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A Note on the Smooth Estimator of the Quantile Function with Left-Truncated Data

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Abstract. This note focuses on estimating the quantile function based on the kernel smooth estimator under a truncated dependent model. The Bahadur-type representation of the kernel smooth estimator is established, and from the Bahadur representation it can be seen that this estimator is strongly consistent.

Keywords. Kiefer process, law of the iterated logarithm, strong Gaussian approximation, strong mixing, truncated data.

MSC: 62G05, 62G20.

1 Introduction

In medical follow-up or in engineering life testing studies, one may not be able to completely observe the variable of interest, referred to hereafter as lifetime. Among the different forms in which incomplete data appear, right-censoring and left-truncation are of the most common. Left-truncation may occur if the time of origin of the lifetime precedes the time of origin of the study. Only subjects that fail after the start of the study are followed, otherwise they are left-truncated. Woodroofe (1985) reviews examples from astronomy and economy where such data may occur.

Let $\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_N}$ be a sequence of the lifetime variables which may not be mutually independent, but have a common unknown continuous distribution function (d.f.) F with a density function $f = F'$. Let $\mathbf{T_1}, \mathbf{T_2}, \ldots, \mathbf{T_N}$ be a

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sequence of independent and identically distributed (i.i.d.) random variables (rv's) with a continuous d.f. G; they are also assumed to be independent of the random variables $\mathbf{X_i}$'s. In the left-truncation model, $(\mathbf{X_i}, \mathbf{T_i})$ is observed only when $\mathbf{X_i} \geq \mathbf{T_i}$. Let $(X_1, T_1), \ldots, (X_n, T_n)$ be the sample actually observed (i.e., $X_i \geq T_i$), and put $\gamma := \mathbf{P}(\mathbf{X_1} \geq \mathbf{T_1}) > 0$, where $\mathbf{P}$ is the absolute probability (related to the N-sample). Note that n itself is a rv and that γ can be estimated by n/N (although this estimator cannot be calculated since N is unknown). For any d.f. L denote the left and right endpoints of its support by $a_L = \inf\{x : L(x) > 0\}$ and $b_L = \sup\{x : L(x) < 1\}$, respectively. Then, under the current model, as discussed by Woodroofe (1985), we assume that $a_G \leq a_F$ and $b_G \leq b_F$, and define

$$C(x) = \mathbf{P}(\mathbf{T_1} \leq x \leq \mathbf{X_1} | \mathbf{T_1} \leq \mathbf{X_1}) = \mathbb{P}(T_1 \leq x \leq X_1) = \gamma^{-1} G(x)(1 - F(x)), \tag{1.1}$$

where $\mathbb{P}(\cdot) = \mathbf{P}(\cdot|n)$ is the conditional probability (related to the n-sample) and consider its empirical estimate

$$C_n(x) = n^{-1} \sum_{i=1}^{n} I(T_i \leq x \leq X_i), \tag{1.2}$$

where $I(\cdot)$ is the indicator function. Thus, the nonparametric maximum likelihood estimate of F originally proposed by Lynden-Bell (1971), is given by

$$\widehat{F}_n(x) = 1 - \prod_{X_i \leq x} \left(1 - \frac{1}{nC_n(X_i)}\right),$$

assuming no ties in the data. Let

$$F^*(x) = \mathbf{P}(\mathbf{X_1} \leq x | \mathbf{T_1} \leq \mathbf{X_1}) = \mathbb{P}(X_1 \leq x) = \gamma^{-1} \int_{a_F}^{x} G(u) dF(u),$$

be the d.f. of the observed lifetimes. Its empirical estimator is given by

$$F_n^*(x) = n^{-1} \sum_{i=1}^{n} I(X_i \leq x).$$

On the other hand, the d.f. of the observed T_i's is given by

$$G^*(x) = \mathbf{P}(\mathbf{T_1} \leq x | \mathbf{T_1} \leq \mathbf{X_1}) = \mathbb{P}(T_1 \leq x) = \gamma^{-1} \int_{a_F}^{\infty} G(x \wedge u) dF(u),$$

and is estimated by

$$G_n^*(x) = n^{-1} \sum_{i=1}^{n} I(T_i \leq x).$$

It can be concluded from (1.1) and (1.2) that

$$C(x) = G^*(x) - F^*(x^-), \quad C_n(x) = G_n^*(x) - F_n^*(x^-). \tag{1.3}$$

The quantile function of the distribution function F is defined as

$$Q(p) = F^{-1}(p) := \inf\{t : F(t) \geq p\} \quad 0 < p < 1.$$

The role of the quantile function in statistical data modeling was emphasized by Parzen (1979). In econometrics, Gastwirth (1971) used the quantile function to give a succinct definition of the Lorenz curve, which measures inequality in distribution of resources and in size distribution.

Several nonparametric estimators of $Q(p)$ for a random (untruncated) sample from F appear in the literature. For example, the sample quantile function is defined by

$$F_n^{-1}(p) := \inf\{t : F_n(t) \geq p\} \quad 0 < p < 1,$$

where F_n is the empirical distribution function based on the sample drawn from the population distribution function F.

In the i.i.d. framework, the properties of the estimator F_n^{-1} have been extensively studied (see e.g., Csörgő, 1983; Shorack and Wellner, 1986). Under a ϕ-mixing condition (see Doukhan, 1996, for the definition), the Bahadur representation was obtained by Sen (1972) and the extension to the α-mixing case (see Definition 1.1) was obtained by Yoshihara (1995). Under an α-mixing condition, the strong approximation of the quantile process $\sqrt{n}f(F^{-1}(\cdot))[F_n^{-1}(\cdot) - F^{-1}(\cdot)]$ by a two-parameter Gaussian process at the rate $O((\log n)^{-\lambda})$ for some $\lambda > 0$, was obtained by Fotopoulos et al. (1994). The strong approximation of Fotopoulos et al. (1994) was obtained by Yu (1996) with a lighter strong mixing decay rate and wider intervals.

For a truncated model with mutually independent X_i's and T_i's, and independent and identically distributed sequences, Gürler et al. (1993) obtained weak and strong representations for the quantile function

$$\widehat{Q}_n(p) := \inf\{t : \widehat{F}_n(t) \geq p\} \quad 0 < p < 1.$$

In the left-truncation and right censorship model (LTRC), Tse (2005), obtained strong Gaussian approximations of the product-limit (PL) quantile process

$$\rho_n(p) := \sqrt{n}f(Q(p))[\widehat{Q}_n(p) - Q(p)], \tag{1.4}$$

using a two-parameter Kiefer-type process at the rate $O((\log n)^{3/2}n^{-1/8})$ (where $\widehat{F}_n(t)$ is replaced by the PL estimator for the LTRC model.)

Under α-mixing and left-truncation, Lemdani et al. (2005) established strong consistency, asymptotic normality and the Bahadur representation of quantile function $\widehat{Q}_n(\cdot)$. The strong Gaussian approximation of the PL-quantile process has been constructed by Bolbolian et al. (2010) at the rate $O((\log n)^{-\lambda})$ for some $\lambda > 0$.

The kernel smooth quantile estimator of $Q(p)$, $0 < p < 1$, can also be defined for the left-truncated data. For $0 < p < 1$, it is given by

$$Q_n(p) := h_n^{-1} \int_0^1 \widehat{Q}_n(t) K\left(\frac{t-p}{h_n}\right) dt,$$

where $\{h_n, n \geq 1\}$ is a bandwidth sequence of positive numbers such that $h_n \to 0$ as $n \to \infty$, and $K(\cdot)$ is a probability density function which is zero outside a finite interval $(-1, 1)$.

The kernel smooth quantile estimator of $Q(p)$, was first mentioned in Parzen (1979), in the i.i.d. framework (with no truncation, where $\widehat{Q}_n$ is replaced by the sample quantile function F_n^{-1}). Subsequently, a number of scholars discussed its properties for independent random variables. Yang (1985) established the asymptotic normality and mean consistency of $Q_n(p)$, and obtained its Bahadur representation. Under an α-mixing condition, Wei et al. (2010) obtained a Bahadur representation and strong consistency of the kernel smooth quantile estimator $Q_n(\cdot)$.

Under random censorship or randomly truncated data, Xiang (1995) and Yong et al. (2006) established a similar Bahadur representation of $Q_n(p)$, the latter also giving strong consistency and asymptotic normality of the estimator.

The main aim of this paper is to derive a Bahadur-type representation of $Q_n(\cdot)$, for the case of truncated data where the underlying lifetimes are assumed to be strongly mixing. As a result, we obtain strong uniform consistency. The counterpart of these results for the censored dependent model was established by Ajami et al. (2011).

We consider the strong mixing dependence, which amounts to a form of asymptotic independence between the past and future as shown by its definition.

Definition 1.1. Let $\{Z_i, i \geq 1\}$ denote a sequence of random variables. Given a positive integer m, set

$$\alpha(m) = \sup_{k \geq 1} \sup_{A,B} \{|P(A \cap B) - P(A)P(B)| \ ; \ A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+m}^{\infty}\},$$

where $\mathcal{F}_i^k$ denote the σ-field of events generated by $\{Z_j; i \leq j \leq k\}$. The sequence is said to be strong mixing (α-mixing) if the mixing coefficient $\alpha(m) \to 0$ as $m \to \infty$.

Among various mixing conditions used in the literature, α-mixing, is reasonably weak and has many practical applications. Many processes and time series exist that fulfill the strong mixing condition. As a simple example we can consider the Gaussian AR(1) process for which

$$Z_t = \rho Z_{t-1} + \varepsilon_t,$$

where $|\rho| < 1$ and ε_t's are i.i.d. random variables with standard normal distribution. It can be shown (see Ibragimov and Linnik, 1971, pp. 312-313) that $\{Z_t\}$ satisfies a strong mixing condition. The stationary autoregressive-moving average (ARMA) processes (Doukhan, 1994), which are widely applied in time series analyses, are α-mixing with an exponential mixing coefficient, i.e., $\alpha(n) = O(e^{-\nu n})$ for some $\nu > 0$. The threshold models, the EXPAR models (see Ozaki, 1979), the simple ARCH models (see Engle, 1982), their extensions (see Diebolt and Guégan, 1993) and the bilinear Markovian models are geometrically strongly mixing under a number of general ergodicity conditions.

2 Main Result

Before we state our result we list all the assumptions used throughout this paper. Let $0 < p_0 < p_1 < 1$ be such that $a_F + \delta < Q(p_0) < Q(p_1) < b_F - \delta$ for some $\delta > 0$.

Assumptions.
A1. $\{X_i\}_{i\geq 1}$ is a sequence of stationary α-mixing rv's with mixing coefficient $\alpha(n) = O(e^{-(\log n)^{1+\nu}})$ for some $\nu > 0$.
A2. F is twice continuously differentiable on $[Q(p_0) - \delta, Q(p_1) + \delta]$ and f is bounded away from zero there.
A3. G is continuously differentiable on $[Q(p_0) - \delta, Q(p_1) + \delta]$ and $g = G'$ is bounded away from zero there. We also assume that $a_G < a_F$.
A4. K is a probability density function with finite support $(-1, 1)$.
A5. $\int_{-\infty}^{\infty} tK(t)dt = 0$.

Our main result is the following theorem:

Theorem 2.1. *Let $\{h_n\}$ be a sequence of positive bandwidths tending to zero as $n \to \infty$ such that*

$$\sum_{n=2}^{\infty} \frac{(\log n)^{1/2}}{h_n n^{\eta}} < \infty \quad for \quad some \quad \eta > 0. \tag{2.1}$$

Under the stated assumptions,

$$Q_n(p) - Q(p) = \frac{p - \widehat{F}_n(Q(p))}{f(Q(p))} + O(a_n) \quad a.s., \tag{2.2}$$

uniformly on $p_0 \leq p \leq p_1$, where

$$a_n = h_n^2 \vee n^{-1/2}(\log n)^{-\lambda} \vee (n^{-1/2}(h_n \log n)^{3/4}),$$

for some $\lambda > 0$ and $a \vee b = \max(a, b)$.

Proof. See the Appendix. □

Strong consistency of $Q_n(p)$ can be stated as a corollary to Theorem 2.1.

Corollary 2.1. *Assume that the conditions of Theorem 2.1 are satisfied, and*

$$\frac{nh_n^4}{\log\log n} \to 0, \quad as \quad n \to \infty. \tag{2.3}$$

Then, we have

$$\limsup_{n\to\infty} \sqrt{\frac{n}{\log\log n}} \sup_{p_0\le p\le p_1} |Q_n(p) - Q(p)| = 0 \quad a.s.$$

Remark 2.1. If the bandwidth h_n is chosen to be $h_n \sim \alpha n^{-\beta}$ with $\alpha > 0$, $\beta \ge \frac{1}{4}$ and for some $\eta > \beta + 1$ such that (2.1) is obtained, then condition (2.3) is satisfied.

Remark 2.2. By choosing $\eta > 2$, we see that the classical kernel-method condition $nh_n \to \infty$ clearly implies (2.1).

3 Appendix

Without loss of generality, under Assumption **A2**, we can assume that our probability space is so rich that the approximation

$$\sup_{p_0\le p\le p_1} |\rho_n(p) - (1-p)B(Q(p), n)| = O((\log n)^{-\lambda}) \quad a.s., \tag{3.1}$$

of Bolbolian et al. (2010) holds, where $\rho_n(\cdot)$ is defined in (1.4) and $\lambda > 0$. Here $B(t, n)$ is a two-parameter zero-mean Gaussian process which is defined by

$$B(t,n) := \frac{k(t,n)/\sqrt{n}}{C(t)} + \int_{a_F}^{t} \frac{k(u,n)/\sqrt{n}}{C(u)^2} dC(u), \quad a_F \le t < b_F, \tag{3.2}$$

where $k(x, n)$ is a generalized Kiefer process with a covariance function

$$E[k(s,t)k(s',t')] = \Gamma(s,s')\min(t,t'),$$

where

$$\Gamma(s,s') = Cov(g_1(s), g_1(s')) + \sum_{j=2}^{\infty}[Cov(g_1(s), g_j(s')) + Cov(g_1(s'), g_j(s))],$$

$g_j(s) = I(X_j \le s) - F^*(s)$. Let $\Gamma^*(s,s',t,t') = \min(t,t')\Gamma(s,s')$. Finally, without loss of generality, we put $a_F = 0$.

In order to prove the main theorem, we need the following lemmas. The first lemma gives the functional law of the iterated logarithm for the generalized Kiefer process $k(x, n)$.

Lemma 3.1. *Let Assumption **A1** is satisfied. Then the sequence*

$$\{(2n\log\log n)k(\cdot, n), n \ge 1\}$$

of functions on $[0,\infty)$ is with probability 1 relatively compact in the supremum norm and has the unit ball B in the reproducing kernel Hilbert space $H(\Gamma^*)$ as its set of limits.

Proof. Theorem 2 of Lai (1974) (see the remark on Page 19) implies that the conclusion holds for the sequence $\{(2n\log\log n)k(\cdot,n), n\geq 1\}$. □

As a consequence of Lemma 3.1, we have

$$\limsup_{n\to\infty}\sup_{0\leq x<\infty}\frac{k(x,n)}{\sqrt{2n\log\log n}}=O(1)\quad a.s., \tag{3.3}$$

we have

$$\limsup_{n\to\infty}\sup_{0\leq x\leq b}\frac{B(x,n)}{\sqrt{\log\log n}}=O(1)\quad a.s., \tag{3.4}$$

where $b<b_F$.

Lemma 3.2. *Let $\{h_n\}$ be a sequence of positive bandwidths tending to zero as $n\to\infty$ such that (2.1) is satisfied. Under Assumptions **A1-A3**, we have*

$$\sup_{p_0\leq p\leq p_1}\sup_{|t|\leq 1}|B(Q(p+h_nt),n)-B(Q(p),n)|=O\left((h_n\log n)^{3/4}\right)\quad a.s. \tag{3.5}$$

Proof. It is easy to see that, for n large enough

$$\begin{aligned}
&|B(Q(p+h_nt),n)-B(Q(p),n)|\\
&\quad\leq \left|\frac{k(Q(p+h_nt),n)/\sqrt{n}}{C(Q(p+h_nt))}-\frac{k(Q(p),n)/\sqrt{n}}{C(Q(p))}\right|\\
&\qquad+\left(\inf_{Q(p_0)-\delta\leq u\leq Q(p_1)+\delta}C(u)\right)^{-2}\\
&\qquad\times\sup_{0\leq u<\infty}\frac{|k(u,n)|}{\sqrt{n}}\,|C(Q(p+h_nt)-C(Q(p))|\\
&\quad=:\ I_1+I_2
\end{aligned} \tag{3.6}$$

Because $f(\cdot)$ is continuous on $[Q(p_0)-\delta, Q(p_1)+\delta]$ and bounded away from zero, it can be concluded from the Mean Value Theorem that, for n large enough

$$\sup_{p_0\leq p\leq p_1}\sup_{|t|\leq 1}|Q(p+h_nt)-Q(p)|\leq Mh_n, \tag{3.7}$$

where M is a positive constant. For n large enough, it can be obtained from Assumption A3, (3.7) and the Mean Value Theorem that

$$\sup_{p_0\leq p\leq p_1}\sup_{|t|\leq 1}|G(Q(p+h_nt))-G(Q(p))|=O(h_n) \tag{3.8}$$

Moreover, with Assumption A2, (1.1) and (3.8), we have,

$$\sup_{p_0\leq p\leq p_1}\sup_{|t|\leq 1}|C(Q(p+h_nt)-C(Q(p))|=O(h_n) \tag{3.9}$$

Now, with (3.3) and (3.9), we have,

$$\sup_{p_0\le p\le p_1}\ \sup_{|t|\le 1} I_2 = O(h_n\sqrt{\log\log n}) \quad a.s. \tag{3.10}$$

To deal with I_1, first note that

$$\begin{aligned} I_1 &\le \frac{n^{-1/2}}{\gamma^{-1}G(Q(p+h_nt))(1-p-h_nt)}\,|k(Q(p+h_nt),n)-k(Q(p),n)| \\ &\quad + \sup_{0\le u<\infty}\frac{|k(u,n)|}{\sqrt{n}}\left|\frac{1}{C(Q(p+h_nt)}-\frac{1}{C(Q(p))}\right| \\ &=: I_3+I_4 \end{aligned}$$

Using the same proof as I_2, we have

$$\sup_{p_0\le p\le p_1}\ \sup_{|t|\le 1} I_4 = O(h_n\sqrt{\log\log n}) \quad a.s. \tag{3.11}$$

Moreover,

$$\begin{aligned} &\sup_{p_0\le p\le p_1}\ \sup_{|t|\le 1}|k(Q(p+h_nt),n)-k(Q(p),n)| \\ &\qquad\le \sup_{Q(p_0)\le x\le Q(p_1)}\ \sup_{0\le y\le Mh_n}|k(x+y,n)-k(x,n)| \end{aligned}$$

Similar to the proof of Theorem 8.2.1 in Csörgő, (1983) and with the use of Lemma 3.4 in Berkes and Philipp (1977), we obtain

$$\sup_{Q(p_0)\le x\le Q(p_1)}\ \sup_{0\le y\le Mh_n}|k(x+y,n)-k(x,n)| = O(n^{1/2}(h_n\log n)^{3/4}) \ \ a.s. \tag{3.12}$$

Therefore,

$$\sup_{p_0\le p\le p_1}\ \sup_{|t|\le 1} I_3 = O\left((h_n\log n)^{3/4}\right) \quad a.s. \tag{3.13}$$

With (3.11) and (3.13) we have

$$\sup_{p_0\le p\le p_1}\ \sup_{|t|\le 1} I_1 = O\left((h_n\log n)^{3/4}\right) \quad a.s. \tag{3.14}$$

Now with (3.6), (3.10) and (3.14) we obtain the result. □

Lemma 3.3. *Under the conditions of Lemma* 3.2*,*

$$\sup_{p_0\le p\le p_1}\ \sup_{|t|\le 1}|\beta_n(p+h_nt)-\beta_n(p)| = O\left((\log n)^{-\lambda}\vee(h_n\log n)^{3/4}\right) \quad a.s.,$$

where $\beta_n(t)=\sqrt{n}\big(\widehat{Q}_n(t)-Q(t)\big)$.

Proof. With the use of triangle inequality, we have

$$
\begin{aligned}
&\sup_{|t|\le 1} |\beta_n(p+h_nt)-\beta_n(p)| \\
&\quad\le \sup_{|t|\le 1}\left|\beta_n(p+h_nt)-\frac{1-p-h_nt}{f(Q(p+h_nt))}B(Q(p+h_nt),n)\right| \\
&\qquad + \left|\beta_n(p)-\frac{1-p}{f(Q(p))}B(Q(p),n)\right| \\
&\qquad + \sup_{|t|\le 1}\left|\frac{1-p-h_nt}{f(Q(p+h_nt))}B(Q(p+h_nt),n)-\frac{1-p}{f(Q(p))}B(Q(p),n)\right| \\
&\quad =: L_1+L_2+L_3,
\end{aligned}
$$

for all $p_0 \le p \le p_1$. Using (3.1), we obtain

$$
L_1 + L_2 = O\left((\log n)^{-\lambda}\right) \quad a.s., \tag{3.15}
$$

uniformly on $p_0 \le p \le p_1$. For L_3, we have for n large enough

$$
\begin{aligned}
L_3 \;\le\; & \sup_{|t|\le 1}\left|\frac{1-p-h_nt}{f(Q(p+h_nt))}-\frac{1-p}{f(Q(p))}\right| |B(Q(p),n)| \\
& + M\sup_{|t|\le 1}|B(Q(p+h_nt),n)-B(Q(p),n)|,
\end{aligned} \tag{3.16}
$$

for all $p_0 \le p \le p_1$, where M is a positive constant. Because $f(t)$ is continuously differentiable on $[Q(p_0)-\delta, Q(p_1)+\delta]$ and bounded away from zero, the first term of (3.16) is not larger than $Ch_n|B(Q(p),n)|$ uniformly on $p_0 \le p \le p_1$, where C is a positive constant. Therefore equations (3.2) and (3.4) imply

$$
\sup_{|t|\le 1}\left|\frac{1-p-h_nt}{f(Q(p+h_nt))}-\frac{1-p}{f(Q(p))}\right| |B(Q(p),n)| = O(h_n\sqrt{\log\log n}) \quad a.s., \tag{3.17}
$$

uniformly on $p_0 \le p \le p_1$. With the use of (3.16), (3.17) and Lemma 3.2, we have

$$
L_3 = O\left((h_n \log n)^{3/4}\right). \tag{3.18}
$$

Therefore, the lemma is proved via (3.15) and (3.18). □

Proof of Theorem 2.1. Considering Assumption **A4,** we have

$$
\begin{aligned}
Q_n(p)-Q(p) \;=\; & n^{-1/2}h_n^{-1}\int_0^1 (\beta_n(t)-\beta_n(p))\, K\left(\frac{t-p}{h_n}\right) dt + n^{-1/2}\beta_n(p) \\
& + \left(h_n^{-1}\int_0^1 Q(t)K\left(\frac{t-p}{h_n}\right) dt - Q(p)\right) \\
=: \;& I_1 + I_2 + I_3.
\end{aligned}
$$

Implementing Lemma 3.3, we have

$$\begin{aligned} |I_1| &\leq n^{-1/2} \sup_{|t|\leq 1} |\beta_n(p + h_n t) - \beta_n(p)| \\ &= O\left(n^{-1/2}(\log n)^{-\lambda} \vee (n^{-1/2}(h_n \log n)^{3/4}\right) \quad a.s., \qquad (3.19) \end{aligned}$$

uniformly on $p_0 \leq p \leq p_1$. Theorem 1 of Lemdani et. al. (2005) implies

$$I_2 = \frac{p - \hat{F}_n(Q(p))}{f(Q(p))} + O(n^{-1/2}(\log n)^{-\lambda}) \quad a.s., \qquad (3.20)$$

where $\lambda > 0$. Using Assumptions **A4**, **A5** and the Taylor expansion of $Q(\cdot)$ about p and Lemma 2.2 in Yong et al. (2006), we get

$$\begin{aligned} I_3 &= \left(\int_{-1}^{1} Q(p + h_n t)K(t)dt - Q(p)\right) \\ &= O(h_n^2), \qquad (3.21) \end{aligned}$$

uniformly on $p_0 \leq p \leq p_1$. The result follows from (3.19)-(3.21).

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