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Residual Lifetimes of k-out-of-n Systems with Exchangeable Components

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Abstract. We consider a $(n-k+1)$-out-of-n system with exchangeable lifetimes of the components. The paper investigates the stochastic ordering properties of the residual lifetime of the system under condition that, at time t, at least $(n-r+1)$ components are alive. Some results are then extended to the case where the system has a coherent structure with exchangeable components.

Keywords. Exchangeable random variables, k-out-of-n system, multivariate life distributions, multivariate stochastic order, multivariate survival function, signatures.

MSC: 62N05, 62G30.

1 Introduction

Consider a technical system having n components which is working if at least $(n-k+1)$ of its n components are operating. The system fails if k or more components fail. In reliability theory such a system is called the $(n-k+1)$-out-of-n system (see Barlow and Proschan, 1975). If the components lifetimes (life lengths) are denoted by $X_1, X_2, ..., X_n$ and the system lifetime denoted by $T \equiv T(X_1, X_2, ..., X_n)$, respectively,

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then the reliability of the system is $P\{T > t\} = P\{X_{k:n} > t\}$, where $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ are the order statistics of random variables $X_1, X_2, ..., X_n$. If $k = 1$ the system is a series system and if $k = n$ the system is a parallel system. In the case of independent and identically distributed (i.i.d.) components lifetimes with common distribution function F, Bairamov et al. (2002) defined the mean residual life (MRL) function of a parallel system under the condition that all components are alive at time t, as $\psi_n(t) = E\{X_{n:n} - t \mid X_{1:n} > t\}$. This function involves joint distributions of order statistics $X_{1:n}$ and $X_{n:n}$ and unlike usual mean residual life function $\Psi_F(t) = E\{X_1 - t \mid X_1 > t\}$ does not characterize the distribution function for a given n. It is shown that the distribution function F can be uniquely determined by $\psi_n(t)$ and $\psi_{n-1}(t)$, for some n. In the case when $X_1, X_2, ..., X_n$ are i.i.d. random variables, Asadi and Bairamov (2005, 2006) studied the monotonicity and ageing properties of the MRL function of a parallel system under the condition that, at time t, at least $(n - r + 1)$ components are alive and the MRL function of $(n - r + 1)$-out-of-n system under the condition that, at time t, all the components are working. In general, the mean residual life function of $(n - k + 1)$-out-of-n system under the condition that, at time t, at least $(n - r + 1)$ components are alive is $\Psi(t) = E(X_{k:n} - t \mid X_{r:n} > t)$, $r \leq k$. For developments and further results for i.i.d. components lifetimes see Khaledi and Shaked (2007), Navarro et al. (2008), and Li and Zhang (2008). Attempts have also been made to investigate the reliability properties of the coherent systems for which the components are not necessarily i.i.d. In this regard, one can refer to Navarro et al. (2005, 2007, 2008, 2013), Gurler and Bairamov (2009), Kochar and Xu (2010), Navarro and Rubio (2010, 2011), Zhang (2010), Belzunce et al. (2011), Sadegh (2008, 2011), Tavangar and Bairamov (2012), Tavangar (2014), Tavangar and Asadi (2015) and Gupta et al. (2015).

This paper investigates the stochastic ordering properties of residual lives and mean residual life function of k-out-of-n system with lifetimes being exchangeable random variables. The results presented in this paper can be used for applications in reliability analysis where the assumption of independence can not be accepted and the exchangeability is a more realistic assumption.

Throughout this paper for any random vector $\boldsymbol{Z}$ and any event A, we denote by $(\boldsymbol{Z} \mid A)$ a random vector having as its distribution the conditional distribution of $\boldsymbol{Z}$ given A. Also we assume that the left end-point of all univariate marginals is zero.

2 Residual Lifetime of *k*-out-of-*n* System

Consider a $(n-k+1)$-out-of-n system consisting of n components. Assume that the components lifetimes $X_1, X_2, \ldots, X_n$ have an arbitrary joint distribution function $F(x_1, x_2, \ldots, x_n)$ with corresponding joint survival function $\bar{F}(x_1, x_2, ..., x_n)$. Let $X_{k:n}, 1 \leq k \leq n$, denote the kth order statistic of the sample $X_1, X_2, \ldots, X_n$. The *residual life* of the system, given that at time $t \geq 0$ at least $(n-r+1)$ components are working, is defined as $(X_{k:n} - t \mid X_{r:n} > t)$. In the following, some representations for the survival function of this conditional random variable are given.

An elementary result in the theory of order statistics shows that the survival function of the rth order statistic can be written as

$$\begin{aligned} P\{X_{r:n} > t\} &= \sum_{i=0}^{r-1} P\{\text{exactly } i \text{ of the } X_i \text{ are less than or equal to } t\} \\ &= \sum_{i=0}^{r-1} \sum_{C_i} \theta^{(C_i)}(t), \end{aligned}$$

where

$$\begin{aligned} \theta^{(C_i)}(t) \quad &= \quad P\{X_{\ell_0} \leq t, X_{\ell_1} \leq t, \ldots, X_{\ell_i} \leq t, X_{\ell_{i+1}} > t, \ldots, X_{\ell_n} > t\}, \\ & \quad\quad (\ell_1, \ell_2, ..., \ell_n) \in C_i, \end{aligned}$$

$X_{\ell_0} = 0$, almost surely, and C_i is the set of all permutations $\{\ell_1, \ell_2, ..., \ell_n\}$ of $\{1, 2, ..., n\}$ for which $1 \leq \ell_1 < \cdots < \ell_i \leq n$ and $1 \leq \ell_{i+1} < \cdots < \ell_n \leq n$. Also one can observe that

$$\begin{aligned} & P\{X_{r:n} > t, X_{k:n} > t + x\} \\ & \quad = \sum_{i=0}^{r-1} \sum_{j=0}^{k-i-1} P\{\text{exactly } i \text{ of the } X_i \text{ are less than or equal to } t, \\ & \quad\quad \text{exactly } j \text{ of the } X_i \text{ are between } t \text{ and } t + x\}. \end{aligned}$$

If, for $(\ell_1, \ell_2, ..., \ell_n) \in C_i$, we denote the event $[X_{\ell_0} \leq t, X_{\ell_1} \leq t, ..., X_{\ell_i} \leq t, X_{\ell_{i+1}} > t, ..., X_{\ell_n} > t]$ by $A^{(t, C_i)}, i = 0, 1, \ldots, r-1$, then the right-hand side of the last expression can be written as

$$\sum_{i=0}^{r-1}\sum_{j=0}^{k-i-1}\sum_{C_i}\sum_{C_{i(j)}} P\{X_{\ell_0}\le t, X_{\ell_1}\le t,\ldots,X_{\ell_i}\le t, t<X_{s_1}$$
$$\le t+x,\ldots,t<X_{s_j}\le t+x, X_{s_{j+1}}>t+x,\ldots,X_{s_{n-i}}>t+x\}$$
$$=\sum_{i=0}^{r-1}\sum_{j=0}^{k-i-1}\sum_{C_i}\sum_{C_{i(j)}} \theta^{(C_i)}(t)P\{X_{s_0}-t\le x, X_{s_1}-t\le x,\ldots,$$
$$X_{s_j}-t\le x,\ldots,X_{s_{j+1}}-t>x, X_{s_{n-i}}-t>x \mid A^{(t,C_i)}\},$$

where $X_{s_0}=t$, almost surely, the summation on C_i has been interpreted as before, and $C_{i(j)}$ is the set of all permutations $\{s_1,s_2,...,s_{n-i}\}$ of $\{\ell_{i+1},...,\ell_n\}$ for which $1\le s_1<\cdots<s_j\le n$ and $1\le s_{j+1}<\cdots<s_{n-i}\le n$. It should be mentioned here that a similar expression is also considered in Zhang (2010). Now, let us define the vector $(X^{(t,C_i)}_{\ell_{i+1},n},...,X^{(t,C_i)}_{\ell_n,n})$ as any random vector which has the same distribution of $(X_{\ell_{i+1}}-t,...,X_{\ell_n}-t\mid A^{(t,C_i)})$. Then we can write

$$P\{X_{r:n}>t, X_{k:n}>t+x\}$$
$$=\sum_{i=0}^{r-1}\sum_{C_i}\theta^{(C_i)}(t)\sum_{j=0}^{k-i-1}\sum_{C_{i(j)}} P\{X^{(t,C_i)}_{s_0,n}\le x, X^{(t,C_i)}_{s_1,n}\le x,\ldots,X^{(t,C_i)}_{s_j,n}\le x,$$
$$X^{(t,C_i)}_{s_{j+1},n}>x,\ldots,X^{(t,C_i)}_{s_{n-i},n}>x\}$$
$$=\sum_{i=0}^{r-1}\sum_{C_i}\theta^{(C_i)}(t)$$
$$\sum_{j=0}^{k-i-1} P\{\text{exactly } j \text{ of the } X^{(t,C_i)}_{i,n} \text{ are less than or equal to } x\}$$
$$=\sum_{i=0}^{r-1}\sum_{C_i}\theta^{(C_i)}(t)P\{X^{(t,C_i)}_{k-i:n-i}>x\}$$
$$=\sum_{i=0}^{r-1}\sum_{C_i}\theta^{(C_i)}(t)\bar{F}^{(t,C_i)}_{k-i,n-i}(x),$$

where $X^{(t,C_i)}_{s_0,n}=0$, almost surely, and $X^{(t,C_i)}_{k-i:n-i}$ is the $(k-i)$th order statistic of the conditional random vector $(X_{\ell_{i+1}}-t,...,X_{\ell_n}-t\mid A^{(t,C_i)})$ with survival function $\bar{F}^{(t,C_i)}_{k-i,n-i}(x)$. This implies that the survival function $\bar{F}_{r,k,n,t}(x)$ of $(X_{k:n} - t \mid X_{r:n} > t)$ can be represented as

$$\bar{F}_{r,k,n,t}(x) = \frac{\sum_{i=0}^{r-1} \sum_{C_i} \theta^{(C_i)}(t) \bar{F}^{(t,C_i)}_{k-i,n-i}(x)}{\sum_{i=0}^{r-1} \sum_{C_i} \theta^{(C_i)}(t)}, \tag{1}$$

which is a mixture representation of survival functions of the $X^{(t,C_i)}_{k-i:n-i}$, $i = 0, 1, \ldots, r-1$.

Remark 2.1. Let the components lifetimes be i.i.d. with the common distribution function $F(x)$ and survival function $\bar{F}(x)$. Then, it is easy to show that $\theta^{(C_i)}(t) = \{\varphi(t)\}^i \bar{F}^n(t)$, where $\varphi(t) = F(t)/\bar{F}(t)$, and $\bar{F}^{(t,C_i)}_{k-i,n-i}(x) = P\{X^{(t)}_{k-i:n-i} > x\}$, where $X^{(t)}_{k-i:n-i}$ is the $(k-i)^{th}$ order statistic from the i.i.d. sample $X_1^{(t)}, X_2^{(t)}, ..., X_n^{(t)}$ with common survival function $\bar{H}_t(x) = \bar{F}(t+x)/\bar{F}(t)$, $x \geq 0$. Now representation (1) can be written as

$$\bar{F}_{r,k,n,t}(x) = \frac{\sum_{i=0}^{r-1} \binom{n}{i} \{\varphi(t)\}^i P\{X^{(t)}_{k-i:n-i} > x\}}{\sum_{i=0}^{r-1} \binom{n}{i} \{\varphi(t)\}^i},$$

which reduces to the expression in Lemma 2.2 of Li and Zhao (2006).

Order statistics have been studied quit extensively in the case where observations are i.i.d. Bairamov and Parsi (2011) considering the distributions of order statistics from mixed exchangeable random variables as a special case obtained the joint distribution of two nonadjacent order statistics from exchangeable random variables. It also follows that

$$P\{X_{k:n} \leq x\} = \sum_{j=k}^{n} (-1)^{j-k} \binom{j-1}{k-1} \sum_{C_j} P\{\max(X_{\ell_1}, ..., X_{\ell_j}) \leq x\},$$

where the summation extends over all permutation $\ell_1, \ell_2, ...\ell_n$ of $1, 2, ..., n$ for which $1 \leq \ell_1 < \cdots < \ell_j \leq n$ and $1 \leq \ell_{j+1} < \cdots < \ell_n \leq n$; see David and Nagaraja (2003, p. 46). Therefore, we have

$$\bar{F}^{(t,C_i)}_{k-i,n-i}(x) = 1 - \sum_{j=k-i}^{n-i} (-1)^{i+j-k} \binom{j-1}{k-i-1} F^{(t,C_i)}_{j:j}(x),$$

where

$$F^{(t,C_i)}_{j:j}(x) = \sum_{C_{i(j)}} P\{\max(X^{(t,C_i)}_{s_1,n}, ..., X^{(t,C_i)}_{s_j,n}) \leq x\},$$

and the summation can be interpreted as before. Thus we get

$$\bar{F}_{r,k,n,t}(x) = 1 - \frac{\sum_{i=0}^{r-1}\sum_{C_i}\theta^{(C_i)}(t)\sum_{j=k-i}^{n-i}(-1)^{i+j-k}\binom{j-1}{k-i-1}F_{j:j}^{(t,C_i)}(x)}{\sum_{i=0}^{r-1}\sum_{C_i}\theta^{(C_i)}(t)},$$

which is an expression for the survival function of $(X_{k:n} - t \mid X_{r:n} > t)$ in terms of the distribution functions of lifetimes of parallel structures.

In a special case when $(X_1, X_2, ..., X_n)$ is a random vector with exchangeable distribution, the representation will be more simpler. If we denote the probability $P\{X_0 \leq t, X_1 \leq t, ..., X_i \leq t, X_{i+1} > t, ..., X_n > t\}$ by $F_{i,n}(t)$, then

$$\sum_{C_i}\theta^{(C_i)}(t) = \binom{n}{i}F_{i,n}(t)$$

and

$$\sum_{C_i}F_{j:j}^{(t,C_i)}(x) = \binom{n}{i}\sum_{N_{i(j)}}P\{\max(X_{s_1,n}^{(t,N_i)}, ..., X_{s_j,n}^{(t,N_i)}) \leq x\},$$

where $N_i = \{1, 2, ..., i\}$ and $N_{i(j)}$ is the set of all permutations $\{s_1, s_2, ..., s_{n-i}\}$ of $\{i+1, ..., n\}$ for which $i + 1 \leq s_1 < \cdots < s_j \leq n$, and $i + 1 \leq s_{j+1} < \cdots < s_n \leq n$. Here we assume that $X_0 = 0$, almost surely. It is easy to see that $(X_{i+1,n}^{(t,N_i)}, ..., X_{n,n}^{(t,N_i)})$ is also a random vector with exchangeable distribution. This implies that

$$(X_{s_1,n}^{(t,N_i)}, ..., X_{s_j,n}^{(t,N_i)}) \stackrel{d}{=} (X_{i+1,n}^{(t,N_i)}, ..., X_{i+j,n}^{(t,N_i)}).$$

Therefore, we obtain the representation

$$\bar{F}_{r,k,n,t}(x) = 1 - \frac{\sum_{i=0}^{r-1}\binom{n}{i}F_{i,n}(t)\sum_{j=k-i}^{n-i}(-1)^{i+j-k}\binom{n-i}{j}\binom{j-1}{k-i-1}F_{i,j,n}^{(t)}(x)}{\sum_{i=0}^{r-1}\binom{n}{i}F_{i,n}(t)},$$

where $F_{i,n}(t) = P\{A_i^{(t)}\}$, $F_{i,j,n}^{(t)}(x) = P\{X_{i+1} \leq t + x, ..., X_{i+j} \leq t + x \mid A_i^{(t)}\}$, and $A_i^{(t)}$ denotes the event $[X_0 \leq t, X_1 \leq t, ..., X_i \leq t, X_{i+1} > t, ..., X_n > t]$, $i = 0, 1, ..., r-1$.

Remark 2.2. We may wish to rewrite a representation for $\bar{F}_{r,k,n,t}(x)$ in terms of the survival functions of series structures. The survival function of the kth order statistic from an arbitrary sample $X_1, X_2, ..., X_n$ is given by

$$\bar{F}_{k:n}(x) = \sum_{j=n-k+1}^{n}(-1)^{n-k+1-j}\binom{j-1}{n-k}\sum_{C_j}P\{\min(X_{\ell_1}, ..., X_{\ell_j}) > x\},$$

(see, for example, David and Nagaraja, 2003, p. 46). Therefore, we obtain

$$\bar{F}^{(t,C_i)}_{k-i:n-i}(x) = \sum_{j=n-k+1}^{n-i} (-1)^{n-k+1-j} \binom{j-1}{n-k} \bar{F}^{(t,C_i)}_{1:j}(x),$$

where

$$\bar{F}^{(t,C_i)}_{1:j}(x) = \sum_{C_j} P\{\min(X_{\ell_1}, ..., X_{\ell_j}) > x\}.$$

Substituting this in representation (1) we get

$$\bar{F}_{r,k,n,t}(x) = \frac{\sum_{i=0}^{r-1} \sum_{C_i} \theta^{(C_i)}(t) \sum_{j=n-k+i}^{n-i} (-1)^{n-k+1-j} \binom{j-1}{n-k} \bar{F}^{(t,C_i)}_{1:j}(x)}{\sum_{i=0}^{r-1} \sum_{C_i} \theta^{(C_i)}(t)}.$$

In the case where $(X_1, X_2, ..., X_n)$ is a random vector with exchangeable distribution, we have

$$\bar{F}_{r,k,n,t}(x) = \frac{\sum_{i=0}^{r-1} \binom{n}{i} F_{i,n}(t) \sum_{j=n-k+i}^{n-i} (-1)^{n-k+1-j} \binom{j-1}{n-k} \binom{n-i}{j} \bar{F}^{(t)}_{i,j,n}(x)}{\sum_{i=0}^{r-1} \binom{n}{i} F_{i,n}(t)},$$

where

$$\bar{F}^{(t)}_{i,j,n}(x) = P\{X_{i+1} > t + x, ..., X_{i+j} > t + x \mid A^{(t)}_i\}.$$

For practical calculations, we may wish to obtain an explicit expression for $\bar{F}_{r,k,n,t}(x)$ in terms of the joint survival function $\bar{F}(x_1, x_2, ..., x_n)$. The following lemma serves for this purpose.

Lemma 2.1. *Let $(X_1, X_2, ..., X_n)$ be a random vector with exchangeable joint survival function $\bar{F}(x_1, x_2, ..., x_n)$. Then*
(a) for $i = 0, 1, ..., n-1$,

$$\begin{aligned} F_{i,n}(t) &= P\{X_0 \le t, X_1 \le t, ..., X_i \le t, X_{i+1} > t, ..., X_n > t\} \\ &= \sum_{j=0}^{i} (-1)^j \binom{i}{j} \bar{F}(\underbrace{t, t, ..., t}_{n-i+j}, \underbrace{0, 0, ..., 0}_{i-j}); \end{aligned}$$

(b) for $i = 0, 1, ..., n-1, \quad j = 1, 2, ..., n, \quad i + j \le n$

$$\begin{aligned} P\{&X_0 \le t, X_1 \le t, ..., X_i \le t, X_{i+1} > t, ..., X_{i+j} > t, \\ &X_{i+j+1} > t + x, ..., X_n > t + x\} \\ &= \sum_{\ell'=0}^{i} (-1)^{\ell'} \binom{i}{\ell'} \bar{F}(\underbrace{t, t, ..., t}_{j+\ell'}, \underbrace{t + x, ..., t + x}_{n-i+j}, \underbrace{0, 0, ..., 0}_{i-\ell'}), \end{aligned}$$

where $X_0 = 0$, almost surely.

Proof. Denoting the event $[X_{i+1} > t, ..., X_n > t]$ by A, and the event $[X_\ell > t]$ by $B_\ell,\ \ell = 1, ..., i$, we have

$$F_{i,n}(t) = P\{A \backslash \bigcup_{\ell=0}^{i} B_\ell\}.$$

It is easily shown that

$$P\left\{A \backslash \bigcup_{\ell=0}^{i} B_\ell\right\} = P\{A\} - P\left\{\bigcup_{\ell=0}^{i} C_\ell\right\}, \tag{2}$$

where $C_\ell = A \cap B_\ell,\ \ell = 1, 2, ..., i.$ By the inclusion-exclusion principle (see Feller, 1968, pp. 98-101), we have

$$\begin{aligned} P\left\{\bigcup_{\ell=0}^{i} C_\ell\right\} &= \sum_{j=1}^{i} (-1)^{j-1} \sum_{1 \le \ell_1 < \cdots < \ell_j \le i} P\{C_{\ell_1} \cap C_{\ell_2} \cap \ldots \cap C_{\ell_j}\} \\ &= \sum_{j=1}^{i} (-1)^{j-1} \binom{i}{j} \bar{F}(\underbrace{t, t, ..., t}_{n-i+j}, \underbrace{0, 0, ..., 0}_{i-j}), \end{aligned}$$

where the second equality follows from the fact that $(X_1, X_2, ..., X_n)$ has exchangeable distribution. Substituting the last expression in (2), we get the required result in part (a).

A similar argument can be used to prove part (b). ■

Theorem 2.1 *Let $(X_1, X_2, ..., X_n)$ be a random vector with exchangeable joint survival function $\bar{F}(x_1, x_2, ..., x_n)$. Then*

$$\bar{F}_{r,k,n,t}(x) = \frac{K_1(t, x)}{K_2(t)}, \tag{3}$$

where

$$\begin{aligned} K_1(t, x) \ &= \ \sum_{i=0}^{r-1} \sum_{\ell'=0}^{i} (-1)^{\ell'} \binom{n}{i} \binom{i}{\ell'} \sum_{j=0}^{k-i-1} \sum_{\ell=0}^{j} (-1)^{\ell} \binom{n-i}{j} \binom{j}{\ell} \\ &\quad \times \bar{F}(\underbrace{t, t, ..., t}_{j-\ell+\ell'}, \underbrace{t+x, ..., t+x}_{n-i-j+\ell}, \underbrace{0, 0, ..., 0}_{i-\ell'}), \end{aligned}$$

and

$$K_2(t) = \sum_{i=0}^{r-1} \sum_{\ell'=0}^{i} (-1)^{\ell'} \binom{n}{i} \binom{i}{\ell'} \bar{F}(\underbrace{t, t, ..., t}_{n-i+\ell'}, \underbrace{0, 0, ..., 0}_{i-\ell'}).$$

Proof. The probability that the random point $(\xi_1, \xi_2, ..., \xi_n)$ falls into parallelepiped $a_i < \xi_i \le b_i,\ i = 1, 2, ..., n$, where a_i and b_i are arbitrary constants, is (see Gnedenko, 1978, p. 135)

$$\begin{aligned} &P\{a_1 < \xi_1 \le b_1, a_2 < \xi_2 \le b_2, ..., a_n < \xi_n \le b_n\} \\ &= P\{\xi_1 > a_1, \xi_2 > a_2, ..., \xi_n > a_n\} \\ &- \sum_{i=1}^{n} p_i + \sum_{i<j} p_{ij} - \cdots + (-1)^n P\{\xi_1 > b_1, \xi_2 > b_2, ..., \xi_n > b_n\} \end{aligned}$$

where $p_{ij...k}$ denotes the probability $P\{\xi_1 > c_1, \xi_2 > c_2, ..., \xi_n > c_n\}$ for $c_i = b_i, c_j = b_j, ..., c_k = b_k$ and for the other indices $c_s = a_s$. One can modify this expression as

$$\begin{aligned} &P\{a_1 < \xi_1 \le b_1, a_2 < \xi_2 \le b_2, ..., a_n < \xi_n \le b_n, A\} \\ &= P\{\xi_1 > a_1, \xi_2 > a_2, ..., \xi_n > a_n, A\} \\ &- \sum_{i=1}^{n} p'_i + \sum_{i<j} p'_{ij} - \cdots + (-1)^n P\{\xi_1 > b_1, \xi_2 > b_2, ..., \xi_n > b_n, A\}, \end{aligned}$$

for any event A, where $p'_{ij...k}$ denotes the probability $P\{\xi_1 > c_1, \xi_2 > c_2, ..., \xi_n > c_n, A\}$ for $c_i = b_i, c_j = b_j, ..., c_k = b_k$ and for the other indices $c_s = a_s$.

Let $A = [X_0 \le t, X_1 \le t, ..., X_i \le t, X_{i+j+1} > t + x, ..., X_n > t + x]$. Then, using Lemma 2.1, we have

$$\begin{aligned} &P\{X_0 \le t, X_1 \le t, ..., X_i \le t, t < X_{i+1} \le t + x, ..., t < X_{i+j} \le t + x, \\ &\qquad X_{i+j+1} > t + x, ..., X_n > t + x\} \\ &= P\{X_0 \le t, X_1 \le t, ..., X_i \le t, X_{i+1} > t, ..., X_{i+j} > t, \\ &\qquad X_{i+j+1} > t + x, ..., X_n > t + x\} \\ &+ \sum_{\ell=1}^{j-1} (-1)^\ell \binom{j}{\ell} P\{X_0 \le t, X_1 \le t, ..., X_i \le t, X_{i+1} > t, ..., X_{i+j-\ell} > t, \\ &\qquad X_{i+j-\ell+1} > t + x, ..., X_n > t + x\} \\ &+ (-1)^j P\{X_0 \le t, X_1 \le t, ..., X_i \le t, X_{i+1} > t + x, ..., X_{i+j} \\ &\qquad > t + x, ..., X_n > t + x\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell'=0}^{i}(-1)^{\ell'}\binom{i}{\ell'}\bar{F}(\underbrace{t,t,...,t}_{j+\ell'},\underbrace{t+x,...,t+x}_{n-i-j},\underbrace{0,0,...,0}_{i-\ell'}) \\
&+ \sum_{\ell=1}^{j-1}(-1)^{\ell}\binom{i}{\ell}\sum_{\ell'=0}^{i}(-1)^{\ell'}\binom{i}{\ell'}\bar{F}(\underbrace{t,t,...,t}_{j-\ell+\ell'},\underbrace{t+x,...,t+x}_{n+\ell-i-j},\underbrace{0,0,...,0}_{i-\ell'}) \\
&+ (-1)^{j}\sum_{\ell'=0}^{i}(-1)^{\ell'}\binom{i}{\ell'}\bar{F}(\underbrace{t,t,...,t}_{\ell'},\underbrace{t+x,...,t+x}_{n-i},\underbrace{0,0,...,0}_{i-\ell'}) \\
&= \sum_{\ell=0}^{j}(-1)^{\ell}\binom{j}{\ell}\sum_{\ell'=0}^{i}(-1)^{\ell'}\binom{i}{\ell'}\bar{F}(\underbrace{t,t,...,t}_{j-\ell+\ell'},\underbrace{t+x,...,t+x}_{n-i-j+\ell},\underbrace{0,0,...,0}_{i-\ell'}).
\end{aligned}
$$

Substituting this to expression (1) and then applying Lemma 2.1 again, the required result follows. ■

3 Some Examples

In this section we study the behavior of the mean residual life (MRL) function of k-out-of-n systems consisting of n components with exchangeable lifetimes. The MRL function of $(n-k+1)$-out-of-n system is defined as the expectation of the residual lifetime of the system described in Section 2. If we denote this conditional expectation by $M_{r,k,n}(t)$, then

$$M_{r,k,n}(t) = E(X_{k:n} - t \mid X_{r:n} > t), \quad t \geq 0.$$

Note that by Theorem 2.1,

$$
\begin{aligned}
M_{r,k,n}(t) &= \int_0^\infty \bar{F}_{r,k,n,t}(x)dx \\
&= \frac{1}{K_2(t)}\int_0^\infty K_1(t,x)dx. \qquad (4)
\end{aligned}
$$

Example 3.1. For a special version of Marshall and Olkin's multivariate exponential distribution with survival function

$$
\begin{aligned}
&F(x_1,x_2,...,x_n) = \exp\left\{-\lambda\sum_{i=1}^{n}x_i - \lambda^*\max(x_1,x_2,...,x_n)\right\}, \\
&\quad x_i > 0, \quad \lambda > 0, \lambda^* \geq 0,
\end{aligned}
$$

(see Kotz et al., 2000), one can easily see that

$$\bar{F}(\underbrace{t,t,...,t}_{n-i+\ell'},\underbrace{0,0,...,0}_{i-\ell'}) = \exp\{-[(n-i+\ell')\lambda+\lambda^*]t\},$$

and

$$\int_0^\infty \bar{F}(\underbrace{t,t,...,t}_{j-\ell+\ell'},\underbrace{t+x,...,t+x}_{n-i-j+\ell},\underbrace{0,0,...,0}_{i-\ell'})dx = \frac{\exp\{-[(n-i+\ell')\lambda+\lambda^*]t\}}{(n-i-j+\ell)\lambda+\lambda^*}.$$

From (4), we obtain

$$M_{r,k,n}(t) = \frac{\sum_{i=0}^{r-1}\sum_{\ell'=0}^{i}(-1)^{\ell'}\binom{n}{i}\binom{i}{\ell'}\left\{\sum_{j=0}^{k-i-1}\sum_{\ell=0}^{j}\frac{(-1)^{\ell}\binom{n-i}{j}\binom{j}{\ell}}{\lambda(n-i-j+\ell)+\lambda^*}\right\}e^{-[(n-i+\ell')\lambda+\lambda^*]t}}{\sum_{i=0}^{r-1}\sum_{\ell'=0}^{i}(-1)^{\ell'}\binom{n}{i}\binom{i}{\ell'}e^{-(\lambda(n-i+\ell')+\lambda^*)t}}.$$

Figure 1 shows the graphs of MRL $M_{r,k,n}(t)$ of Marshall and Olkin's multivariate exponential distribution with parameters $\lambda = 1$ and $\lambda^* = 5$, for the case where $n = 6$ and $k = 5$.

Example 3.2. Mardia's multivariate Pareto distribution of the first kind with equal parameters has the joint survival function,

$$\bar{F}(x_1,x_2,...,x_n) = \left(\theta^{-1}\sum_{i=1}^{n}x_i - n + 1\right)^{-a}, \quad x_i > \theta > 0, \quad a > 1;$$

see Kotz et al. (2000). We have, for $t > \theta$,

$$\int_0^\infty \bar{F}(\underbrace{t,t,...,t}_{j-\ell+\ell'},\underbrace{t+x,...,t+x}_{n-i-j+\ell},\underbrace{\theta,\theta,...,\theta}_{i-\ell'})dx$$
$$= \int_0^\infty \left\{\theta^{-1}(n-i+j+\ell)x + \theta^{-1}(n-i+\ell')t - (n-i+\ell') + 1\right\}^{-a} dx$$
$$= \frac{\theta}{(a-1)(n-i-j+\ell)}\left\{\theta^{-1}(n-i+\ell')t - (n-i+\ell') + 1\right\}^{1-a}.$$

Since $n - i + j + \ell \geq 1$, one can conclude that

$$M_{r,k,n}(t) = \frac{\theta}{a-1}\frac{\sum_{i=0}^{r-1}\sum_{\ell'=0}^{i}(-1)^{\ell'}\binom{n}{i}\binom{i}{\ell'}\left\{\theta^{-1}(n-i+\ell')t-(n-i+\ell')+1\right\}^{1-a}c_{i,j,k,\ell}}{\sum_{i=0}^{r-1}\sum_{\ell'=0}^{i}(-1)^{\ell'}\binom{n}{i}\binom{i}{\ell'}\left\{\theta^{-1}(n-i+\ell')t-(n-i+\ell')+1\right\}^{a}},$$

where

$$c_{i,j,k,\ell} = \sum_{j=0}^{k-i-1}\sum_{\ell=0}^{j}\frac{(-1)^{\ell}\binom{n-i}{j}\binom{j}{\ell}}{n-i-j+\ell}.$$

Figure 2 shows the graphs of MRL $M_{r,k,n}(t)$ of Mardia's multivariate Pareto distribution with parameters $a = 6$ and $\theta = 1$, for the case where $n = 6$ and $k = 5$.

Example 3.3. Warmuth (1988) extended the well-known Fréchet bivariate bounds for n-dimensional distributions (see also Kotz et al., 2000, p. 45). Based on n univariate distributions $F_1(x_1), F_2(x_2), ..., F_n(x_n)$, the upper bound is $\min\{F_1(x_1), F_2(x_2), ..., F_n(x_n)\}$, a n-dimensional distribution function. In a similar way, one can obtain the n-dimensional survival function $\min\{\bar{F}_1(x_1), \bar{F}_2(x_2), ..., \bar{F}_n(x_n)\}$ as an upper bound for any joint survival function with marginal survival functions $\bar{F}_1(x_1), \bar{F}_2(x_2), ..., \bar{F}_n(x_n)$. Consider the exchangeable survival function

$$\bar{F}(x_1, x_2, ..., x_n) = \min\{\bar{F}(x_1), \bar{F}(x_2), ..., \bar{F}(x_n)\},$$

where $\bar{F}(x)$ is a lifetime survival function. We have

$$\int_0^\infty \bar{F}(\underbrace{t, t, ..., t}_{j-\ell+\ell'}, \underbrace{t+x, ..., t+x}_{n-i-j+\ell}, \underbrace{0, 0, ..., 0}_{i-\ell'})dx = \int_0^\infty \bar{F}(t+x)dx = m(t)\bar{F}(t),$$

(since $n - i - j + \ell \geq 1$), where $m(t) = E(X_1 - t \mid X_1 > t)$ is the MRL function corresponding to $\bar{F}(x)$. From (4), we obtain $M_{r,k,n}(t) = m(t)$.

4 Stochastic Comparisons among Residual Lives of k-out-of-n Systems

Before giving the main results of this section, we first recall some aging concepts and stochastic orders that are pertinent to the developments of the paper.

Definition 4.1. Let X and Y be two random variables with distribution functions F and G and survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively. Then X is said to be less than Y in stochastic order (denoted by $X \leq_{st} Y$) if $\bar{G}(x) \geq \bar{F}(x)$.

For a comprehensive discussion on other univariate stochastic orders, we refer the reader to Shaked and Shanthikumar (2007) and Müller and Stoyan (2002).

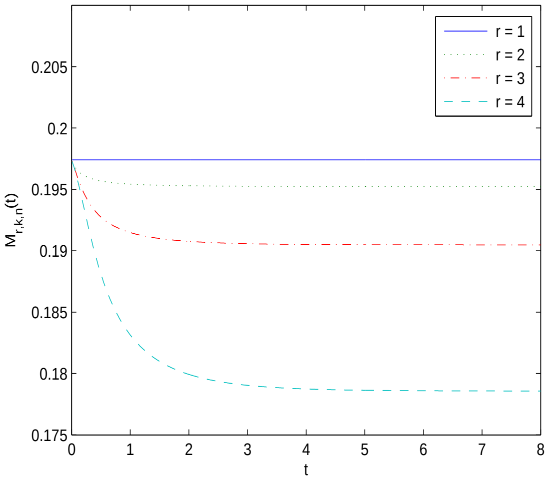

Figure 1: The MRL of a 2-out-of-6 system with Marshall and Olkin's multivariate exponential components lifetimes

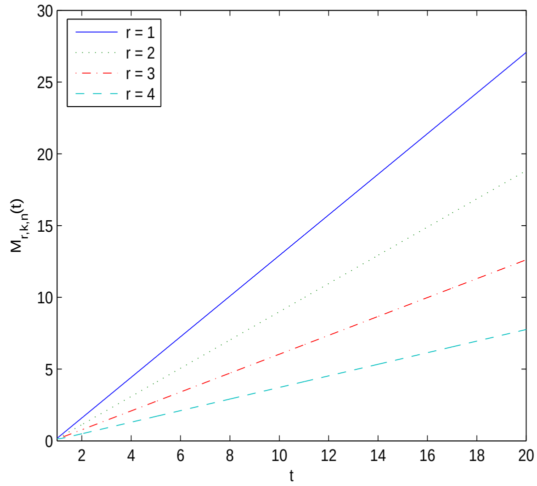

Figure 2: The MRL of a 2-out-of-6 system with Mardia's multivariate Pareto components lifetimes

Definition 4.2. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors with joint density functions f and g, respectively. Then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the multivariate likelihood ratio order (denoted by $\boldsymbol{X} \leq_{\text{lr}} \boldsymbol{Y}$) if ($\wedge$ and $\vee$ denote, respectively, the minimum and the maximum operations)

$$\begin{aligned} &f(x_1, x_2, ..., x_n)g(y_1, y_2, ..., y_n) \\ &\quad \leq f(x_1 \wedge y_1, x_2 \wedge y_2, ..., x_n \wedge y_n)g(x_1 \vee y_1, x_2 \vee y_2, ..., x_n \vee y_n), \end{aligned}$$

for all $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$ (see Shaked and Shanthikumar (2007) and references therein).

The following theorem from Shaked and Shanthikumar (2007) will be essential for our derivations.

Theorem 4.1. *Let $\{X_1, X_2, \ldots\}$ and $\{Y_1, Y_2, \ldots\}$ be two sequences of (possibly dependent) random variables such that $(X_1, X_2, \ldots, X_k) \leq_{st} (Y_1, Y_2, \ldots, Y_k)$, $k \geq 1$. Then $X_{i:m} \leq_{st} Y_{j:n}$, whenever $i \leq j$ and $m - i \geq n - j$.*

Corollary 4.1. *Let $\{X_1, X_2, \ldots\}$ be a sequence of (not necessarily independent) random variables. Then $X_{i:m} \leq_{st} X_{j:n}$, whenever $i \leq j$ and $m - i \geq n - j$.*

The next result reveals that, for any fixed r, n and t, the residual lifetime $(X_{k:n} - t \mid X_{r:n} > t)$ is stochastically increasing in k. The proof is trivial from the definition of order statistics.

Theorem 4.2. *For any integers r, k and n such that $1 \leq r < k < n$,*

$$(X_{k:n} - t \mid X_{r:n} > t) \leq_{st} (X_{k+1:n} - t \mid X_{r:n} > t).$$

It seems that more results are possible only under simplifying the assumptions such as exchangeability of the components lifetimes. Hence we suppose that $\boldsymbol{X} = (X_1, X_2, ..., X_n)$ be an exchangeable random vector. Then the survival function of $(X_{k:n} - t \mid X_{r:n} > t)$ in (1) can be expressed as

$$\bar{F}_{r,k,n,t}(x) = \frac{\sum_{i=0}^{r-1} \binom{n}{i} P\{A_i^{(t)}\} \bar{F}_{k-i,n-i}^{(i,t)}(x)}{\sum_{i=0}^{r-1} \binom{n}{i} P\{A_i^{(t)}\}}, \tag{5}$$

where $A_i^{(t)} = [X_0 \leq t, X_1 \leq t, ..., X_i \leq t, X_{i+1} > t, ..., X_n > t]$, $\bar{F}^{(i,t)}_{k-i,n-i}(x)$ $= P\{X^{(i,t)}_{k-i:n-i} > x\}$ and $X^{(i,t)}_{k-i:n-i}$ is the $(k-i)$th order statistic corresponding to the conditional random vector $(X_{i+1} - t, ..., X_n - t \mid A_i^{(t)})$.

The following theorem proves that when $\boldsymbol{X}$ satisfies the multivariate totally positive of order 2 (MTP$_2$) property (i.e. if $\boldsymbol{X} \leq_{lr} \boldsymbol{X}$), $(X_{k:n} - t \mid X_{r:n} > t)$ is decreasing in r in the sense of stochastic order. We refer the reader to Karlin (1968) and Karlin and Rinott (1980) for some properties of MTP$_2$ random vectors.

Theorem 4.3. *Let $(X_1, X_2, ..., X_n)$ be an absolutely continuous and exchangeable random vector which satisfies the MTP$_2$ property. Then for any integers r and k with $1 \leq r-1 < k \leq n$,*

$$(X_{k:n} - t \mid X_{r:n} > t) \leq_{st} (X_{k:n} - t \mid X_{r-1:n} > t),$$

for all $t \geq 0$.

Proof. Using (5), it can be shown, after some simplifications, that $\bar{F}_{r,k,n,t}(x) - \bar{F}_{r+1,k,n,t}(x)$ has the same sign as

$$\sum_{i=0}^{r-1} \binom{n}{i} P\{A_i^t\}\{\bar{F}^{(i,t)}_{k-i,n-i}(x) - \bar{F}^{(r,t)}_{k-r,n-r}(x)\}.$$

Let $\boldsymbol{x} = (x_1, x_2, ..., x_n)$, $A = \{\boldsymbol{x} : x_1 \leq t, ..., x_r \leq t, x_{r+1} > t, ..., x_n > t\}$ and $B = \{\boldsymbol{x} : x_0 \leq t, x_1 \leq t, ..., x_i \leq t, x_{i+1} > t, ..., x_n > t\}$, where $i = 0, 1, ..., r-1$. Then we have $A \vee B = \{\boldsymbol{x} \vee \boldsymbol{y} : \boldsymbol{x} \in A, \boldsymbol{y} \in B\} = B$ and $A \wedge B = \{\boldsymbol{x} \wedge \boldsymbol{y} : \boldsymbol{x} \in A, \boldsymbol{y} \in B\} = A$, where $\wedge$ and $\vee$ denote, respectively, the minimum and the maximum operators. Since $\boldsymbol{X}$ satisfies the MTP$_2$ property, from Theorem 6.E.2 in Shaked and Shanthikumar (2007), we have $(\boldsymbol{X} \mid \boldsymbol{X} \in A) \leq_{lr} (\boldsymbol{X} \mid \boldsymbol{X} \in B)$. But since the multivariate likelihood ratio order is closed under marginalization (see Theorem 6.E.4 in Shaked and Shanthikumar (2007)),

$$(X_n - t, ..., X_{i+1} - t \mid \boldsymbol{X} \in A) \leq_{lr} (X_n - t, ..., X_{i+1} - t \mid \boldsymbol{X} \in B).$$

Using Theorem 6.E.8 in Shaked and Shanthikumar (2007), we obtain

$$(X_n - t, ..., X_{i+1} - t \mid \boldsymbol{X} \in A) \leq_{st} (X_n - t, ..., X_{i+1} - t \mid \boldsymbol{X} \in B).$$

Now since $i < r$, from Theorem 4.1, we have $\bar{F}^{(i,t)}_{k-i,n-i}(x) - \bar{F}^{(r,t)}_{k-r,n-r}(x) \geq 0$, and hence $\bar{F}_{r,k,n,t}(x) \geq \bar{F}_{r+1,k,n,t}(x)$ for $x \geq 0$. This completes the proof of the theorem. ■

The next example shows that if we remove the MTP$_2$ assumption in Theorem 4.3, then the conclusion of the theorem does not remain valid.

Example 4.1. Consider the bivariate exponential distribution of Gumbel (1960) with joint reliability function

$$\bar{F}(x_1, x_2) = \exp\left\{-\frac{x_1}{\theta_1} - \frac{x_2}{\theta_2} - \frac{\alpha x_1 x_2}{\theta_1\theta_2}\right\}, x_1, x_2 \geq 0, \theta_1, \theta_2 > 0, \alpha \in (0,1),$$

see also Kotz et al. (2000) and Castillo et al. (1997). It is known that the joint density function of the Gumbel's bivariate exponential distribution does not satisfy the MTP_2 property. On the other hand, we have

$$P\{X_{2:2} - t > x \mid X_{1:2} > t\} = \frac{(e^{-\frac{x}{\theta_1}} + e^{-\frac{x}{\theta_2}})e^{-\frac{\alpha t(t+x)}{\theta_1\theta_2}} - \exp\left\{-(\frac{1}{\theta_1} + \frac{1}{\theta_2})x - \frac{\alpha(t+x)^2}{\theta_1\theta_2}\right\}}{e^{-\frac{\alpha t^2}{\theta_1\theta_2}}},$$

and

$$P\{X_{2:2} - t > x \mid X_{2:2} > t\} = \frac{e^{-\frac{t+x}{\theta_1}} + e^{-\frac{t+x}{\theta_2}} - \exp\left\{-(\frac{1}{\theta_1} + \frac{1}{\theta_2})(t+x) - \frac{\alpha(t+x)^2}{\theta_1\theta_2}\right\}}{e^{-\frac{t}{\theta_1}} + e^{-\frac{t}{\theta_2}} - \exp\left\{-(\frac{1}{\theta_1} + \frac{1}{\theta_2})t - \frac{\alpha t^2}{\theta_1\theta_2}\right\}},$$

for each $x \geq 0$. Figure 3 shows the graphs of the survival functions $P\{X_{2:2} - t > x \mid X_{r:2} > t\}$, $r = 1, 2$, for $\alpha = 0.2$ and $\theta_1 = \theta_2 = 1$ at a fixed point $t = 3$. It is obvious that $(X_{2:2} - t \mid X_{2:2} > t) \nleq_{\mathrm{st}} (X_{2:2} - t \mid X_{1:2} > t)$.

Remark 4.1. The behavior of $(X_{k:n} - t \mid X_{r:n} > t)$ in terms of n is considered in Rezapour et al. (2013). They have shown that the above residual lifetime is stochastically decreasing in n when the components lifetimes following an Archimedean copula. We do not know whether a similar result also holds for exchangeable or arbitrary dependent lifetimes.

Theorem 4.4. *Let $\mathcal{S}_1$, and $\mathcal{S}_2$ be two $(n-k+1)$-out-of-n systems with exchangeable vectors of component lifetimes $\boldsymbol{X} = (X_1, X_2, ..., X_n)$ and $\boldsymbol{Y} = (Y_1, Y_2, ..., Y_n)$, respectively. If $(\boldsymbol{X}|\boldsymbol{X} \in E_i(t)) \leq_{st} (\boldsymbol{Y}|\boldsymbol{Y} \in E_i(t))$, for all $i = 0, 1, ..., r-1$, and for $E_i(t) = [0,t]^i \times (t,\infty)^{n-i}$ or $[0,t]^i \times [0,\infty) \times (t,\infty)^{n-i-1}$, then*

$$(X_{k:n} - t \mid X_{r:n} > t) \leq_{st} (Y_{k:n} - t \mid Y_{r:n} > t), \ t \geq 0.$$

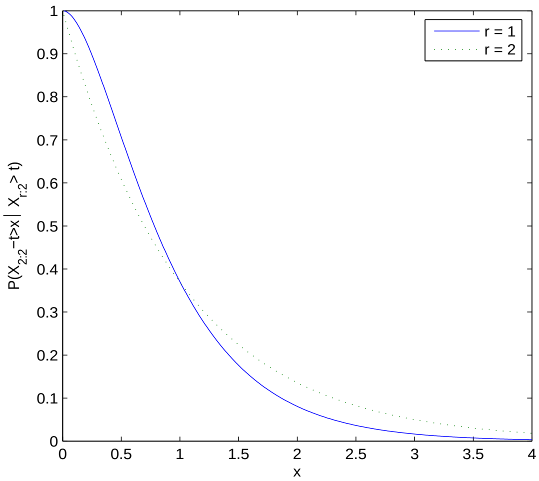

Figure 3: The survival function $P\{X_{2:2} - t > x \mid X_{r:2} > t\}$ for Example 4.1.

Proof. Suppose that an observer observes that exactly first i components of the system with lifetime $\boldsymbol{X}$ have already failed by time t. Thus, a history that the observer has observed is of the form

$$A_i^{(t)} = [\boldsymbol{X} \in E_i(t)] = [X_1 \le t, ..., X_i \le t, X_{i+1} > t, ..., X_n > t].$$

Similarly, one can consider the history

$$B_i^{(t)} = [\,\boldsymbol{Y} \in E_i(t)] = [Y_1 \le t, ..., Y_i \le t, Y_{i+1} > t, ..., Y_n > t],$$

for another system with component lifetimes $\boldsymbol{Y}$. Let $\bar{G}_{r,k,n,t}(x)$ be the survival function of $(Y_{k:n} - t \mid Y_{r:n} > t)$ and $\bar{G}^{(i,t)}_{k-i,n-i}(x) = P\{Y^{(i,t)}_{k-i:n-i} > x\}$, where $Y^{(i,t)}_{k-i:n-i}$ denotes the $(k-i)$th order statistic of the conditional random vector $(Y_{i+1} - t, ..., Y_n - t \mid B_i^{(t)})$. Using (5), we have

$$\begin{aligned}
&\bar{G}_{r,k,n,t}(x) - \bar{F}_{r,k,n,t}(x) \qquad (6)\\
&= \frac{\sum_{i=0}^{r-1}\binom{n}{i}P\{B_i^{(t)}\}\bar{G}^{(i,t)}_{k-i,n-i}(x)}{\sum_{j=0}^{r-1}\binom{n}{j}P\{B_j^{(t)}\}} - \frac{\sum_{i=0}^{r-1}\binom{n}{i}P\{A_i^{(t)}\}\bar{F}^{(i,t)}_{k-i,n-i}(x)}{\sum_{j=0}^{r-1}\binom{n}{j}P\{A_j^{(t)}\}}\\
&= \frac{\sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\binom{n}{i}\binom{n}{j}\left\{P\{A_j^{(t)}\}P\{B_i^{(t)}\})\bar{G}^{(i,t)}_{k-i,n-i}(x) - P\{A_i^{(t)}\})P\{B_j^{(t)}\}\bar{F}^{(i,t)}_{k-i,n-i}(x)\right\}}{\left\{\sum_{j=0}^{r-1}\binom{n}{j}P\{A_j^{(t)}\}\right\}\left\{\sum_{j=0}^{r-1}\binom{n}{j}P\{B_j^{(t)}\}\right\}}.
\end{aligned}$$

The numerator in the right-hand side of the equality can be rewritten as

$$\sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\binom{n}{i}\binom{n}{j}P\{A_j^{(t)}\}P\{B_i^{(t)}\}\left\{\bar{G}_{k-i,n-i}^{(i,t)}(x)-\bar{F}_{k-i,n-i}^{(i,t)}(x)\right\}$$
$$+\sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\binom{n}{i}\binom{n}{j}\left\{P\{A_j^{(t)}\}P\{B_i^{(t)}\}-P\{A_i^{(t)}\}P\{B_j^{(t)}\}\right\}\bar{F}_{k-i,n-i}^{(i,t)}(x). \tag{7}$$

We show that each term of this expression is nonnegative.

It is known that the usual multivariate stochastic ratio order is closed under marginalization. Hence, from the assumption of the theorem, we have for $j=i+1,...,n$,

$$(X_{i+1}-t,...,X_j-t \mid A_i^{(t)}) \leq_{st} (Y_{i+1}-t,...,Y_j-t \mid B_i^{(t)}).$$

Now, we get from Theorem 4.1 that $\bar{F}_{k-i,n-i}^{(i,t)}(x) \leq \bar{G}_{k-i,n-i}^{(i,t)}(x)$, $i = 0,1,...,r-1$, $x \geq 0$. Therefore the first term in the right-hand side of (7) is nonnegative. The second term can be rephrased as

$$\sum_{i=0}^{r-1}\sum_{j=0}^{i}\binom{n}{i}\binom{n}{j}\left\{P\{A_j^{(t)}\}P\{B_i^{(t)}\}-P\{A_i^{(t)}\}P\{B_j^{(t)}\}\right\}\bar{F}_{k-i,n-i}^{(i,t)}(x)$$
$$+\sum_{i=0}^{r-1}\sum_{j=i}^{r-1}\binom{n}{i}\binom{n}{j}\left\{P\{A_j^{(t)}\}P\{B_i^{(t)}\}-P\{A_i^{(t)}\}P\{B_j^{(t)}\}\right\}\bar{F}_{k-i,n-i}^{(i,t)}(x)$$
$$=\sum_{i=0}^{r-1}\sum_{j=0}^{i}\binom{n}{i}\binom{n}{j}\left\{P\{A_j^{(t)}\}P\{B_i^{(t)}\}-P\{A_i^{(t)}\}P\{B_j^{(t)}\}\right\}\bar{F}_{k-i,n-i}^{(i,t)}(x)$$
$$+\sum_{j=0}^{r-1}\sum_{i=0}^{j}\binom{n}{i}\binom{n}{j}\left\{P\{A_j^{(t)}\}P\{B_i^{(t)}\}-P\{A_i^{(t)}\}P\{B_j^{(t)}\}\right\}\bar{F}_{k-i,n-i}^{(i,t)}(x)$$
$$=\sum_{i=0}^{r-1}\sum_{j=0}^{i}\binom{n}{i}\binom{n}{j}\left\{P\{A_j^{(t)}\}P\{B_i^{(t)}\}-P\{A_i^{(t)}\}P\{B_j^{(t)}\}\right\}\bar{F}_{k-i,n-i}^{(i,t)}(x)$$
$$+\sum_{i=0}^{r-1}\sum_{j=0}^{i}\binom{n}{i}\binom{n}{j}\left\{P\{A_i^{(t)}\}P\{B_j^{(t)}\}-P\{A_j^{(t)}\}P\{B_i^{(t)}\}\right\}\bar{F}_{k-j,n-j}^{(j,t)}(x)$$
$$=\sum_{i=0}^{r-1}\sum_{j=0}^{i}\binom{n}{i}\binom{n}{j}\left\{P\{A_i^{(t)}\}P\{B_j^{(t)}\}-P\{A_j^{(t)}\}P\{B_i^{(t)}\}\right\}\{\bar{F}_{k-j,n-j}^{(j,t)}(x)$$
$$-\bar{F}_{k-i,n-i}^{(i,t)}(x)\}.$$

By Corollary 4.1, we have $\bar{F}^{(i,t)}_{k-i,n-i}(x) \leq \bar{F}^{(j,t)}_{k-j,n-j}(x)$, for $j \leq i$. It is not difficult to show that under the assumption of the theorem,

$$\begin{aligned}&(X_{i+1} \mid X_0 \leq t, X_1 \leq t, ..., X_i \leq t, X_{i+2} > t, ..., X_n > t)\\ &\quad \leq_{st} (Y_{i+1} \mid Y_0 \leq t, Y_1 \leq t, ..., Y_i \leq t, Y_{i+2} > t, ..., Y_n > t).\end{aligned}$$

This implies that

$$\begin{aligned}&\frac{P\{Y_{i+1} > t \mid Y_0 \leq t, Y_1 \leq t, ..., Y_i \leq t, Y_{i+2} > t, ..., Y_n > t\}}{P\{X_{i+1} > t \mid X_0 \leq t, X_1 \leq t, ..., X_i \leq t, X_{i+2} > t, ..., X_n > t\}}\\ &\quad \geq \frac{P\{Y_{i+1} \leq t \mid Y_0 \leq t, Y_1 \leq t, ..., Y_i \leq t, Y_{i+2} > t, ..., Y_n > t\}}{P\{X_{i+1} \leq t \mid X_0 \leq t, X_1 \leq t, ..., X_i \leq t, X_{i+2} > t, ..., X_n > t\}},\end{aligned}$$

which, in turn, implies that

$$\begin{aligned}&\frac{P\{Y_0 \leq t, Y_1 \leq t, ..., Y_i \leq t, Y_{i+1} > t, ..., Y_n > t\}}{P\{X_0 \leq t, X_1 \leq t, ..., X_i \leq t, X_{i+1} > t, ..., X_n > t\}}\\ &\quad \geq \frac{P\{Y_1 \leq t, ..., Y_{i+1} \leq t, Y_{i+2} > t, ..., Y_n > t\}}{P\{X_1 \leq t, ..., X_{i+1} \leq t, X_{i+2} > t, ..., X_n > t\}}.\end{aligned}$$

That is

$$\frac{P\{B_i^{(t)}\}}{P\{A_i^{(t)}\}} \geq \frac{P\{B_{i+1}^{(t)}\}}{P\{A_{i+1}^{(t)}\}}.$$

Therefore $P\{B_i^{(t)}\}/P\{A_i^{(t)}\}$ is decreasing in i. Hence for $j \leq i$,

$$\frac{P\{B_j^{(t)}\}}{P\{A_j^{(t)}\}} \geq \frac{P\{B_i^{(t)}\}}{P\{A_i^{(t)}\}}.$$

Thus, the second term in the right-hand side of (7) is also nonnegative, and we obtain the desired result. That is $\bar{G}_{r,k,n,t}(x) \geq \bar{F}_{r,k,n,t}(x)$, for all $x, t \geq 0$, and all r, k, and n such that $1 \leq r - 1 < k \leq n$. ■

Remark 4.2. There are some results, in the i.i.d. case, describing the properties and behavior of the residual lifetime $(X_{k:n} - t \mid X_{r:n} > t)$ in terms of the time $t \geq 0$. For example, it is known that when the common distribution of the observations has increasing failure rate, then the residual lifetime of the system is stochastically decreasing in t; see, for example, Asadi and Bairamov (2005, 2006) and Li and Zhao (2006). The natural question that arises here is to what extent these results can be generalized to the case of exchangeable components.

5 Extensions to General Coherent Systems

According to Barlow and Proschan (1975), a coherent system can be defined as a structure function which is nondecreasing in each vector argument and such that each component is relevant (a component is irrelevant if it does not matter whether or not it is working). The k-out-of-n systems described in previous sections are special cases of coherent systems.

Samaniego (1985) introduced a signature of coherent system, a useful concept that can be used to express the distribution function of the lifetime of a coherent system with independent and identically distributed continuous components lifetimes $Z_1, Z_2, \dots, Z_n$ as a mixture of the distribution functions of the order statistics $Z_{1:n}, Z_{2:n}, \dots, Z_{n:n}$ of the Z_i. Kochar et al. (1999) further studied and developed this concept. More precisely, the *signature* of the lifetime $\tau(Z_1, Z_2, \dots, Z_n)$ of a coherent system is a probability vector $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ where $p_i = P\{\tau(Z_1, Z_2, \dots, Z_n) = Z_{i:n}\}$. The probabilities $p_1, p_2, \dots, p_n$ can be computed by

$$p_i = \frac{n_i}{n!}, \quad i = 1, 2, \dots, n,$$

where n_i is the number of ways that distinct $Z_1, Z_2, \dots, Z_n$ can be ordered such that $\tau(z_1, z_2, \dots, z_n) = z_{i:n}$, $i = 1, 2, \dots, n$, where $z_{i:n}$ is the value of $Z_{i:n}$. Because the vector $\boldsymbol{p}$ is a probability vector, and does not depend on the common distribution function of the Z_i, the survival function of $\tau(Z_1, Z_2, \dots, Z_n)$ is a mixture of the survival functions of $Z_{1:n}, Z_{2:n}, \dots, Z_{n:n}$ with weights $p_1, p_2, \dots, p_n$, respectively. That is

$$P\{\tau(Z_1, Z_2, \dots, Z_n) > x\} = \sum_{i=1}^{n} p_i P\{Z_{i:n} > x\}. \tag{8}$$

Navarro et al. (2005) proved that the representation (8) also holds in the case of absolutely continuous exchangeable distributions.

Now consider a coherent system with the property that, with probability 1, it is operating as long as $(n - s + 1)$ components operate. Such systems must have a signature of the form $(0, 0, \dots, 0, p_s, p_{s+1}, \dots, p_n)$. Let $(X_1, X_2, \dots, X_n)$ be a nonnegative random vector representing components lifetimes of the system. If we denote the lifetime of the system by $T = \tau(X_1, X_2, \dots, X_n)$, then, under the condition that all components of the system are alive, the residual lifetime of the system is $(T - t \mid X_{r:n} > t)$, $r = 1, 2, \dots, s - 1$. Khaledi and Shaked (2007) showed that the survival function of the residual lifetime of the system can be expressed as

$$P\{T - t > x \mid X_{r:n} > t\} = \sum_{i=s}^{n} p_i P\{X_{i:n} - t > x \mid X_{r:n} > t\}. \quad (9)$$

The next theorem may be useful for system operators to be able to stochastically compare the residual lives of systems with different types of components. We mention here that for any two probability vectors $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$, $\boldsymbol{p}$ is said to be smaller than $\boldsymbol{q}$ in the usual stochastic order (denoted by $\boldsymbol{p} \leq_{st} \boldsymbol{q}$) if $\sum_{i=j}^{n} p_i \leq \sum_{i=j}^{n} q_i$, $j = 1, 2, \ldots, n$.

Theorem 5.1. *Let $X_1, X_2, \ldots, X_n$ be the exchangeable components lifetimes of a coherent system with life function τ_1, and let $Y_1, Y_2, \ldots, Y_n$ be the exchangeable components lifetimes of another coherent system with life function τ_2. Denote $T_1 = \tau_1(X_1, X_2, \ldots, X_n)$ and $T_2 = \tau_2(Y_1, Y_2, \ldots, Y_n)$. For some $2 \leq s \leq n$, suppose that the signatures of τ_1, and τ_2 are of the forms $(0, 0, \ldots, 0, p_s, p_{s+1}, \ldots, p_n)$, and $(0, 0, \ldots, 0, q_s, q_{s+1}, \ldots, q_n)$, respectively. If $\boldsymbol{p} \leq_{st} \boldsymbol{q}$ and $(\boldsymbol{X} | \boldsymbol{X} \in E_i(t)) \leq_{st} (\boldsymbol{Y} | \boldsymbol{Y} \in E_i(t))$, for all $i = 0, 1, ..., r-1$, and for $E_i(t) = [0, t]^i \times (t, \infty)^{n-i}$ or $[0, t]^i \times [0, \infty) \times (t, \infty)^{n-i-1}$, then*

$$(T_1 - t \mid X_{r:n} > t) \leq_{st} (T_2 - t \mid Y_{r:n} > t), \quad r = 1, 2, ..., s-1.$$

Proof. From (9), we have

$$\begin{aligned} P\{T_1 - t > x \mid X_{r:n} > t\} &= \sum_{i=s}^{n} p_i P\{X_{i:n} - t > x \mid X_{r:n} > t\} \\ &\leq \sum_{i=s}^{n} p_i P\{Y_{i:n} - t > x \mid Y_{r:n} > t\} \\ &\leq \sum_{i=s}^{n} q_i P\{Y_{i:n} - t > x \mid Y_{r:n} > t\} \\ &= P\{T_2 - t > x \mid Y_{r:n} > t\}, \end{aligned}$$

where the first inequality follows from Theorem 4.4 and the second one follows from the fact that $P\{Y_{i:n} - t > x \mid Y_{r:n} > t\}$ is increasing in $i = s, ..., n$ (see Theorem 4.2), and from Theorem 1.A.6 in Shaked and Shanthikumar (2007). This completes the proof of the theorem. ■

The next result follows directly from Theorem 4.3 and equality in (9), and shows that the residual lifetime $(T - t \mid X_{r:n} > t)$ of a coherent system is decreasing in r in the sense of stochastic order.

Corollary 5.1. *Let $(X_1, X_2, ..., X_n)$ be the exchangeable components lifetimes of a coherent system with life function τ and lifetime $T = \tau(X_1, X_2, ..., X_n)$. For some $1 \leq s \leq n$, suppose that the signature of τ is of the form $(0, 0, ..., 0, p_s, p_{s+1}, ...p_n)$. Let $(X_1, X_2, ..., X_n)$ has the MTP$_2$ property. Then for any integer r such that $2 \leq r \leq s + 1$,*

$$(T - t \mid X_{r:n} > t) \leq_{st} (T - t \mid X_{r-1:n} > t),$$

for all $t \geq 0$.

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