Theta}^{1/2}$ be the positive definite square root of $\mathbf{\Theta}$ and define $\mathbf{V} = \mathbf{\Theta}^{-1/2}\mathbf{X}$. Hence, $\mathbf{V} \sim \mathcal{K}\Re^{\beta,I}_{n\times m}(\kappa,\mathbf{0},\mathbf{I}_n,\mathbf{\Sigma})$. Moreover, its density is of the form

$$\frac{\beta^{mn\beta/2+\sum_{i=1}^m k_i}\Gamma^\beta_m[n\beta/2]}{\pi^{mn\beta/2}\Gamma^\beta_m[n\beta/2,\kappa]|\mathbf{\Sigma}|^{n\beta/2}q_\kappa(\mathbf{\Sigma})} \operatorname{etr}\left\{-\beta \operatorname{tr}\mathbf{\Sigma}^{-1}\mathbf{V}^*\mathbf{V}\right\}$$
$$q_\kappa\left[\mathbf{V}^*\mathbf{V}\right](d\mathbf{V}).$$

Finally, defining $\mathbf{Y} = \mathbf{V}^*\mathbf{V}$, the desired result is obtained by applying the Lemma 3.3 and denoting $a = n\beta/2$.

2. It is obtained in the similar way to 1.

□

Note that (4.5) and (4.6) are the density functions of Riesz distributions of type I and II, respectively, which were obtained on a special case of the *Riesz measure* by Faraut and Korányi (1994), Hassairi and Lajmi (2001) and Díaz-García (2015).

Below, the characteristic functions of the Kotz-Riesz distribution of type I are found. However, in order to apply Theorem 3.1, it is necessary to find the expected value of $C^\beta_\tau(\mathbf{AY})$ where $\mathbf{Y} = \mathbf{X}^*\mathbf{X}$ has a Riesz distribution.

Theorem 4.3. *Assuming that* $\mathbf{Z} \sim \mathcal{K}\Re^{\beta,I}_{n\times m}(\kappa,\mathbf{0},\mathbf{I}_n,\mathbf{I}_m)$*,*

$$\operatorname{E}(C^\beta_\tau(\mathbf{AY})) = \frac{\beta^{mn\beta/2+\sum_{i=1}^m k_i}\Gamma^\beta_m[n\beta/2,\tau+\kappa]}{\Gamma^\beta_m[n\beta/2,\kappa]}C^\beta_\tau(\mathbf{A}),$$

where $\mathbf{Y} \stackrel{d}{=} \mathbf{Z}^*\mathbf{Z}$, $\tau = (t_1, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, $t_1, t_2, \dots, t_m$ *are nonnegative integers,* $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, *and* $k_1, k_2, \dots, k_m$ *are nonnegative integers.*

Proof. By (4.5), $\mathbf{Y} = \mathbf{Z}^*\mathbf{Z} \sim \mathfrak{R}_m^{\beta,I}(\beta n/2, \kappa, \mathbf{I}_m)$. Then $\mathrm{E}(C_\tau^\beta(\mathbf{AY}))$ is obtained as

$$\begin{aligned} &= \frac{\beta^{am+\sum_{i=1}^m k_i}}{\Gamma_m^\beta[\beta n/2, \kappa]} \int_{\mathbf{Y}\in\mathfrak{P}_m^\beta} C_\tau^\beta(\mathbf{AY}) \operatorname{etr}\{-\beta\mathbf{Y}\}|\mathbf{Y}|^{(n-m-1)\beta/2-1} q_\kappa(\mathbf{Y})(d\mathbf{Y}) \\ &\qquad \text{by Lemma 3.2} \\ &= \frac{\beta^{am+\sum_{i=1}^m k_i}}{\Gamma_m^\beta[\beta n/2, \kappa]} \frac{J(\mathbf{I}_m)}{C_\tau^\beta(\mathbf{I}_m)} C_\tau^\beta(\mathbf{A}), \end{aligned}$$

where by (2.17),

$$\begin{aligned} J(\mathbf{I}_m) &= \int_{\mathbf{Y}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\beta\mathbf{Y}\}|\mathbf{Y}|^{(n-m-1)\beta/2-1} q_\kappa(\mathbf{Y}) C_\tau^\beta(\mathbf{Y})(d\mathbf{Y}) \\ &= \int_{\mathbf{Y}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\beta\mathbf{Y}\}|\mathbf{Y}|^{(n-m-1)\beta/2-1} q_\kappa(\mathbf{Y}) \\ &\qquad\qquad \times C_\tau^\beta(\mathbf{I}_m) \int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} q_\tau(\mathbf{H}^*\mathbf{YH})(d\mathbf{H})(d\mathbf{Y}). \end{aligned}$$

Thus, proceeding as in the proof of Theorem 5.9 in Gross and Richards (1987, p. 802) and by applying (2.11),

$$\begin{aligned} J(\mathbf{I}_m) &= C_\tau^\beta(\mathbf{I}_m) \int_{\mathbf{Y}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\beta\mathbf{Y}\}|\mathbf{Y}|^{(n-m-1)\beta/2-1} q_\kappa(\mathbf{Y}) q_\tau(\mathbf{Y})(d\mathbf{Y}) \\ &= C_\tau^\beta(\mathbf{I}_m) \int_{\mathbf{Y}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\beta\mathbf{Y}\}|\mathbf{Y}|^{(n-m-1)\beta/2-1} q_{\kappa+\tau}(\mathbf{Y})(d\mathbf{Y}) \\ &= C_\tau^\beta(\mathbf{I}_m)\Gamma_m^\beta[\beta n/2, \kappa+\tau]. \end{aligned}$$

From where the desired result is achieved. □

The following result gives the characteristic function of the spherical Kotz-Riesz distribution of type I.

Theorem 4.4. *Assume that* $\mathbf{Z} \sim \mathcal{K}\mathfrak{R}_{n\times m}^{\beta,I}(\kappa, \mathbf{0}, \mathbf{I}_n, \mathbf{I}_m)$. *Then the characteristic function of* $\mathbf{Z}$ *can be expressed as*

$$\phi_{\mathbf{Z}}(\mathbf{T}) = \frac{\beta^{mn\beta/2+\sum_{i=1}^m k_i}}{\Gamma_m^\beta[n\beta/2, \kappa]} \sum_{t=0}^\infty \sum_\tau \frac{\Gamma_m^\beta[n\beta/2, \tau+\kappa]}{[\beta n/2]_\tau^\beta} \frac{C_\tau^\beta(-\mathbf{TT}^*/4)}{t!},$$

where $\operatorname{Re}(\beta n/2) > (m-1)\beta/2 - t_m$, $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots t_m \geq 0$, $\sum_{i=1}^m t_i = t$, $t_1, t_2, \dots, t_m$ *are nonnegative integers,* $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, *and* $k_1, k_2, \dots, k_m$ *are nonnegative integers.*

Proof. This is an immediately obtained from Theorem 3.1 and Theorem 4.3. □

Finally, the characteristic function of the elliptical Kotz-Riesz distribution of type I is given.

Corollary 4.1. Assume that $\mathbf{X} \sim \mathcal{K}\mathfrak{R}_{n\times m}^{\beta,I}(\kappa, \boldsymbol{\mu}, \boldsymbol{\Theta}, \boldsymbol{\Sigma})$. Then the characteristic function of $\mathbf{X}$ can be expressed as

$$\phi_{\mathbf{X}}(\mathbf{T}) = \frac{\beta^{mn\beta/2+\sum_{i=1}^m k_i}\operatorname{etr}\{i\boldsymbol{\mu}\mathbf{T}^*\}}{\Gamma_m^\beta[n\beta/2,\kappa]}\sum_{t=0}^{\infty}\sum_{\tau}\frac{\Gamma_m^\beta[n\beta/2,\tau+\kappa]}{[\beta n/2]_\tau^\beta} \times \frac{C_\tau^\beta(-\boldsymbol{\Theta}\mathbf{T}\boldsymbol{\Sigma}\mathbf{T}^*/4)}{t!},$$

where $\operatorname{Re}(\beta n/2) > (m-1)\beta/2 - t_m$, $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots t_m \geq 0$, $\sum_{i=1}^m t_i = t$, $t_1, t_2, \dots, t_m$ are nonnegative integers, $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, $k_1, and k_2, \dots, k_m$ are nonnegative integers.

Proof. Let $\boldsymbol{\Sigma} \in \boldsymbol{\Phi}_m^\beta$, $\boldsymbol{\Theta} \in \boldsymbol{\Phi}_n^\beta$, $\boldsymbol{\mu} \in \mathfrak{L}_{n,m}^\beta$, and $\mathbf{Z} \sim \mathcal{K}\mathfrak{R}_{n\times m}^{\beta,I}(\kappa, \mathbf{0}, \mathbf{I}_n, \mathbf{I}_m)$. Define $\mathbf{X} = \boldsymbol{\mu} + \boldsymbol{\Theta}^{1/2}\mathbf{Z}\boldsymbol{\Sigma}^{1/2}$, where $\mathbf{B}^{1/2}$ is the positive definite square root of $\mathbf{B}$, such that $\mathbf{B}^{1/2}\mathbf{B}^{1/2} = \mathbf{B}$. Then $\mathbf{X} \sim \mathcal{K}\mathfrak{R}_{n\times m}^{\beta,I}(\kappa, \boldsymbol{\mu}, \boldsymbol{\Theta}, \boldsymbol{\Sigma})$. Therefore,

$$\begin{aligned}
\phi_{\mathbf{X}}(\mathbf{T}) &= \operatorname{E}(\operatorname{etr}\{i\mathbf{X}\mathbf{T}^*\}) = \operatorname{E}(\operatorname{etr}\{i\left(\boldsymbol{\mu} + \boldsymbol{\Theta}^{1/2}\mathbf{Z}\boldsymbol{\Sigma}^{1/2})\mathbf{T}^*\}\right) \\
&= \operatorname{etr}\{i\boldsymbol{\mu}\mathbf{T}^*\}\operatorname{E}(\operatorname{etr}\{i\boldsymbol{\Theta}^{1/2}\mathbf{Z}\boldsymbol{\Sigma}^{1/2}\mathbf{T}^*\}) \\
&= \operatorname{etr}\{i\boldsymbol{\mu}\mathbf{T}^*\}\operatorname{E}(\operatorname{etr}\{i\mathbf{Z}\left(\boldsymbol{\Theta}^{1/2}\mathbf{T}\boldsymbol{\Sigma}^{1/2})^*\}\right) \\
&= \operatorname{etr}\{i\boldsymbol{\mu}\mathbf{T}^*\}\phi_{\mathbf{Z}}\left(\boldsymbol{\Theta}^{1/2}\mathbf{T}\boldsymbol{\Sigma}^{1/2}\right).
\end{aligned}$$

The conclusion follows as the immediate consequence of Theorem 4.4. □

5 Conclusions

There is no doubt about the importance of the Riesz distribution from a theoretical point of view and, by establishing a generalisation of the Wishart distribution in imminent, from a practical point of view. For a while, statisticians have been trying to extend the theory of sampling in multivariate analysis for samples taken from a non-normal population. The introduction of Kotz-Riesz distribution and the distribution of Riesz are a progress in that direction. Furthermore, now these results, combined with those obtained in Díaz-García (2015) and Díaz-García (2016), allow us to generalise diverse techniques in multivariate analysis, assuming a Riesz distribution instead of a Wishart distribution, or equivalently, assuming a Kotz-Riesz distribution instead of a matrix multivariate normal distribution. Finally we emphasize that the Kotz-Riesz distribution belongs to the family of left-elliptical distributions. Then, all the results obtained for the latter family, as moments, estimation, hypothesis testing, etc., can be quite *easily* particularised for the Kotz-Riesz distribution.

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