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Generalized Ridge Regression Estimator in Semiparametric Regression Models

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Abstract. In the context of ridge regression, the estimation of ridge (shrinkage) parameter plays an important role in analyzing data. Many efforts have been put to develop skills and methods of computing shrinkage estimators for different full-parametric ridge regression approaches, using eigenvalues. However, the estimation of shrinkage parameter is neglected for semiparametric regression models. The main focus of this paper is to develop necessary tools for computing the risk function of regression coefficient based on the eigenvalues of design matrix in semiparametric regression model, making use of differencing methodology. In this respect, some new estimators for shrinkage parameter are also proposed. It is shown that one of these estimators which is constructed based on well-known harmonic mean, performs better for large values of signal to noise ratio. For our proposal, the Monte Carlo simulation studies and a real application related to housing attributes are conducted to illustrate the efficiency of shrinkage estimators based on minimum risk criteria.

Keywords. Differencing methodology, generalized ridge estimator, ker-

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1 Introduction

Semiparametric regression models (SRMs) have received considerable attention in statistics and econometrics, because of their flexibility in modeling real events. A SRM has form

$$y_i = \boldsymbol{x}_i\boldsymbol{\beta} + f(t_i) + \epsilon_i, \quad i = 1, ..., n, \tag{1.1}$$

where $\boldsymbol{x}_i = (x_{i1}, x_{i2}, ..., x_{ip})$ is a vector of explanatory variables, $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_p)'$ is an unknown p-dimensional parameter vector, the t_i is known (non-stochastic) in some bounded domain $D \in \mathbb{R}$, $f(t_i)$ is an unknown smooth function and ϵ_i's are independent and identically distributed random errors with mean 0 and variance σ^2.

Most of approaches for SRMs are based on different nonparametric regression procedures. There have been several methods to estimate $\boldsymbol{\beta}$ and $f(.)$. Extensive study regarding estimation and application of the SRM (1.1) can be found in the monograph of Härdle et al. (2000). An alternative approach to nonparametric procedure, is differencing methodology. This incoming, uses differences to remove the trend in the data that arises from the function $f(\cdot)$ and does not require an estimator of $f(\cdot)$. It is often called difference-based procedure. Provided that $f(\cdot)$ is differentiable and t_i's are closely spaced, it is possible to remove the effect of the function $f(\cdot)$ by differencing the data, appropriately. In the model (1.1), Yatchew (1997) concentrated on estimation of the linear component and used differencing to eliminate bias induced from the presence of the nonparametric component. The difference-based estimation procedure is optimal in the sense that the estimator of linear component is asymptotically efficient and the estimator of nonparametric component is asymptotically minimax (Wang *et al.* 2011). Thus, differencing allows one to perform inference on β as if there was no nonparametric component $f(\cdot)$ in the model SRM (1.1). Once $\boldsymbol{\beta}$ is estimated, a variety of nonparametric techniques could be applied to estimate $f(\cdot)$ as if $\boldsymbol{\beta}$ was known. Wang et al. (2011) used higher order differences for optimal efficiency in estimating the linear part, by using a special class of difference sequences.

Now, consider a SRM in the presence of multicollinearity. For the purpose of this study, we only employ the ridge regression concept due to

Hoerl and Kennard (1970), to combat multicollinearity. There are a lot of work adopting ridge regression methodology to overcome the multicollinearity problem. To mention a few recent studies in full-parametric regression, see Saleh and Kibria (1993), Saleh (2006), Ozkale and Kaciranlar (2007), Muniz and Kibria (2009), Ozkale (2009), Hassanzadeh Bashtian et al. (2011a, b), Kaciranlar et al. (2011), Kibria and Saleh (2004, 2011), Roozbeh et al. (2011, 2012), Akdeniz Duran et al. (2012) and Kibria (2012). Akdeniz and Tabakan (2009) and Roozbeh and Arashi (2013) employed this methodology in facing with SRM. The main focus of this approach is to develop necessary tools for computing the risk function of regression coefficient in a SRM incorporating eigenvalues of design matrix. To this end, the differencing methodology will be applied. We are also seeking for some new estimators of shrinkage parameter.

The organization of the paper is as follows: Section 2 contains a nutshell of the difference-based methodology. In section 3, difference-based generalized ridge estimator of linear part is introduced. Some of estimation methods usually used for estimating the ridge parameter in full-parametric regression model and along proposing some new ones will be reviewed in section 4. Properties of the proposed estimator are exactly derived in section 5. Section 6 is devoted to a Monte Carlo simulation study and an application in housing attributes. Finally, we conclude our paper by giving some remarks in section 7.

2 Differencing Approach

In this section, we use a difference-based technique to estimate the linear regression coefficient vector $\boldsymbol{\beta}$. This technique has been used to remove the nonparametric component in SRM by various authors (e.g. Yatchew, 1997 and 2003; Klipple and Eubank, 2007, and Brown and Levine, 2007). In a full matrix notation, the model (1.1) can be represented as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{f}(t) + \boldsymbol{\epsilon}, \tag{2.1}$$

where $\boldsymbol{y} = (y_1, ..., y_n)'$, $\boldsymbol{X} = (\boldsymbol{x}_1, ..., \boldsymbol{x}_n)'$ is the $n \times p$ non-stochastic design matrix of full column rank, $\boldsymbol{f}(t) = (f(t_1), ..., f(t_n))'$, and $\boldsymbol{\epsilon} = (\epsilon_1, ..., \epsilon_n)'$. In general, we assume that $\boldsymbol{\epsilon}$ is a vector of disturbances (errors) distributed with $E(\boldsymbol{\epsilon}) = \boldsymbol{0}$ and $E(\boldsymbol{\epsilon}\boldsymbol{\epsilon}') = \sigma^2 \boldsymbol{I}_n$, where $\boldsymbol{I}_n$ is an identity matrix of order n and σ^2 is an unknown parameter.

Yatchew (1997) suggested to estimate $\boldsymbol{\beta}$ on the basis of the m^{th} order differencing equation

$$\sum_{j=0}^{m} d_j y_{i-j} = \Big(\sum_{j=0}^{m} d_j x_{i-j}\Big)\beta + \sum_{j=0}^{m} d_j f(t_{i-j}) + \sum_{j=0}^{m} d_j \epsilon_{i-j}, \qquad (2.2)$$

where $d_0, d_1, ..., d_m$ are differencing weights.

In the forthcoming section, we adopt an explanation from Akdeniz et al. (2013), to show how the approximation works.

How does the Approximation work?

Suppose t_i's are equally spaced on the unit interval and $f'(.) \leq L$. By the mean value theorem, for some $t_i^* \in [t_{i-1}, t_i]$ we have

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1}) \leq \frac{L}{n}.$$

Note that with $m = p = 1$ from (2.2) we have that

$$\begin{aligned} y_i - y_{i-1} &= (x_i - x_{i-1})\beta + f(t_i) - f(t_{i-1}) + \epsilon_i - \epsilon_{i-1} \\ &= (x_i - x_{i-1})\beta + O\big(\frac{1}{n}\big) + \epsilon_i - \epsilon_{i-1} \\ &\cong (x_i - x_{i-1})\beta + \epsilon_i - \epsilon_{i-1}. \end{aligned}$$

We then estimate the linear regression coefficient β by the ordinary least-squares estimator based on the differences. Then, the least-squares estimate will be obtained as $\hat{\beta}_{diff} = \frac{\sum_{i=2}^{n}(x_i - x_{i-1})(y_i - y_{i-1})}{\sum_{i=2}^{n}(x_i - x_{i-1})^2}$.

Now, let $\boldsymbol{d} = (d_0, ..., d_m)$ be a $(m+1)$-vector, where m is the order of differencing and $d_0, d_1, ..., d_m$ are differencing weights minimizing the variance of linear estimators, i.e.,

$$min_{d_0,...,d_m} \sum_{l=1}^{m} \Big(\sum_{j=0}^{m} d_j d_{l+j}\Big)^2,$$

satisfying the conditions

$$\sum_{j=0}^{m} d_j = 0, \quad \sum_{j=0}^{m} d_j^2 = 1. \qquad (2.3)$$

The role of constraints (2.3) are now evident. The first condition ensures that as the t's become close, the non-parametric effect is removed and the second one ensures that the variance of sum of weighted residuals remains equals to σ^2 in (2.2).

Now, we define the $(n-m)\times n$ differencing matrix $\boldsymbol{D}$ whose elements satisfy (2.3) as

$$\boldsymbol{D} = \begin{pmatrix} d_0 & d_1 & ... & d_m & 0 & 0 & ... & 0 \\ 0 & d_0 & d_1 & ... & d_m & 0 & ... & 0 \\ \vdots & \ddots & & & & & & \vdots \\ 0 & 0 & ... & 0 & d_0 & d_1 & ... & d_m \end{pmatrix}.$$

This and related matrices are given, for example, in Yatchew(2003).

Incorporating the differencing matrix in model (2.1) permits direct estimation of the parametric effect. As a result of developments in Speckman (1988), it is known that the parameter vector $\boldsymbol{\beta}$ in (2.1) can be estimated with parametric efficiency. We now show the difference-based estimators can be used for this purpose. In as much as the data have been ordered so that the values of the nonparametric variable(s) are close, the application of the differencing matrix $\boldsymbol{D}$ in model (2.1) removes the nonparametric effect in large samples. If $f(.)$ is an unknown function that is an inferential object and has a bounded first derivative, then $\boldsymbol{Df}(t)$ is close to $\boldsymbol{0}$, so that by applying the differencing matrix, we may rewrite (2.1) as

$$\boldsymbol{Dy} \doteq \boldsymbol{DX\beta} + \boldsymbol{D\epsilon},$$

or

$$\boldsymbol{y_D} \doteq \boldsymbol{X_D\beta} + \boldsymbol{\epsilon_D}, \tag{2.4}$$

where $\boldsymbol{y_D} = \boldsymbol{Dy}$, $\boldsymbol{X_D} = \boldsymbol{DX}$, $\boldsymbol{\epsilon_D} = \boldsymbol{D\epsilon}$.

For arbitrary differencing coefficients satisfying (2.3), the difference-based ordinary least square estimator (DOLSE) of the regression coefficient $\boldsymbol{\beta}$ is obtained in the SRM as (see Yatchew, 1997)

$$\hat{\boldsymbol{\beta}}_{\boldsymbol{D}} = \boldsymbol{S_D}^{-1}\boldsymbol{X'_D y_D}, \ \boldsymbol{S_D} = \boldsymbol{X'_D X_D}. \tag{2.5}$$

Thus, differencing allows one to perform inferences on $\boldsymbol{\beta}$ as if there was no nonparametric component $f(.)$ in the model (2.1) (Yatchew 2003). Once $\boldsymbol{\beta}$ is estimated, a variety of nonparametric techniques could be applied to estimate $f(\cdot)$ as if $\boldsymbol{\beta}$ was known.

To account the parameter $\boldsymbol{\beta}$ in (2.4), we introduce the modified estimator of σ^2 as

$$\hat{\sigma}^2_{\boldsymbol{D}} = \frac{\boldsymbol{y'_D}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{y_D}}{tr\big(\boldsymbol{D'}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{D}\big)}, \tag{2.6}$$

where $tr(.)$ is the trace function for a square matrix and $\boldsymbol{P}$ is the projection matrix defined as

$$\boldsymbol{P} = \boldsymbol{X_D}(\boldsymbol{X'_D X_D})^{-1}\boldsymbol{X'_D}.$$

3 Difference-based Generalized Ridge Estimator

In this section, we will be discussing a biased estimation technique under multicollinearity, for SRM. The covariance matrix of $\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}$ is equal to $\sigma^2\boldsymbol{S}_{\boldsymbol{D}}^{-1}$. As it can be seen, both difference based ordinary least squares estimate (DOLSE) and its covariance matrix heavily depend on the characteristics of the matrix $\boldsymbol{S}_{\boldsymbol{D}}$. If $\boldsymbol{S}_{\boldsymbol{D}}$ is ill-conditioned, the DOLS estimators are sensitive to a number of errors. For example, some of the regression coefficients may be statistically insignificant or have wrong signs, and they may result in wide confidence intervals for individual parameters (which are called unstable estimators). With these errors, it is difficult to make valid statistical inference.

The problem of multicollinearity can be solved by collecting additional data, re-parameterizing the model and reselecting variables. There are two well-known mathematical methods to overcome multicollinearity: the principal components regression method and the ridge regression method. In this article, we will discuss the ridge regression method. A brief review of the literature reveals an abundance of works related to the ridge regression method. Hoerl and Kennard (1970) first proposed this method to solve the multicollinearity problem. They suggested a small positive number to be added to the diagonal elements of $\boldsymbol{S}_{\boldsymbol{D}}$ matrix; and the resulting estimator has the following form

$$\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(k) = \boldsymbol{S}_{\boldsymbol{D}}^{-1}(k)\boldsymbol{X}'_{\boldsymbol{D}}\boldsymbol{Y}_{\boldsymbol{D}},\ \boldsymbol{S}_{\boldsymbol{D}}(k) = \boldsymbol{S}_{\boldsymbol{D}} + k\boldsymbol{I}_p, \tag{3.1}$$

which is known as a difference-based ridge estimator (DRE). For a positive value of k, this estimator provides a smaller mean squared error (MSE) compared to the LSE. The constant k ($k \geq 0$) is called the "ridge" or "shrinkage" parameter, and it must be estimated using the real data. Although, the ridge estimator is the most popular method for dealing with multicollinearity, it has some drawbacks. Dependency on the ridge parameter k tends to result in either instability or bias. However, as $k \to \infty$, $\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(k) \to \mathbf{0}$ and one obtains a stable, but biased estimator of $\boldsymbol{\beta}$. As $k \to 0$, $\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(k) \to \hat{\boldsymbol{\beta}}_{\boldsymbol{D}}$ and one obtains an unbiased, but unstable estimator of $\boldsymbol{\beta}$. The expected distance between $\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(k)$ and $\boldsymbol{\beta}$ must decrease as k increases from the origin. The value of k that produces the best estimator, however, is not clear. It is realized that the estimator $\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(k)$ is a complicated function of k.

It is clear that for the semi-positive definite matrix $\boldsymbol{S}_{\boldsymbol{D}}$, there exists an orthogonal matrix $\boldsymbol{\Gamma}$ such that $\boldsymbol{S}_{\boldsymbol{D}} = \boldsymbol{\Gamma\Lambda\Gamma}'$, where $\boldsymbol{\Lambda} = diag(\lambda_1, ..., \lambda_p)$

contains the eigenvalues of matrix $\boldsymbol{S_D}$. Therefore, the orthogonal (canonical) version of the model (2.4) is given by

$$\boldsymbol{y_D} = \boldsymbol{X_D^*}\boldsymbol{\alpha} + \boldsymbol{\epsilon_D}, \tag{3.2}$$

where $\boldsymbol{X_D^*} = \boldsymbol{X_D}\boldsymbol{\Gamma}$ and $\boldsymbol{\alpha} = \boldsymbol{\Gamma}'\boldsymbol{\beta}$.

But, when the matrix $\boldsymbol{S_D}$ is ill-conditioned (in the sense that there is a near linear dependency among the columns of matrix), the DOLSE of $\boldsymbol{\beta}$ has a large variance, and multicollinearity is said to be present. If multicollinearity is present, at least for one eigenvalue, $\lambda_i \doteq 0$. More closeness of the smallest eigenvalue to the origin, the more strength of linear multicollinearity. To make the behavior of $\boldsymbol{S_D}$ matrix more like the canonical form, we need to increase the eigenvalues. Ridge regression replaces $\boldsymbol{S_D}$ with $\boldsymbol{S_D}(k) = \boldsymbol{S_D} + k\boldsymbol{I}_p$, $(k > 0)$, which is the same as replacing the λ_i by $\lambda_i + k$. This replacement counters the damaging effect of the smallest eigenvalue.

Then, the canonical difference-based generalized ridge estimator (CDGRE) will have from

$$\hat{\boldsymbol{\alpha}}_{\boldsymbol{D}}(\boldsymbol{K}) = (\boldsymbol{S_D^*} + \boldsymbol{K})^{-1}\boldsymbol{X_D^{*\prime}}\boldsymbol{y_D} = \boldsymbol{T_D^*}(\boldsymbol{K})\hat{\boldsymbol{\alpha}}_{\boldsymbol{D}},\ \boldsymbol{T_D^*}(\boldsymbol{K}) = (\boldsymbol{K}\boldsymbol{S_D^{*-1}} + \boldsymbol{I}_p)^{-1},$$

where $\boldsymbol{S_D^*} = \boldsymbol{X_D^{*\prime}}\boldsymbol{X_D^*}$, $\boldsymbol{K} = diag(k_1, k_2, ..., k_p)$, $k_i \geq 0$ and $\hat{\boldsymbol{\alpha}}_{\boldsymbol{D}} = \boldsymbol{\Lambda}^{-1}\boldsymbol{X_D^{*\prime}}\boldsymbol{y_D}$ is the canonical difference-based ordinary least squares (CDOLS) estimates of $\boldsymbol{\alpha}$.

Hoerl and Kennard (1970) showed that for known optimal

$$k_i = \frac{\sigma^2}{\alpha_i^2},\ i = 1, ..., p, \tag{3.3}$$

the generalized ridge regression estimator is superior to all other estimators within the class of biased estimators, where σ^2 is the error variance of model (2.4) and α_i^2 is the i^{th} element of $\boldsymbol{\alpha}$. However, the optimal value of k_i fully depends on the unknown σ^2 and α_i, and they must be estimated from the observed data. Hoerl and Kennard (1970), suggested to replace σ^2 and α_i^2 by their corresponding unbiased estimators. That is,

$$\hat{k}_i = \frac{\hat{\sigma}_{\boldsymbol{D}}^2}{\hat{\alpha}_i^2}\ i = 1, ..., p, \tag{3.4}$$

where $\hat{\sigma}_{\boldsymbol{D}}^2$ is an unbiased and efficient estimator of σ^2 and $\hat{\alpha}_i$ is the i^{th} element of $\hat{\boldsymbol{\alpha}}_{\boldsymbol{D}}$, which is an unbiased estimator of α.

4 Ridge Parameter Estimators

In this Section, we will review and propose some new methods of estimating k based on (3.4) as follows:

1. Hoerl and Kennard (1970), suggested the following estimator for k

$$\hat{k}_{HK} = \frac{\hat{\sigma}^2_{\boldsymbol{D}}}{\hat{\alpha}^2_{\max}}, \tag{4.1}$$

where $\hat{\alpha}_{\max}$ is the maximum element of $\hat{\boldsymbol{\alpha}}_{\boldsymbol{D}}$. If σ^2 and α are known, then $\hat{\boldsymbol{\alpha}}_{\boldsymbol{D}}(\hat{k}_{HK})$ will give smaller risk than $\hat{\boldsymbol{\alpha}}_{\boldsymbol{D}}$ (Hoerl and Kennard, 1970).

2. Hoerl et al. (1975), suggested the following estimator for k

$$\hat{k}_{HKB} = \frac{p\hat{\sigma}^2_{\boldsymbol{D}}}{\sum_{i=1}^{p}\hat{\alpha}_i^2} = \frac{p\hat{\sigma}^2_{\boldsymbol{D}}}{\hat{\boldsymbol{\alpha}}'_{\boldsymbol{D}}\hat{\boldsymbol{\alpha}}_{\boldsymbol{D}}} = \frac{p\hat{\sigma}^2_{\boldsymbol{D}}}{\hat{\boldsymbol{\beta}}'_{\boldsymbol{D}}\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}}. \tag{4.2}$$

3. Kibria (2003), proposed the following estimators by using the arithmetic mean, geometric mean and median of $\hat{k}_i$ in (3.4) as

$$\hat{k}_{AM} = \frac{1}{p}\sum_{i=1}^{p}\frac{\hat{\sigma}^2_{\boldsymbol{D}}}{\hat{\alpha}_i^2}, \tag{4.3}$$

$$\hat{k}_{GM} = \frac{\hat{\sigma}^2_{\boldsymbol{D}}}{(\prod_{i=1}^{p}\hat{\alpha}_i^2)^{\frac{1}{p}}}, \tag{4.4}$$

$$\hat{k}_{Med} = Median\left\{\frac{\hat{\sigma}^2_{\boldsymbol{D}}}{\hat{\alpha}_i^2}\right\}, \quad i = 1, 2, ..., p. \tag{4.5}$$

4. New estimator: We propose to estimate k by using the harmonic mean of $\hat{k}_i^{-1}$ in (3.4), which produces the following new estimator:

$$\hat{k}_{HM} = \frac{p}{\hat{\sigma}^2_{\boldsymbol{D}}\sum_{i=1}^{p}\hat{\alpha}_i^{-2}}. \tag{4.6}$$

5. Khalaf and Shukur (2005), suggested a new method to estimate the ridge parameter k as a modification of k_{HK} which has form

$$\hat{k}_{KS} = \frac{\lambda_{max}\hat{\sigma}^2_{\boldsymbol{D}}}{(n-m-p)\hat{\sigma}^2_{\boldsymbol{D}} + \lambda_{max}\hat{\alpha}^2_{max}}, \tag{4.7}$$

where λ_{max} is the maximum eigenvalue of $\boldsymbol{S}_{\boldsymbol{D}}$.

6. Alkhamisi et al. (2006), proposed the following estimators:

$$\hat{k}_{AKS1} = \frac{1}{p}\sum_{i=1}^{p}\left(\frac{\lambda_i\hat{\sigma}^2_{\boldsymbol{D}}}{(n-m-p)\hat{\sigma}^2_{\boldsymbol{D}}+\lambda_i\hat{\alpha}_i^2}\right), \tag{4.8}$$

$$\hat{k}_{AKS2} = \max_{i=1,\dots,p}\left\{\frac{\lambda_i\hat{\sigma}^2_{\boldsymbol{D}}}{(n-m-p)\hat{\sigma}^2_{\boldsymbol{D}}+\lambda_i\hat{\alpha}_i^2}\right\}, \tag{4.9}$$

$$\hat{k}_{AKS3} = Median\left\{\frac{\lambda_i\hat{\sigma}^2_{\boldsymbol{D}}}{(n-m-p)\hat{\sigma}^2_{\boldsymbol{D}}+\lambda_i\hat{\alpha}_i^2}\right\}. \tag{4.10}$$

7. Muniz and Kibria (2009), by using the geometric mean of $\frac{\lambda_i\hat{\sigma}^2_{\boldsymbol{D}}}{(n-p)\hat{\sigma}^2_{\boldsymbol{D}}+\lambda_i\hat{\sigma}^2_{\boldsymbol{D}}}$, proposed the following estimator:

$$\hat{k}_{MK1} = \left(\prod_{i=1}^{p}\frac{\lambda_i\hat{\sigma}^2_{\boldsymbol{D}}}{(n-m-p)\hat{\sigma}^2_{\boldsymbol{D}}+\lambda_i\hat{\alpha}_i^2}\right)^{\frac{1}{p}}. \tag{4.11}$$

8. Muniz and Kibria (2009), following Alkhamisi and Shukur (2008), proposed the following estimators based on $m_i = \dfrac{\hat{\sigma}_{\boldsymbol{D}}}{|\hat{\alpha}_i|}$:

$$\begin{aligned}
\hat{k}_{MK2} &= \max_{i=1,\dots,p}\left\{\frac{1}{m_i}\right\},\\
\hat{k}_{MK3} &= \max_{i=1,\dots,p}\left\{m_i\right\},\\
\hat{k}_{MK4} &= \left(\prod_{i=1}^{p}\frac{1}{m_i}\right)^{\frac{1}{p}},\\
\hat{k}_{MK5} &= \left(\prod_{i=1}^{p}m_i\right)^{\frac{1}{p}},\\
\hat{k}_{MK6} &= Median\left\{\frac{1}{m_i}\right\},\\
\hat{k}_{MK7} &= Median\left\{m_i\right\}.
\end{aligned} \tag{4.12}$$

9. New estimators: We propose to estimate k by using the harmonic mean of $\frac{\lambda_i\hat{\sigma}^2_{\boldsymbol{D}}}{(n-p)\hat{\sigma}^2_{\boldsymbol{D}}+\lambda_i\hat{\alpha}_i^2}$, m_i^{-1} and m_i, which produce the following new

estimators:

$$\hat{k}_{RAK1} = \frac{p\hat{\sigma}^2_{\boldsymbol{D}}}{\sum_{i=1}^{p}\left((n-p)\hat{\sigma}^2_{\boldsymbol{D}} + \lambda_i\hat{\alpha}_i^2\right)}, \tag{4.13}$$

$$\hat{k}_{RAK2} = HM\left\{\frac{1}{m_i}\right\} = \frac{p}{\hat{\sigma}_{\boldsymbol{D}}\sum_{i=1}^{p}\frac{1}{|\hat{\alpha}_i|}}, \tag{4.14}$$

$$\hat{k}_{RAK3} = HM\{m_i\} = \frac{p\hat{\sigma}_{\boldsymbol{D}}}{\sum_{i=1}^{p}|\hat{\alpha}_i|}. \tag{4.15}$$

5 Computing Risk Function

For any particular estimator $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$, the risk function associated with the square error loss is measured by

$$\mathrm{R}(\hat{\boldsymbol{\beta}}, \boldsymbol{\beta}) = E\big[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\big].$$

Lemma 5.1. *(Roozbeh et al., 2011) The bias, covariance matrix and risk functions of difference-based generalized ridge estimator (DGRE) can be evaluated as follows:*

$$\begin{aligned}
\boldsymbol{b}\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(\boldsymbol{K})\big) &= E\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(\boldsymbol{K}) - \boldsymbol{\beta}\big) \\
&= -\boldsymbol{K}\boldsymbol{S}_{\boldsymbol{D}}^{-1}(\boldsymbol{K})\boldsymbol{\beta}, &(5.1)\\
Cov\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(\boldsymbol{K})\big) &= \sigma^2\boldsymbol{S}_{\boldsymbol{D}}^{-1}(\boldsymbol{K})\boldsymbol{S}_{\boldsymbol{D}}\boldsymbol{S}_{\boldsymbol{D}}^{-1}(\boldsymbol{K}), &(5.2)\\
R\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(\boldsymbol{K}), \boldsymbol{\beta}\big) &= \sigma^2 tr\big(\boldsymbol{S}_{\boldsymbol{D}}^{-1}(\boldsymbol{K})\boldsymbol{S}_{\boldsymbol{D}}\boldsymbol{S}_{\boldsymbol{D}}^{-1}(\boldsymbol{K})\big) + \boldsymbol{\beta}'\boldsymbol{S}_{\boldsymbol{D}}^{-1}(\boldsymbol{K})\boldsymbol{K}^2\boldsymbol{S}_{\boldsymbol{D}}^{-1}(\boldsymbol{K})\boldsymbol{\beta}, &(5.3)
\end{aligned}$$

where $\boldsymbol{S}_{\boldsymbol{D}}(\boldsymbol{K}) = \boldsymbol{S}_{\boldsymbol{D}} + \boldsymbol{K}$.

Then, the properties of $\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}$ (DOLSE) is obtained by letting $\boldsymbol{K} = \boldsymbol{0}$ in the above Lemma as follows:

$$\begin{aligned}
\boldsymbol{b}\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}\big) &= \boldsymbol{0}, &(5.4)\\
Cov\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}\big) &= \sigma^2\boldsymbol{S}_{\boldsymbol{D}}^{-1}, &(5.5)\\
R\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}, \boldsymbol{\beta}\big) &= \sigma^2 tr\big(\boldsymbol{S}_{\boldsymbol{D}}^{-1}\big). &(5.6)
\end{aligned}$$

Theorem 5.1. *The risk function of the DGRE is given by*

$$R\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(\boldsymbol{K}),\boldsymbol{\beta}\big) = \sigma^2\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k_i)^2}+\sum_{i=1}^{p}\frac{k_i^2\alpha_i^2}{(\lambda_i+k_i)^2}. \tag{5.7}$$

Proof. By using $\boldsymbol{S}_{\boldsymbol{D}}^{-1}(\boldsymbol{K})=\boldsymbol{\Gamma}(\boldsymbol{\Lambda}+\boldsymbol{K})^{-1}\boldsymbol{\Gamma}'$, we have

$$\begin{aligned} tr\Big(Cov\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(\boldsymbol{K})\big)\Big) &= \sigma^2 tr\left[\boldsymbol{S}_{\boldsymbol{D}}^{-1}(k)\boldsymbol{S}_{\boldsymbol{D}}\boldsymbol{S}_{\boldsymbol{D}}^{-1}(k)\right] \\ &= \sigma^2 tr\left[\boldsymbol{\Gamma}(\boldsymbol{\Lambda}+k\boldsymbol{I})^{-1}\boldsymbol{\Gamma}'\boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{\Gamma}'\boldsymbol{\Gamma}(\boldsymbol{\Lambda}+k\boldsymbol{I})^{-1}\boldsymbol{\Gamma}'\right] \\ &= \sigma^2 tr\left[\boldsymbol{\Lambda}(\boldsymbol{\Lambda}+k\boldsymbol{I})^{-2}\right] \\ &= \sigma^2\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^2}. \end{aligned} \tag{5.8}$$

Also,

$$\begin{aligned} &QB\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(\boldsymbol{K})\big) \\ &= \boldsymbol{\beta}'\boldsymbol{S}_{\boldsymbol{D}}^{-1}(\boldsymbol{K})\boldsymbol{K}^2\boldsymbol{S}_{\boldsymbol{D}}^{-1}(\boldsymbol{K})\boldsymbol{\beta} \\ &= \boldsymbol{\alpha}'\boldsymbol{\Gamma}'\boldsymbol{\Gamma}(\boldsymbol{\Lambda}+\boldsymbol{K})^{-1}\boldsymbol{\Gamma}'\boldsymbol{K}^2\boldsymbol{\Gamma}(\boldsymbol{\Lambda}+\boldsymbol{K})^{-1}\boldsymbol{\Gamma}'\boldsymbol{\Gamma}\boldsymbol{\alpha} \\ &= \boldsymbol{\alpha}'(\boldsymbol{\Lambda}+\boldsymbol{K})^{-1}\ diag(k_1^2,k_2^2,...,k_p^2)\ (\boldsymbol{\Lambda}+\boldsymbol{K})^{-1}\boldsymbol{\alpha} \\ &= \boldsymbol{\alpha}'\ diag\Big(k_1^2(\lambda_1+k_1)^{-2},k_2^2(\lambda_2+k_2)^{-2},...,k_p^2(\lambda_p+k_p)^{-2}\Big)\ \boldsymbol{\alpha} \\ &= \sum_{i=1}^{p}\frac{k_i^2\alpha_i^2}{(\lambda_i+k_i)^2}, \end{aligned} \tag{5.9}$$

which QB(.) is the quadratic bias of an estimator.

By adding (5.8) into (5.9), we have that

$$\begin{aligned} \mathrm{R}\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(\boldsymbol{K}),\boldsymbol{\beta}\big) &= tr\Big(Cov\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(\boldsymbol{K})\big)\Big)+QB\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(\boldsymbol{K})\big) \\ &= \sigma^2\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k_i)^2}+\sum_{i=1}^{p}\frac{k_i^2\alpha_i^2}{(\lambda_i+k_i)^2}. \end{aligned} \tag{5.10}$$

The proof is complete. □

6 Numerical Results

In this section we conduct some numerical computations as proofs of our assertions. First, we will be considering a Monte Carlo simulation study to evaluate the performance of the newly proposed harmonic mean ridge estimator.

6.1 Monte Carlo Simulation

We evaluate the risk function performance of the proposed estimators comparatively, in this section. To achieve different degrees of collinearity, following McDonald and Galarneau (1975) and Gibbons (1981), the explanatory variables were generated using the following device for $n = 500$:

$$x_{ij} = (1-\gamma^2)^{\frac{1}{2}} z_{ij} + \gamma z_{ip}, \qquad i = 1, 2, ..., n, \quad j = 1, 2, ..., p, \tag{6.1}$$

where z_{ij} are independent standard normal pseudo-random numbers, and γ is specified so that the correlation between any two explanatory variables is given by γ^2. These variables are then standardized so that $\boldsymbol{X}'\boldsymbol{X}$ and $\boldsymbol{X}'\boldsymbol{Y}$ are in correlation forms. Three different sets of correlation corresponding to $\gamma = 0.80, 0.90$ and 0.99 are considered. Then, n observations for the dependent variable are generated according to the following scheme:

$$y_i = \sum_{j=1}^{6} x_{ij}\beta_j + f(t_i) + \epsilon_i, i = 1, ..., n, \tag{6.2}$$

where

$$\boldsymbol{\beta} = (3, 1, -3, 2, -5, 4)',$$
$$f(t) = \frac{1}{3}\Big[\phi\big(t; -3, 0.81\big) + \phi\big(t; 0, 0.36\big) + \phi\big(t; 3, 0.81\big)\Big],$$

is a mixture of normal densities for $t \in [-5, 5]$ and $\phi(x; \mu, \sigma^2)$ is a normal density function with mean μ and variance σ^2. The main reason of selecting such structure for nonlinear part is to check the efficiency of nonparametric estimation for wavy functions. Moreover, $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2\boldsymbol{I}_n)$. Four values of σ^2 are investigated in this study, which are $0.01,\ 0.25,\ 1$ and 4.

We consider a relationship between σ^2 and the signal to noise ratio as,

$$\rho^2 = \frac{\boldsymbol{\beta}'\boldsymbol{\beta}}{\sigma^2}. \tag{6.3}$$

The values of ρ^2 corresponding to σ^2 are 6400, 256, 64 and 16 respectively.

In model (6.2), the parametric effect $\boldsymbol{\beta}$ is estimated by a differencing procedure. Optimal differencing weights do not have analytic expressions but may be calculated easily using an optimization routine. Hall et al. (1990) presents weights to order $m = 10$. These contain some minor errors. We use a fourth-order differencing coefficients, $d_0 = 0.8873$, $d_1 = -0.3099$, $d_2 = -0.2464$, $d_3 = -0.1901$, and $d_4 = -0.1409$ in which $m = 4$.

All computations were conducted using the statistical package R 3.0.0. and consequent results are presented in Tables 1 to 12. We numerically estimated the trace of covariance matrix and risk function of estimators, which heavily depend on k, γ and ρ^2. For estimating the non-linear part, we simulated responses from model (6.2) for $n = 1000$ observations. The $\hat{R}(.)$ and $\delta = \hat{R}(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}, \boldsymbol{\beta}) - \hat{R}(\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(k), \boldsymbol{\beta})$ are plotted in Figures 1 and 2, respectively. In Figure 3, the nonparametric part of the model (6.2) is plotted. This function is difficult to be estimated and provides a good test case for the nonparametric regression method. In the continuation, this figure shows the fitted function by kernel smoothing after estimating the linear part of the model by $\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(\hat{k}_{RAK1})$, that is, $\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(\hat{k}_{RAK1})$, for $\gamma = 0.99$ and different values of ρ^2 (we realized that results for other values of γ will not make significant changes).

6.2 Real Data Example

To motivate the problem of linearly constrained estimation in a SRM, we consider the hedonic prices of housing attributes. Housing prices are very much affected by lot size. The SRM that follows was estimated by Ho (1995) using semiparametric least squares. The data consist of 92 detached homes in Ottowa area that were sold during 1987. The variables are defined as follows: The dependent variable y is sale price (SP), the independent variable include lot size (lot area = LT), square footage of housing (SFH), average neighborhood income (ANI), distance to highway (DHW), presence of garage (GAR), fireplace (FP). We first consider the pure parametric model:

$$
\begin{aligned}
(SP)_i &= \beta_0 + \beta_1(LT)_i + \beta_2(SFH)_i + \beta_3(FP)_i \\
&\quad + \beta_4(DHW)_i + \beta_5(GAR)_i + \beta_6(ANI)_i + \epsilon_i. \qquad (6.4)
\end{aligned}
$$

Estimation details of the above model are summarized in Table 15. According to this table, model (6.4) does not fit to the given data adequately. An appropriate approach is to replace the pure parametric model with a semiparametric model and considering one of explanatory variables as a non-parametric component instead. For this aim, we plotted the dependent variable (SP) versus explanatory variables (except for the binary ones) to find the type of relation (linear or non linear)

Table 1: Evaluation of DGRE at different k estimators in model (6.2) with $\rho^2 = 6400$, $\gamma = 0.80$ and $\lambda_1/\lambda_6 = 46.04$

k Coefficients	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{Med}$	$\hat{k}_{HM}$
$\hat{\beta}_1$	3.004356	3.004355	3.004352	3.004293	3.004346	3.004350	2.514849
$\hat{\beta}_2$	1.001238	1.001241	1.001246	1.001357	1.001258	1.001250	1.127386
$\hat{\beta}_3$	-3.005162	-3.005151	-3.005134	-3.004742	-3.005092	-3.005121	-1.810047
$\hat{\beta}_4$	2.009390	2.009391	2.009394	2.009450	2.009400	2.009396	1.895689
$\hat{\beta}_5$	-5.004895	-5.004880	-5.004854	-5.004289	-5.004795	-5.004836	-3.190825
$\hat{\beta}_6$	3.995257	3.995240	3.995209	3.994540	3.995139	3.995187	2.356858
$tr(\hat{Cov})$	0.0003777216	0.0003777189	0.0003777142	0.0003776122	0.0003777035	0.0003777108	0.000151663
$\hat{R}$	0.0003777216	0.0003777195	0.0003777191	0.0003786921	0.0003777330	0.0003777212	2.793306
k Coefficients	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$
$\hat{\beta}_1$	3.004355	3.004305	3.004084	3.004351	3.004347	2.631451	3.003164
$\hat{\beta}_2$	1.001241	1.001334	1.001750	1.001249	1.001256	1.134754	1.003456
$\hat{\beta}_3$	-3.005151	-3.004823	-3.003358	-3.005124	-3.005099	-1.999930	-2.997327
$\hat{\beta}_4$	2.009391	2.009438	2.009647	2.009395	2.009399	1.950808	2.010500
$\hat{\beta}_5$	-5.004880	-5.004405	-5.002291	-5.004840	-5.004805	-3.489788	-4.993585
$\hat{\beta}_6$	3.995240	3.994678	3.992179	3.995192	3.995150	2.566833	3.981902
$tr(\hat{Cov})$	0.0003777189	0.0003776333	0.0003772522	0.0003777116	0.0003777053	0.0001784408	0.0003756868
$\hat{R}$	0.0003777195	0.0003783373	0.0003971309	0.0003777205	0.0003777293	2.341915	0.0007482647
k Coefficients	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
$\hat{\beta}_1$	2.870722	3.004149	2.809949	3.004200	3.004352	2.927269	3.004199
$\hat{\beta}_2$	1.106150	1.001628	1.122254	1.001532	1.001246	1.080266	1.001535
$\hat{\beta}_3$	-2.498696	-3.003790	-2.349657	-3.004126	-3.005134	-2.665261	-3.004118
$\hat{\beta}_4$	2.030268	2.009586	2.016734	2.009538	2.009394	2.034842	2.009539
$\hat{\beta}_5$	-4.257040	-5.002915	-4.030750	-5.003400	-5.004855	-4.506684	-5.003388
$\hat{\beta}_6$	3.205116	3.992916	3.000441	3.993489	3.995209	3.448649	3.993475
$tr(\hat{Cov})$	0.0002633842	0.0003773645	0.0002355272	0.000377452	0.0003777143	0.0002973988	0.0003774499
$\hat{R}$	0.941015	0.0003888697	1.368764	0.000384013	0.0003777191	0.4989723	0.0003841134

Table 2: Evaluation of DGRE at different k estimators in model (6.2) with $\rho^2 = 6400$, $\gamma = 0.90$ and $\lambda_1/\lambda_6 = 112.09$

k Coefficients	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{Med}$	$\hat{k}_{HM}$
$\hat{\beta}_1$	2.972062	2.972056	2.972053	2.971736	2.972001	2.971965	2.551066
$\hat{\beta}_2$	0.993511	0.993518	0.993523	0.993987	0.993600	0.993652	1.073340
$\hat{\beta}_3$	-2.999750	-2.999716	-2.999696	-2.997630	-2.999353	-2.999123	-1.880966
$\hat{\beta}_4$	2.000925	2.000930	2.000933	2.001236	2.000983	2.001017	1.901762
$\hat{\beta}_5$	-5.020892	-5.020839	-5.020808	-5.017594	-5.020274	-5.019916	-3.268602
$\hat{\beta}_6$	4.034303	4.034237	4.034198	4.030195	4.033533	4.033087	2.326446
$tr(\hat{Cov})$	0.0007962156	0.0007961966	0.0007961853	0.0007950288	0.0007959932	0.0007958642	0.0003411472
$\hat{R}$	0.0007962156	0.0007962049	0.0007962065	0.0008276470	0.0007971395	0.0007987253	2.7399526655
k Coefficients	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$
$\hat{\beta}_1$	2.972056	2.971856	2.971331	2.971979	2.972013	2.083282	2.968637
$\hat{\beta}_2$	0.993518	0.993812	0.994573	0.993632	0.993582	0.981399	0.998327
$\hat{\beta}_3$	-2.999716	-2.998411	-2.995016	-2.999210	-2.999433	-1.199159	-2.978050
$\hat{\beta}_4$	2.000930	2.001122	2.001618	2.001004	2.000971	1.620867	2.004018
$\hat{\beta}_5$	-5.020839	-5.018809	-5.013526	-5.020051	-5.020398	-2.187627	-4.987127
$\hat{\beta}_6$	4.034237	4.031707	4.025133	4.033255	4.033687	1.658107	3.992422
$tr(\hat{Cov})$	0.0007961966	0.0007954658	0.0007935676	0.0007959129	0.0007960378	0.0001755045	0.0007841445
$\hat{R}$	0.0007962049	0.0008084911	0.0009557303	0.0007980365	0.0007967706	3.5079397572	0.0041231945
k Coefficients	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
$\hat{\beta}_1$	2.794510	2.971334	2.817214	2.971177	2.972053	2.876379	2.971661
$\hat{\beta}_2$	1.078670	0.994569	1.075416	0.994795	0.993523	1.060850	0.994096
$\hat{\beta}_3$	-2.370985	-2.995034	-2.428243	-2.994026	-2.999697	-2.596006	-2.997148
$\hat{\beta}_4$	2.007444	2.001615	2.013764	2.001761	2.000933	2.024517	2.001307
$\hat{\beta}_5$	-4.039098	-5.013555	-4.128812	-5.011986	-5.020809	-4.391303	-5.016843
$\hat{\beta}_6$	2.966570	4.025170	3.051735	4.023219	4.034199	3.314758	4.029260
$tr(\hat{Cov})$	0.0005045339	0.0007935781	0.0005267583	0.0007930151	0.0007961855	0.0005963682	0.0007947591
$\hat{R}$	1.4226460829	0.0009544667	1.2489691832	0.0010297939	0.0007962065	0.7479770423	0.0008438797

Table 3: Evaluation of DGRE at different k estimators in model (6.2) with $\rho^2 = 6400$, $\gamma = 0.99$ and $\lambda_1/\lambda_6 = 1421.50$

k Coefficients	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{Med}$	$\hat{k}_{HM}$
$\hat{\beta}_1$	3.049552	3.049607	3.049678	3.050472	3.049811	3.049727	0.651822
$\hat{\beta}_2$	0.975267	0.975352	0.975461	0.976686	0.975665	0.975536	0.459765
$\hat{\beta}_3$	-2.907825	-2.907577	-2.907257	-2.903677	-2.906662	-2.907039	-0.194824
$\hat{\beta}_4$	1.988085	1.988120	1.988165	1.988669	1.988250	1.988196	0.530299
$\hat{\beta}_5$	-4.987981	-4.987641	-4.987201	-4.982282	-4.986384	-4.986901	-0.039708
$\hat{\beta}_6$	3.902959	3.902288	3.901420	3.891722	3.899807	3.900827	0.511742
$tr(\hat{Cov})$	0.0087623038	0.0087599894	0.0087569951	0.0087235894	0.0087514359	0.0087549522	0.0000229503
$\hat{R}$	0.0087623038	0.0087606276	0.0087603522	0.0089016301	0.0087654991	0.0087613892	0.2237553665
k Coefficients	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$
$\hat{\beta}_1$	3.049607	3.050101	3.051108	3.049725	3.049784	1.049561	3.067550
$\hat{\beta}_2$	0.975352	0.976112	0.977677	0.975533	0.975623	0.574500	1.007175
$\hat{\beta}_3$	-2.907577	-2.905358	-2.900766	-2.907049	-2.906785	-0.094210	-2.806860
$\hat{\beta}_4$	1.988120	1.988433	1.989072	1.988195	1.988232	0.749305	1.999194
$\hat{\beta}_5$	-4.987641	-4.984593	-4.978282	-4.986916	-4.986552	-0.478778	-4.848288
$\hat{\beta}_6$	3.902288	3.896274	3.883855	3.900857	3.900139	0.639113	3.637512
$tr(\hat{Cov})$	0.0087599896	0.0087392614	0.0086965360	0.0087550543	0.0087525799	0.0001312755	0.0078709256
$\hat{R}$	0.0087606277	0.0088024154	0.0092091713	0.0087613138	0.0087638392	1.0928604116	0.0954177767
k Coefficients	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
$\hat{\beta}_1$	1.585404	3.055222	1.375080	3.054134	3.049670	1.852262	3.053972
$\hat{\beta}_2$	0.725780	0.984272	0.666076	0.982493	0.975448	0.802196	0.982230
$\hat{\beta}_3$	-0.532420	-2.881032	-0.354918	-2.886418	-2.907295	-0.769547	-2.887209
$\hat{\beta}_4$	1.052536	1.991663	0.931753	1.990980	1.988160	1.209500	1.990878
$\hat{\beta}_5$	-1.242849	-4.951112	-0.935504	-4.958535	-4.987253	-1.649032	-4.959625
$\hat{\beta}_6$	0.842954	3.830879	0.755942	3.845274	3.901523	0.972066	3.847391
$tr(\hat{Cov})$	0.0004656052	0.0085154908	0.0003049538	0.0085644936	0.0087573518	0.0007358490	0.0085717141
$\hat{R}$	2.5232665780	0.0156244733	1.9892476387	0.0131501171	0.0087602730	3.0435274818	0.0128313082

Table 4: Evaluation of DGRE at different k estimators in model (6.2) with $\rho^2 = 256$, $\gamma = 0.80$ and $\lambda_1/\lambda_6 = 48.71$

k Coefficients	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{Med}$	$\hat{k}_{HM}$
$\hat{\beta}_1$	3.049869	3.049808	3.049787	3.049598	3.049854	3.049859	2.971958
$\hat{\beta}_2$	0.974882	0.975103	0.975182	0.975869	0.974936	0.974918	1.123622
$\hat{\beta}_3$	-3.082859	-3.082381	-3.082211	-3.080727	-3.082744	-3.082781	-2.714722
$\hat{\beta}_4$	1.981817	1.981902	1.981933	1.982196	1.981838	1.981831	2.021890
$\hat{\beta}_5$	-5.012251	-5.011645	-5.011429	-5.009546	-5.012105	-5.012152	-4.531041
$\hat{\beta}_6$	4.026698	4.025845	4.025541	4.022893	4.026492	4.026559	3.408198
$tr(\hat{Cov})$	0.01010803	0.01010464	0.01010343	0.01009291	0.01010721	0.01010747	0.00774156
$\hat{R}$	0.01010803	0.01010602	0.01010597	0.01012039	0.01010729	0.01010751	0.55748932
k Coefficients	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$
$\hat{\beta}_1$	3.049808	3.049510	3.048701	3.049656	3.049705	2.984263	3.037034
$\hat{\beta}_2$	0.975103	0.976184	0.979069	0.975657	0.975480	1.108641	1.014213
$\hat{\beta}_3$	-3.082383	-3.080044	-3.073788	-3.081184	-3.081567	-2.757539	-2.995349
$\hat{\beta}_4$	1.981902	1.982317	1.983416	1.982115	1.982047	2.019906	1.995984
$\hat{\beta}_5$	-5.011647	-5.008680	-5.000738	-5.010127	-5.010612	-4.588495	-4.900471
$\hat{\beta}_6$	4.025847	4.021674	4.010525	4.023709	4.024392	3.476063	3.872607
$tr(\hat{Cov})$	0.01010465	0.01008808	0.01004385	0.01009615	0.01009887	0.00799294	0.00950182
$\hat{R}$	0.01010602	0.01013594	0.01053735	0.01011312	0.01010896	0.45728659	0.05160637
k Coefficients	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
$\hat{\beta}_1$	3.034813	3.048745	3.026321	3.049002	3.049792	3.043895	3.049169
$\hat{\beta}_2$	1.019888	0.978914	1.039526	0.978005	0.975162	0.994827	0.977410
$\hat{\beta}_3$	-2.982263	-3.074124	-2.935975	-3.076097	-3.082254	-3.039143	-3.077389
$\hat{\beta}_4$	1.997859	1.983358	2.003976	1.983012	1.981925	1.989247	1.982785
$\hat{\beta}_5$	-4.883617	-5.001165	-4.823709	-5.003671	-5.011483	-4.956612	-5.005311
$\hat{\beta}_6$	3.849936	4.011124	3.770527	4.014639	4.025617	3.949183	4.016941
$tr(\hat{Cov})$	0.00941361	0.01004622	0.00910656	0.01006016	0.01010373	0.00980161	0.01006929
$\hat{R}$	0.06425470	0.01050400	0.12022715	0.01033505	0.01010595	0.02083032	0.01024943

Table 5: Evaluation of DGRE at different k estimators in model (6.2) with $\rho^2 = 256$, $\gamma = 0.90$ and $\lambda_1/\lambda_6 = 99.51$

k Coefficients	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{Med}$	$\hat{k}_{HM}$
$\hat{\beta}_1$	3.057420	3.057315	3.057270	3.057149	3.057226	3.057251	2.709857
$\hat{\beta}_2$	0.933302	0.933604	0.933733	0.934076	0.933858	0.933787	1.125468
$\hat{\beta}_3$	-2.979465	-2.978614	-2.978250	-2.977283	-2.977898	-2.978099	-2.020710
$\hat{\beta}_4$	1.977590	1.977722	1.977779	1.977928	1.977833	1.977802	1.906043
$\hat{\beta}_5$	-4.856843	-4.855670	-4.855168	-4.853836	-4.854684	-4.854960	-3.463893
$\hat{\beta}_6$	3.914905	3.913274	3.912575	3.910722	3.911902	3.912286	2.419762
$tr(\hat{Cov})$	0.01961847	0.01960567	0.01960019	0.01958566	0.01959491	0.01959793	0.00898770
$\hat{R}$	0.01961847	0.01961055	0.01961013	0.01961768	0.01961143	0.01961048	2.17724891
k Coefficients	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$
$\hat{\beta}_1$	3.057315	3.057182	3.056615	3.057282	3.057256	2.961360	3.054950
$\hat{\beta}_2$	0.933603	0.933984	0.935578	0.933697	0.933771	1.053626	0.940066
$\hat{\beta}_3$	-2.978616	-2.977543	-2.973036	-2.978351	-2.978142	-2.562245	-2.960251
$\hat{\beta}_4$	1.977722	1.977888	1.978578	1.977763	1.977795	1.999516	1.980483
$\hat{\beta}_5$	-4.855673	-4.854194	-4.847983	-4.855307	-4.855019	-4.267633	-4.830341
$\hat{\beta}_6$	3.913278	3.911221	3.902594	3.912769	3.912368	3.180938	3.878203
$tr(\hat{Cov})$	0.01960570	0.01958957	0.01952197	0.01960171	0.01959857	0.01415444	0.01933130
$\hat{R}$	0.01961055	0.01961442	0.01979831	0.01961007	0.01961035	0.73068934	0.02176133
k Coefficients	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
$\hat{\beta}_1$	2.994734	3.056264	2.986837	3.056352	3.057288	3.000304	3.056330
$\hat{\beta}_2$	1.027874	0.936548	1.034757	0.936306	0.933681	1.022651	0.936365
$\hat{\beta}_3$	-2.670291	-2.970285	-2.642926	-2.970971	-2.978396	-2.690433	-2.970804
$\hat{\beta}_4$	2.002022	1.978995	2.001922	1.978891	1.977756	2.001863	1.978917
$\hat{\beta}_5$	-4.422934	-4.844189	-4.383774	-4.845135	-4.855369	-4.451680	-4.844904
$\hat{\beta}_6$	3.358331	3.897336	3.312567	3.898646	3.912855	3.392378	3.898326
$tr(\hat{Cov})$	0.01542474	0.01948080	0.01509422	0.01949106	0.01960238	0.01567194	0.01948855
$\hat{R}$	0.46028387	0.02004235	0.52586524	0.01997224	0.01961008	0.41362480	0.01998881

Table 6: Evaluation of DGRE at different k estimators in model (6.2) with $\rho^2 = 256$, $\gamma = 0.99$ and $\lambda_1/\lambda_6 = 1473.93$

k Coefficients	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{Med}$	$\hat{k}_{HM}$
$\hat{\beta}_1$	3.135295	3.133190	3.133680	3.130079	3.132806	3.133725	1.663473
$\hat{\beta}_2$	0.881898	0.890264	0.888413	0.900936	0.891677	0.888239	0.844627
$\hat{\beta}_3$	-2.539220	-2.521684	-2.525597	-2.498722	-2.518684	-2.525965	-0.566291
$\hat{\beta}_4$	1.953212	1.955981	1.955381	1.959295	1.956434	1.955324	1.200373
$\hat{\beta}_5$	-5.067491	-5.042568	-5.048137	-5.009801	-5.038295	-5.048661	-1.673165
$\hat{\beta}_6$	3.650347	3.605083	3.615143	3.546528	3.597387	3.616091	0.905304
$tr(\hat{Cov})$	0.22922477	0.22502655	0.22595646	0.21965040	0.22431644	0.22604410	0.01951292
$\hat{R}$	0.22922477	0.22800917	0.22776938	0.23490124	0.22838430	0.22776157	2.89394540
k Coefficients	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$
$\hat{\beta}_1$	3.133200	3.132077	3.115393	3.134612	3.134345	1.939792	3.084587
$\hat{\beta}_2$	0.890224	0.894280	0.937006	0.884737	0.885813	0.926650	0.981837
$\hat{\beta}_3$	-2.521769	-2.513125	-2.415291	-2.533313	-2.531062	-0.804007	-2.294093
$\hat{\beta}_4$	1.955968	1.957258	1.968357	1.954168	1.954526	1.373283	1.973194
$\hat{\beta}_5$	-5.042689	-5.030372	-4.889442	-5.059105	-5.055907	-2.144064	-4.710892
$\hat{\beta}_6$	3.605301	3.583163	3.340575	3.635047	3.629231	1.065505	3.060350
$tr(\hat{Cov})$	0.22504669	0.22300681	0.20121884	0.22780157	0.22726164	0.03013339	0.17727353
$\hat{R}$	0.22800094	0.22950812	0.32402803	0.22814732	0.22791839	3.21936787	0.56609848
k Coefficients	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
$\hat{\beta}_1$	2.392624	3.118746	2.232959	3.122640	3.134732	2.503086	3.121432
$\hat{\beta}_2$	1.036026	0.930154	1.002132	0.921348	0.884248	1.055245	0.924190
$\hat{\beta}_3$	-1.238733	-2.431926	-1.077072	-2.452725	-2.534333	-1.358124	-2.446080
$\hat{\beta}_4$	1.648114	1.966923	1.552945	1.964870	1.954004	1.712270	1.965557
$\hat{\beta}_5$	-2.962581	-4.913605	-2.664676	-4.943699	-5.060554	-3.177603	-4.934098
$\hat{\beta}_6$	1.436590	3.380788	1.284756	3.431662	3.637685	1.561223	3.415336
$tr(\hat{Cov})$	0.05599204	0.20476037	0.04532551	0.20928024	0.22804667	0.06477722	0.20782489
$\hat{R}$	2.99492195	0.29959686	3.18336256	0.27326593	0.22828377	2.79756056	0.28113523

Table 7: Evaluation of DGRE at different k estimators in model (6.2) with $\rho^2 = 64$, $\gamma = 0.80$ and $\lambda_1/\lambda_6 = 37.13$

k Coefficients	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{Med}$	$\hat{k}_{HM}$
$\hat{\beta}_1$	2.950732	2.950688	2.950646	2.950550	2.950609	2.950567	2.941361
$\hat{\beta}_2$	1.025936	1.026176	1.026401	1.026902	1.026595	1.026815	1.050219
$\hat{\beta}_3$	-2.966258	-2.965034	-2.963888	-2.961327	-2.962899	-2.961774	-2.828627
$\hat{\beta}_4$	2.000264	2.000334	2.000400	2.000545	2.000457	2.000520	2.004693
$\hat{\beta}_5$	-5.059252	-5.057530	-5.055918	-5.052315	-5.054527	-5.052943	-4.864470
$\hat{\beta}_6$	4.045204	4.043164	4.041257	4.036996	4.039611	4.037739	3.821437
$tr(\hat{Cov})$	0.03712300	0.03709150	0.03706204	0.03699626	0.03703662	0.03700772	0.03371666
$\hat{R}$	0.03712300	0.03710018	0.03709452	0.03713649	0.03710181	0.03712376	0.12910300
k Coefficients	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$
$\hat{\beta}_1$	2.950688	2.950600	2.950375	2.950631	2.950640	2.942686	2.950083
$\hat{\beta}_2$	1.026174	1.026643	1.027768	1.026477	1.026434	1.048061	1.029114
$\hat{\beta}_3$	-2.965045	-2.962652	-2.956879	-2.963498	-2.963719	-2.842269	-2.949905
$\hat{\beta}_4$	2.000334	2.000471	2.000792	2.000422	2.000410	2.004572	2.001163
$\hat{\beta}_5$	-5.057546	-5.054179	-5.046053	-5.055370	-5.055680	-4.883881	-5.036231
$\hat{\beta}_6$	4.043183	4.039199	4.029602	4.040608	4.040975	3.843123	4.018035
$tr(\hat{Cov})$	0.03709178	0.03703026	0.03688220	0.03705202	0.03705768	0.03404239	0.03670398
$\hat{R}$	0.03710031	0.03710539	0.03738731	0.03709605	0.03709497	0.11253179	0.03822854
k Coefficients	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
$\hat{\beta}_1$	2.946963	2.950362	2.946941	2.950322	2.950664	2.947438	2.950402
$\hat{\beta}_2$	1.039266	1.027830	1.039322	1.028023	1.026303	1.038028	1.027638
$\hat{\beta}_3$	-2.894652	-2.956559	-2.894333	-2.955564	-2.964390	-2.901664	-2.957550
$\hat{\beta}_4$	2.003457	2.000809	2.003467	2.000863	2.000371	2.003230	2.000755
$\hat{\beta}_5$	-4.958204	-5.045603	-4.957751	-5.044201	-5.056624	-4.968126	-5.046998
$\hat{\beta}_6$	3.927400	4.029072	3.926881	4.027419	4.042092	3.938802	4.030718
$tr(\hat{Cov})$	0.03531712	0.03687402	0.03530922	0.03684854	0.03707494	0.03549067	0.03689940
$\hat{R}$	0.06293020	0.03741396	0.06316034	0.03750436	0.03709513	0.05812306	0.03733508

Table 8: Evaluation of DGRE at different k estimators in model (6.2) with $\rho^2 = 64$, $\gamma = 0.90$ and $\lambda_1/\lambda_6 = 109.04$

k Coefficients	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{Med}$	$\hat{k}_{HM}$
$\hat{\beta}_1$	3.148393	3.147808	3.146978	2.948187	3.143685	3.146536	3.147142
$\hat{\beta}_2$	1.061974	1.062869	1.064127	1.212565	1.069005	1.064793	1.063879
$\hat{\beta}_3$	-2.901293	-2.898666	-2.894962	-2.316232	-2.880502	-2.892999	-2.895695
$\hat{\beta}_4$	1.777067	1.777552	1.778233	1.826976	1.780847	1.778592	1.778099
$\hat{\beta}_5$	-4.914891	-4.911986	-4.907888	-4.230865	-4.891860	-4.905714	-4.908698
$\hat{\beta}_6$	3.817779	3.813308	3.807010	2.914636	3.782491	3.803675	3.808254
$tr(\hat{Cov})$	0.07810714	0.07796692	0.07776957	0.05151994	0.07700304	0.07766514	0.07780855
$\hat{R}$	0.07810714	0.07800354	0.07798141	1.05325701	0.07925289	0.07802797	0.07797436
k Coefficients	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$
$\hat{\beta}_1$	3.147812	3.146099	3.140474	3.147363	3.147095	3.067991	3.004999
$\hat{\beta}_2$	1.062863	1.065447	1.073592	1.063546	1.063951	1.148208	1.187776
- $\hat{\beta}_3$	2.898683	-2.891067	-2.866746	-2.896676	-2.895482	-2.614794	-2.446521
$\hat{\beta}_4$	1.777549	1.778944	1.783269	1.777918	1.778137	1.816367	1.826240
- $\hat{\beta}_5$	4.912005	-4.903574	-4.876576	-4.909784	-4.908463	-4.589595	-4.390040
$\hat{\beta}_6$	3.813337	3.800394	3.759273	3.809922	3.807893	3.352145	3.099570
$tr(\hat{Cov})$	0.07796783	0.07756246	0.07627977	0.07786081	0.07779723	0.06399015	0.05670751
$\hat{R}$	0.07800398	0.07811291	0.08240057	0.07797371	0.07797583	0.38704689	0.74363470
k Coefficients	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
$\hat{\beta}_1$	3.122379	3.139840	3.102144	3.143185	3.147523	3.139930	3.143045
$\hat{\beta}_2$	1.096789	1.074478	1.118542	1.069730	1.063302	1.074354	1.069933
$\hat{\beta}_3$	-2.794616	-2.864071	-2.722429	-2.878339	-2.897392	-2.864449	-2.877733
$\hat{\beta}_4$	1.794942	1.783733	1.804871	1.781232	1.777787	1.783668	1.781339
$\hat{\beta}_5$	-4.795794	-4.873599	-4.713861	-4.889459	-4.910576	-4.874019	-4.888787
$\hat{\beta}_6$	3.639190	3.754769	3.521840	3.778833	3.811140	3.755404	3.777809
$tr(\hat{Cov})$	0.07257897	0.07613978	0.06902507	0.07688893	0.07789897	0.07615950	0.07685699
$\hat{R}$	0.12662656	0.08322487	0.20982989	0.07962493	0.07797963	0.08310454	0.07973745

Table 9: Evaluation of DGRE at different k estimators in model (6.2) with $\rho^2 = 64$, $\gamma = 0.99$ and $\lambda_1/\lambda_6 = 1614.55$

k Coefficients	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{Med}$	$\hat{k}_{HM}$
$\hat{\beta}_1$	2.913694	2.912974	2.910501	2.570757	2.893840	2.901656	2.885996
$\hat{\beta}_2$	1.097572	1.102359	1.113406	1.163427	1.145785	1.134569	1.154056
$\hat{\beta}_3$	-3.319342	-3.301389	-3.257923	-2.240961	-3.106917	-3.164352	-3.059353
$\hat{\beta}_4$	2.288878	2.292934	2.302044	2.230638	2.325792	2.318213	2.330712
$\hat{\beta}_5$	-4.846115	-4.821688	-4.762476	-3.350579	-4.555977	-4.634662	-4.490683
$\hat{\beta}_6$	3.842328	3.798265	3.693250	1.887756	3.346863	3.475227	3.243706
$tr(\hat{Cov})$	0.91370911	0.89910859	0.86478062	0.35790139	0.75607707	0.79557007	0.72498702
$\hat{R}$	0.91370911	0.90194441	0.89573146	2.72783313	1.04730599	0.96531918	1.12999313
k Coefficients	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$
$\hat{\beta}_1$	2.912977	2.907335	2.837249	2.913171	2.912972	2.064368	2.416801
$\hat{\beta}_2$	1.102342	1.122825	1.179560	1.101174	1.102370	1.012095	1.124633
$\hat{\beta}_3$	-3.301456	-3.218209	-2.851222	-3.305881	-3.301347	-1.500591	-1.989828
$\hat{\beta}_4$	2.292919	2.309483	2.337829	2.291935	2.292943	1.891465	2.137692
$\hat{\beta}_5$	-4.821780	-4.708286	-4.203593	-4.827802	-4.821631	-2.301121	-2.995842
$\hat{\beta}_6$	3.798431	3.599372	2.825531	3.809253	3.798162	1.192780	1.607586
$tr(\hat{Cov})$	0.89916329	0.83464348	0.60444339	0.90273858	0.89907469	0.17868002	0.28681087
$\hat{R}$	0.90197803	0.91338954	1.56386082	0.90434433	0.90192362	3.47835190	3.08670043
k Coefficients	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
$\hat{\beta}_1$	2.674272	2.870463	2.610506	2.880547	2.913378	2.801847	2.889527
$\hat{\beta}_2$	1.181896	1.165635	1.171494	1.158695	1.099834	1.186260	1.150606
$\hat{\beta}_3$	-2.436321	-2.981369	-2.312612	-3.030047	-3.310921	-2.740060	-3.079842
$\hat{\beta}_4$	2.284456	2.336121	2.252357	2.333132	2.290802	2.332141	2.328743
$\hat{\beta}_5$	-3.625235	-4.383373	-3.451461	-4.450393	-4.834660	-4.049383	-4.518824
$\hat{\beta}_6$	2.144955	3.080720	1.977819	3.181563	3.821611	2.623863	3.287794
$tr(\hat{Cov})$	0.42337975	0.67699347	0.38074734	0.70652800	0.90682967	0.54912879	0.73820517
$\hat{R}$	2.39923579	1.28403137	2.61210869	1.18574096	0.90746320	1.80380021	1.09303675

Table 10: Evaluation of DGRE at different k estimators in model (6.2) with $\rho^2 = 16$, $\gamma = 0.80$ and $\lambda_1/\lambda_6 = 42.80$

k Coefficients	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{Med}$	$\hat{k}_{HM}$
$\hat{\beta}_1$	3.219828	3.219017	3.218011	3.217332	3.217772	3.217851	3.211745
$\hat{\beta}_2$	0.826672	0.828063	0.829751	0.830865	0.830145	0.830015	0.839412
$\hat{\beta}_3$	-2.951549	-2.946267	-2.939816	-2.935533	-2.938305	-2.938802	-2.901981
$\hat{\beta}_4$	2.047202	2.047645	2.048170	2.048509	2.048290	2.048251	2.050890
$\hat{\beta}_5$	-4.796018	-4.788833	-4.780058	-4.774231	-4.778003	-4.778679	-4.728570
$\hat{\beta}_6$	3.738609	3.730440	3.720488	3.713895	3.718161	3.718926	3.662664
$tr(\hat{Cov})$	0.15375578	0.15318305	0.15248630	0.15202533	0.15232357	0.15237706	0.14845942
$\hat{R}$	0.15375578	0.15333138	0.15321306	0.15337328	0.15324801	0.15323391	0.16091014
k Coefficients	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$
$\hat{\beta}_1$	3.219045	3.218358	3.214759	3.218922	3.218833	3.209074	3.214445
$\hat{\beta}_2$	0.828016	0.829174	0.834934	0.828226	0.828376	0.843151	0.835415
$\hat{\beta}_3$	-2.946445	-2.942028	-2.919716	-2.945648	-2.945074	-2.886894	-2.917830
$\hat{\beta}_4$	2.047631	2.047992	2.049691	2.047697	2.047744	2.051805	2.049825
$\hat{\beta}_5$	-4.789075	-4.783067	-4.752710	-4.787991	-4.787210	-4.708026	-4.750142
$\hat{\beta}_6$	3.730714	3.723897	3.689654	3.729483	3.728597	3.639863	3.686773
$tr(\hat{Cov})$	0.15320226	0.15272487	0.15033447	0.15311600	0.15305390	0.14688152	0.15013395
$\hat{R}$	0.15334082	0.15320459	0.15556862	0.15330104	0.15327656	0.16772493	0.15599499
k Coefficients	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
$\hat{\beta}_1$	3.213386	3.216713	3.213229	3.216778	3.219043	3.213695	3.216854
$\hat{\beta}_2$	0.837010	0.831867	0.837243	0.831763	0.828020	0.836548	0.831640
$\hat{\beta}_3$	-2.911541	-2.931665	-2.910616	-2.932067	-2.946430	-2.913368	-2.932544
$\hat{\beta}_4$	2.050260	2.048808	2.050323	2.048777	2.047632	2.050136	2.048740
$\hat{\beta}_5$	-4.741583	-4.768969	-4.740323	-4.769515	-4.789055	-4.744069	-4.770165
$\hat{\beta}_6$	3.677187	3.707952	3.675778	3.708569	3.730692	3.679968	3.709302
$tr(\hat{Cov})$	0.14946740	0.15161021	0.14936953	0.15165324	0.15320070	0.14966070	0.15170445
$\hat{R}$	0.15766261	0.15367907	0.15793975	0.15364028	0.15334004	0.15713947	0.15359626

Table 11: Evaluation of DGRE at different k estimators in model (6.2) with $\rho^2 = 16$, $\gamma = 0.90$ and $\lambda_1/\lambda_6 = 104.85$

k Coefficients	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{Med}$	$\hat{k}_{HM}$
$\hat{\beta}_1$	3.118199	3.117047	3.115523	2.016381	3.105248	3.110729	3.118129
$\hat{\beta}_2$	1.322290	1.324053	1.326231	1.093622	1.337722	1.332172	1.322401
$\hat{\beta}_3$	-2.753157	-2.742672	-2.729443	-0.802076	-2.653540	-2.691613	-2.752504
$\hat{\beta}_4$	1.844819	1.846026	1.847491	1.359887	1.854616	1.851314	1.844896
$\hat{\beta}_5$	-5.260253	-5.246011	-5.227999	-2.089762	-5.123754	-5.176236	-5.259368
$\hat{\beta}_6$	3.787183	3.768557	3.745147	1.400041	3.612808	3.678767	3.786022
$tr(\hat{Cov})$	0.32333398	0.32091972	0.31789438	0.05041252	0.30097406	0.30936900	0.32318325
$\hat{R}$	0.32333398	0.32157874	0.32121806	3.52310551	0.35507958	0.33087045	0.32318583
k Coefficients	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$
$\hat{\beta}_1$	3.117079	3.115714	3.106113	3.117304	3.117089	3.098233	2.806701
$\hat{\beta}_2$	1.324005	1.325967	1.336917	1.323669	1.323990	1.343486	1.347024
$\hat{\beta}_3$	-2.742962	-2.731065	-2.659252	-2.744975	-2.743048	-2.610386	-1.808756
$\hat{\beta}_4$	1.845994	1.847315	1.854155	1.845765	1.845984	1.857690	1.783053
$\hat{\beta}_5$	-5.246404	-5.230209	-5.131653	-5.249142	-5.246521	-5.063796	-3.855234
$\hat{\beta}_6$	3.769070	3.748011	3.622649	3.772643	3.769223	3.539081	2.372777
$tr(\hat{Cov})$	0.32098620	0.31826400	0.30222179	0.32144883	0.32100595	0.29167855	0.15463652
$\hat{R}$	0.32160952	0.32115366	0.35059180	0.32185110	0.32161885	0.39787729	2.31356246
k Coefficients	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
$\hat{\beta}_1$	3.112554	3.108547	3.108497	3.111418	3.117462	3.116720	3.112833
$\hat{\beta}_2$	1.330052	1.334515	1.334567	1.331390	1.323430	1.324534	1.329715
$\hat{\beta}_3$	-2.705424	-2.675900	-2.675544	-2.696753	-2.746403	-2.739774	-2.707590
$\hat{\beta}_4$	1.849981	1.852743	1.852775	1.850826	1.845602	1.846353	1.849765
$\hat{\beta}_5$	-5.195178	-5.154623	-5.154134	-5.183292	-5.251082	-5.242069	-5.198145
$\hat{\beta}_6$	3.702904	3.651441	3.650825	3.687737	3.775177	3.763421	3.706701
$tr(\hat{Cov})$	0.31245991	0.30588195	0.30580349	0.31051649	0.32177713	0.32025512	0.31294706
$\hat{R}$	0.32558556	0.33920414	0.33942014	0.32867540	0.32205168	0.32132537	0.32493581

Table 12: Evaluation of DGRE at different k estimators in model (6.2) with $\rho^2 = 16$, $\gamma = 0.99$ and $\lambda_1/\lambda_6 = 1541.17$

k Coefficients	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{Med}$	$\hat{k}_{HM}$
$\hat{\beta}_1$	3.231443	3.239172	3.242055	3.184787	3.236818	3.239318	3.204426
$\hat{\beta}_2$	1.407247	1.450885	1.512975	1.624897	1.549284	1.537292	1.609214
$\hat{\beta}_3$	-3.753810	-3.664677	-3.519848	-3.119839	-3.419778	-3.454541	-3.201763
$\hat{\beta}_4$	1.615007	1.651080	1.703705	1.806734	1.735489	1.724885	1.790963
$\hat{\beta}_5$	-5.732655	-5.613253	-5.416868	-4.858993	-5.279432	-5.327337	-4.975073
$\hat{\beta}_6$	4.972183	4.713847	4.313489	3.340830	4.051481	4.141115	3.523665
$tr(\hat{Cov})$	3.65446888	3.39520714	3.01165875	2.16175629	2.77197741	2.85300757	2.31369281
$\hat{R}$	3.65446888	3.47797283	3.46880406	3.99697775	3.57135973	3.52974638	3.88358515
k Coefficients	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$
$\hat{\beta}_1$	3.239073	3.238285	2.952711	3.233053	3.234136	2.922205	3.032423
$\hat{\beta}_2$	1.450142	1.542760	1.638300	1.414876	1.420298	1.631908	1.649668
$\hat{\beta}_3$	-3.666267	-3.438929	-2.568662	-3.738803	-3.727998	-2.515838	-2.719432
$\hat{\beta}_4$	1.650460	1.729706	1.838557	1.621270	1.625731	1.834905	1.842653
$\hat{\beta}_5$	-5.615392	-5.305843	-4.056755	-5.712628	-5.698190	-3.978249	-4.279470
$\hat{\beta}_6$	4.718377	4.100676	2.325688	4.928070	4.896465	2.246826	2.567604
$tr(\hat{Cov})$	3.39967149	2.81632771	1.36231578	3.60952182	3.57749208	1.30196410	1.54816038
$\hat{R}$	3.47970402	3.54781277	4.49781678	3.61214615	3.58513003	4.52986665	4.39695685
k Coefficients	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
$\hat{\beta}_1$	3.169311	3.213757	3.159140	3.218259	3.232538	3.197062	3.226226
$\hat{\beta}_2$	1.633358	1.598660	1.637645	1.592445	1.412380	1.615931	1.578718
$\hat{\beta}_3$	-3.064753	-3.248232	-3.031645	-3.273384	-3.743737	-3.169009	-3.324539
$\hat{\beta}_4$	1.815742	1.780768	1.820534	1.774877	1.619219	1.797607	1.762096
$\hat{\beta}_5$	-4.780435	-5.040507	-4.733029	-5.075798	-5.719216	-4.928773	-5.147301
$\hat{\beta}_6$	3.222669	3.631152	3.153494	3.690469	4.942547	3.449546	3.813572
$tr(\hat{Cov})$	2.06519504	2.40455451	2.00921464	2.45521163	3.62424153	2.25171475	2.56155868
$\hat{R}$	4.06745733	3.81592453	4.10730447	3.77877812	3.62543263	3.93000095	3.70329289

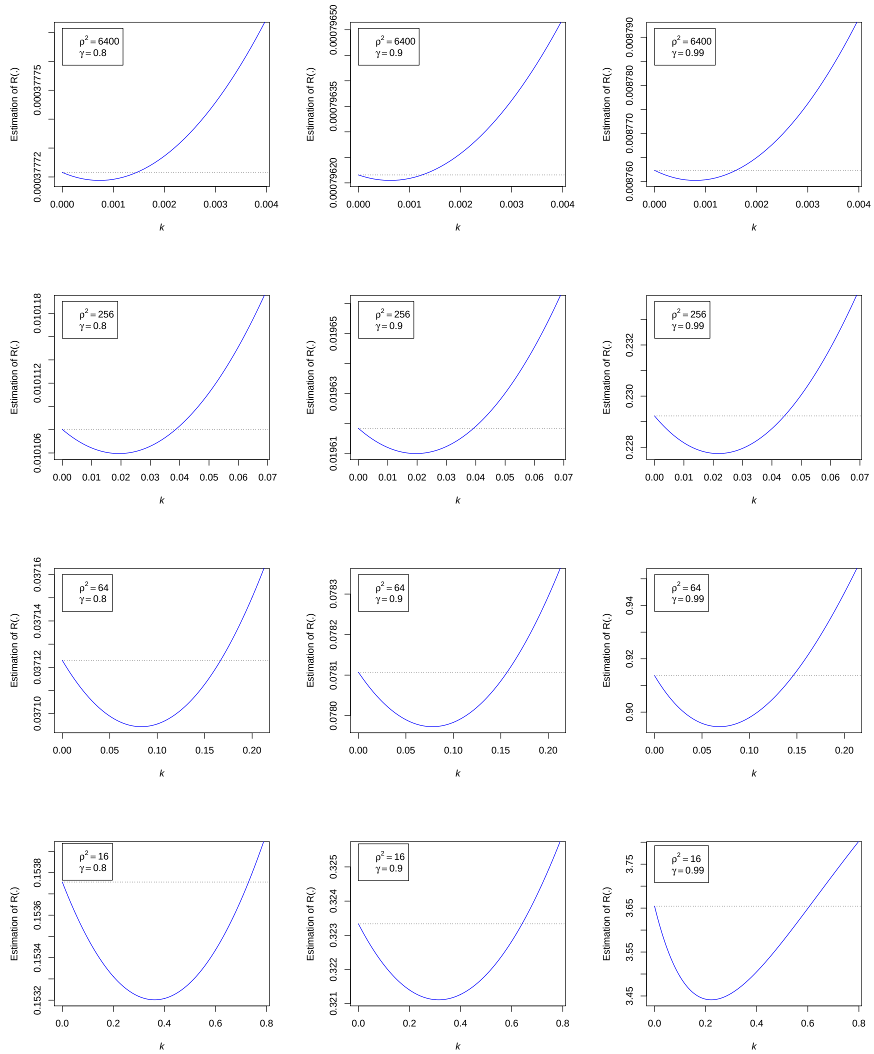

Figure 1: The diagrams of $\hat{R}(.)$ versus k for different values of ρ^2 and γ.

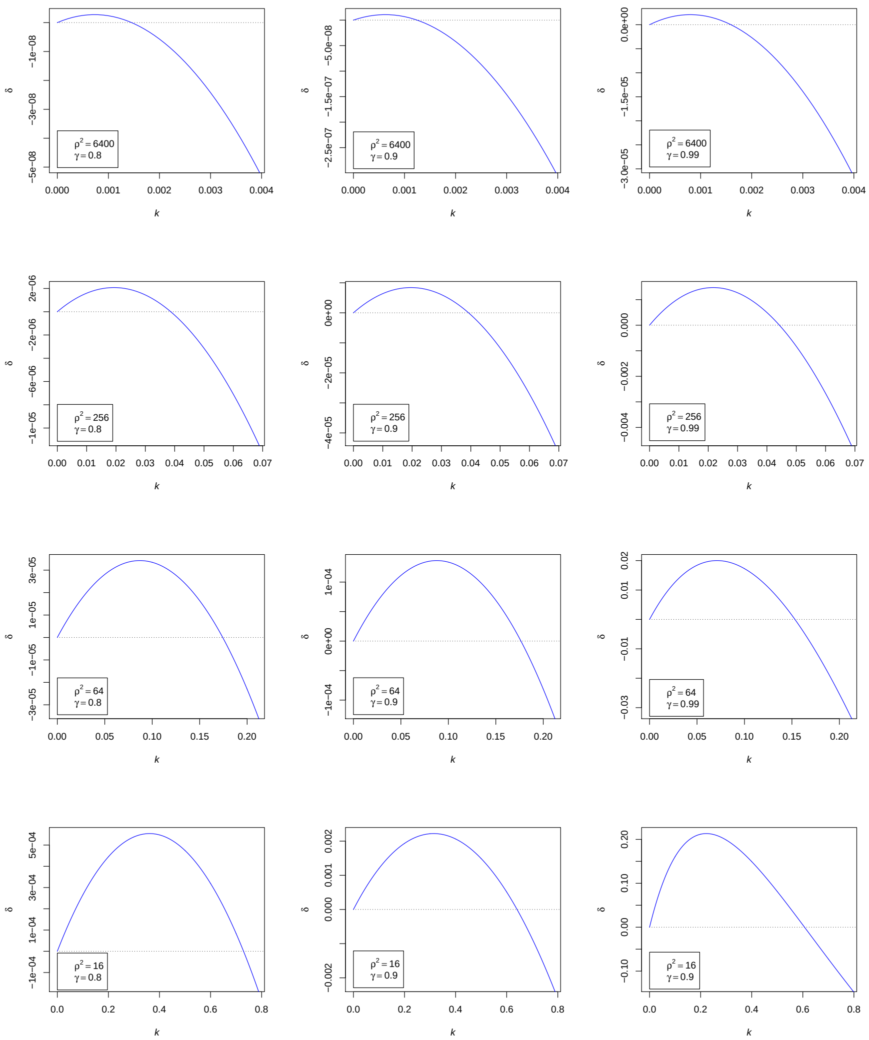

Figure 2: The diagrams of δ versus k for different values of ρ^2 and γ.

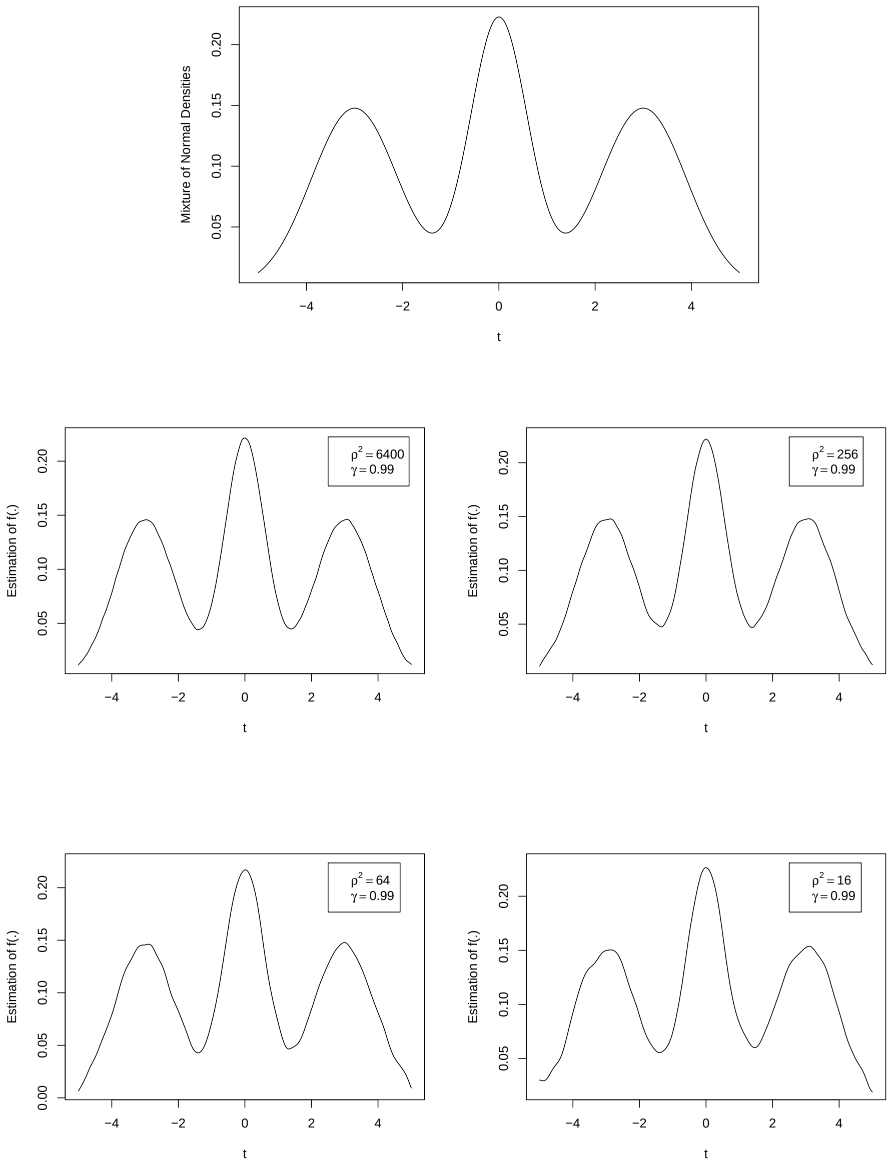

Figure 3: The function under study (Mixtures of normal densities) and estimation of it by kernel approach for $n = 1000$ and $\gamma = 0.99$.

Table 13: Obtained results of simulation study: the ridge estimator which is leading to minimum risk (+), the ridge estimator which is leading to maximum risk (−).

		k																				
ρ^2	γ	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{MED}$	$\hat{k}_{HM}$	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
6400	0.8			+				−												+		
	0.9		+						+					−								
	0.99															−				+		
256	0.8							−												+		
	0.9							−				+										
	0.99						+							−								
64	0.8			+				−														
	0.9				−							+										
	0.99			+										−								
16	0.8									+				−								
	0.9									+					−							
	0.99			+										−								

Table 14: The values of test statistics (6.5)

Variable	Z_0
intercept	-
LT	3.15
SFH	0.61
FP	3.74
DHW	3.84
GAR	3.31
ANI	3.93*

between them in Figure 4. By Yatchew (2003), the test statistic for the null hypothesis that the regression function has the parametric form, i.e., $H_0 : f(t) = h(t;\beta)$ for a known function $h(.)$, against the nonparametric alternative $f(t)$ when one uses optimal differencing weights, is

$$Z_0 = \sqrt{nm}\frac{\hat{\sigma}^2 - \hat{\sigma}^2_{\boldsymbol{D}}}{\hat{\sigma}^2} \xrightarrow{D} N(0,1), \tag{6.5}$$

where

$$\begin{aligned}
\hat{\sigma}^2 &= \frac{1}{n-p}\sum_{i=1}^{n}\big(y_i - h(t;\hat{\boldsymbol{\beta}})\big)^2,\\
\hat{\sigma}^2_{\boldsymbol{D}} &= \frac{\boldsymbol{y}'_{\boldsymbol{D}}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{y}_{\boldsymbol{D}}}{tr\big(\boldsymbol{D}'(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{D}\big)},\\
\boldsymbol{P} &= \boldsymbol{X}_{\boldsymbol{D}}(\boldsymbol{X}'_{\boldsymbol{D}}\boldsymbol{X}_{\boldsymbol{D}})^{-1}\boldsymbol{X}'_{\boldsymbol{D}}.
\end{aligned}$$

We incorporated a fourth-order optimal differencing coefficients.

We consider the average neighborhood income (ANI) as a non-parametric part, because, it has the largest value of nonparametric significance test statistics among those of other independent variables. The statistics of linearity of $h(.)$ for all explanatory variables can be found in Table 14. The nonparametric significance test of the ANI effect using (6.5) yields the value 3.93, which indicates that non linear relation between the ANI and dependent variables is significant. Moreover, we plotted the dependent variable (SP) versus explanatory variables (except for the binary ones) to find the type of relation (linear or non linear) between them in Figure 4. Consequently, the underlying SRM is specified as

$$\begin{aligned}
(SP)_i &= \beta_0 + \beta_1(LT)_i + \beta_2(SFH)_i + \beta_3(FP)_i\\
&\quad + \beta_4(DHW)_i + \beta_5(GAR)_i + f(ANI)_i + \epsilon_i.
\end{aligned} \tag{6.6}$$

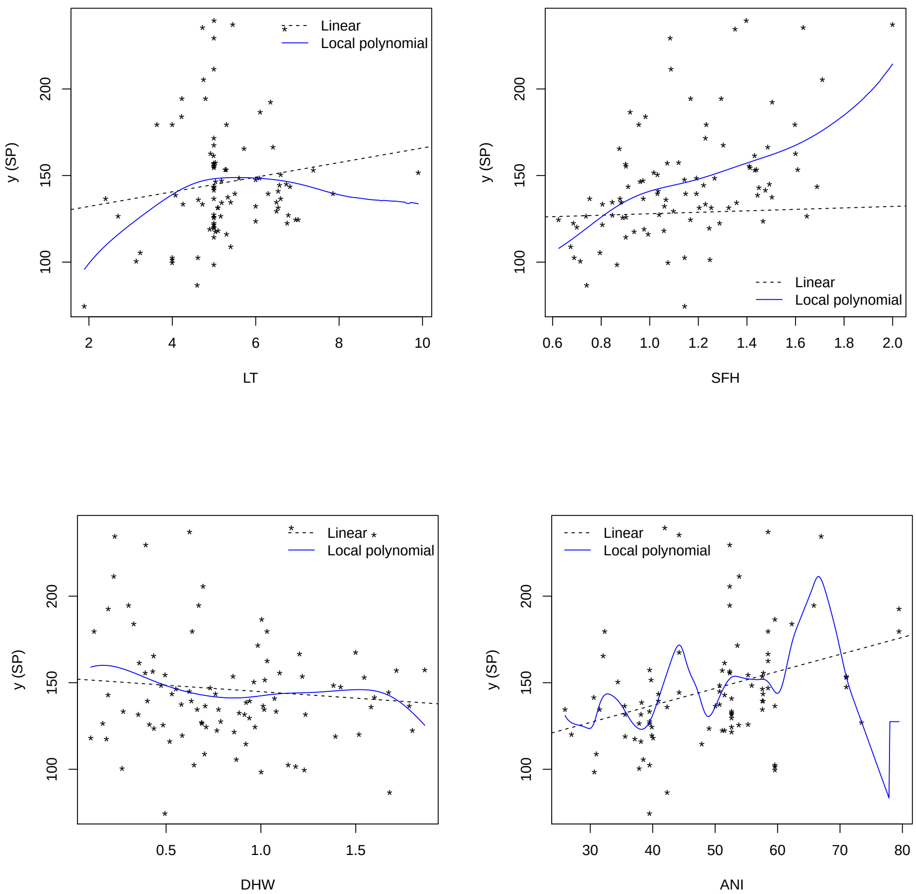

Figure 4: Plots of individual explanatory variables vs. dependent variable, linear fit (black dash line) and local polynomial fit (blue solid line).

Table 15 summarizes the results. The "parametric estimates" refers to a model in which ANI enters. The "differencing estimates" uses fourth-order differencing ($m = 4$). We have also used the kernel regression procedure with bandwidth equal to $h_n = 2.5$ for estimation $f(ANI)$. For the estimation of non parametric effect, first we estimated the parametric effects by differencing method and then, a kernel approach was applied to fit $\boldsymbol{Z_D}_i = SP_i - \boldsymbol{x}_i\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}$ on ANI_i for $i = 1, ..., n$, where $\boldsymbol{x}_i = (LT_i, SFH_i, FP_i, DHW_i, GAR_i)$ (see Figure 5).

The ratio of largest eigenvalue to smallest eigenvalue for new design matrix in model (6.6) is approximately $\lambda_5/\lambda_1 = 427.9926$. Hence, there

Table 15: Fitting of parametric and semiparametric model to housing prices data

Variable	Parametric estimates		Differencing estimates	
	Coef	SE	Coef	SE
intercept	62.6695	20.6901	-	-
LT	0.8146	2.6487	4.7999	2.3973
SFH	39.2694	11.9940	47.2357	11.2128
FP	6.0551	7.6806	5.9293	6.9047
DHW	-8.2151	6.7718	-0.9962	7.1555
GAR	14.4684	6.3840	9.8129	5.9156
ANI	0.5600	0.2829	-	-
s^2	798.0625		277.3467	
RSS	33860.06		78692.63	
R^2	0.3329		0.7736	

exists a potential multicollinearity between the columns of design matrix. Now, in order to overcome the multicollinearity, we use the proposed ridge estimators for model (6.6). The DGRE for proposed estimators of ridge parameter is given in Table 16. As it can be understood, $\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(k_{AM})$ is the best estimator for linear part of the SRM in the sense of having smaller risk. Finally, we estimated the non parametric effect ($f(ANI)$) after estimating the linear part by $\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(k)$ for $k = 0$ and all proposed $k = \hat{k}_{HK}, \hat{k}_{HKB}, \ldots, \hat{k}_{RAK3}$ by kernel regression method in Figure 6, i.e., we used kernel fit to regress $\boldsymbol{Z}_{\boldsymbol{D}}(k) = SP - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(k)$ on ANI.

7 Summary

In this paper, we proposed difference-based generalized ridge estimator in a semiparametric regression model, in the presence of multicollinearity. We extended some methods based on the works of Hoerl and Kennard (1970), Kibria (2003), Alkhamisi et al. (2006), Alkhamisi and Shukur (2008) and Muniz and Kibia (2009) for the estimation of ridge parameter. Since theoretical comparison was not possible, a simulation study conducted to compare the performance of the proposed estimators numerically. Based on the numerical results, we found that, the

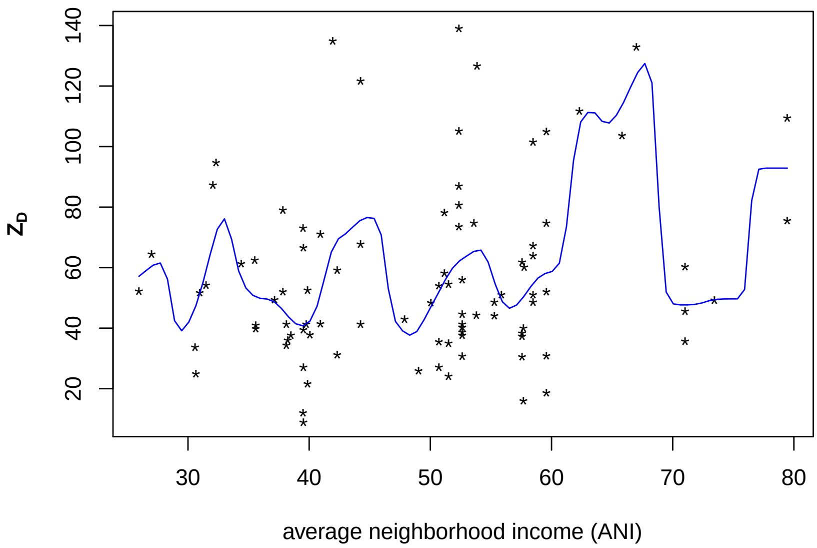

Figure 5: Estimation of non-parametric effect f(ANI).

proposed ridge regression estimators perform better than the difference-based ordinary least squares estimator in estimating the parametric part and can be recommended for practitioners. In conclusion the following point raised:

In the Monte Carlo study for $n = 500$, $P = 6$ and different ρ^2 and γ (based on the results of Tables 1 to 13 and Figures 1 to 3), it can be realized that the factors affecting the performance of the estimators are the degree of correlation (γ) and the signal to noise ratio (ρ^2). It can be concluded that $\hat{k}_{RAK1}$ is leading to be the best estimator of the parametric part ($\hat{\boldsymbol{\beta}}_{\boldsymbol{D}}(\hat{k}_{RAK1})$ has minimum risk in all proposed estimators) of the model for large values of ρ^2 and $\hat{k}_{HKB}$, $\hat{k}_{AKS1}$ and $\hat{k}_{AKS3}$ are the best estimators of ridge parameter for other values of ρ^2. Furthermore, $\hat{k}_{HM}$, $\hat{k}_{MK2}$ and $\hat{k}_{MK3}$ are the worst estimators of ridge parameter for different values of ρ^2 and γ in this example. In general, the values of ρ^2 have negative effect on the estimated risks. Also from Figure 3, we found that the estimators of the wavy function are reasonable using the differencing method by applying the kernel smoothing and cross-validation criterion.

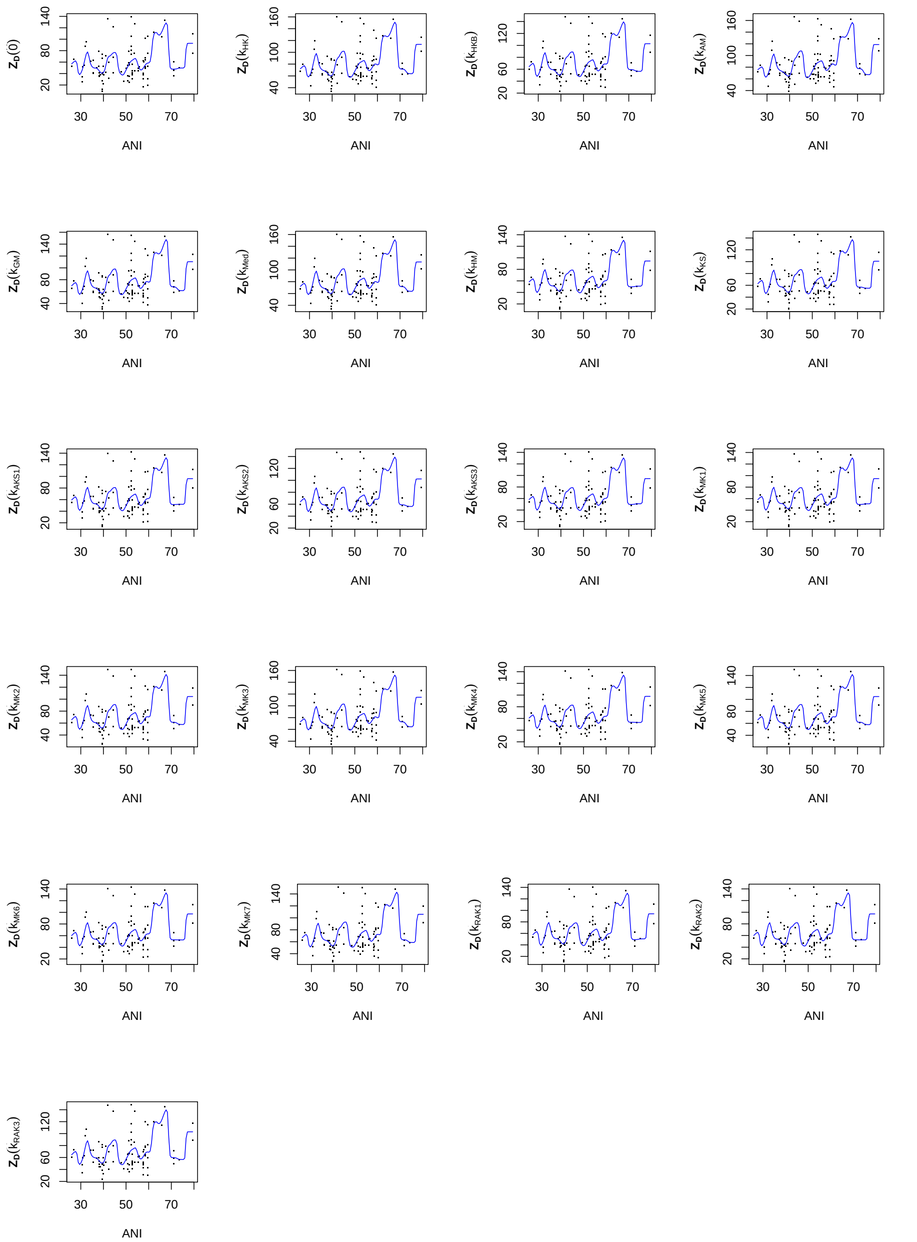

Figure 6: Estimation of the nonparametric part by kernel regression after removing the linear part by proposed estimators in housing prices data.

Table 16: Evaluation of DGRE at different k estimators in housing prices data

k Coefficients	0.0	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{GM}$	$\hat{k}_{Med}$	$\hat{k}_{HM}$
intercept	-	-	-	-	-	-	-
LT	4.7999333	4.761249	4.776731	4.751570	4.765573	4.761249	4.795167
SFH	47.2357403	28.654851	37.984975	24.489319	31.302716	28.654851	45.649816
FP	5.9292984	8.673551	7.637363	8.797355	8.472159	8.673551	6.257137
DHW	-0.9961543	-2.136362	-1.747433	-2.152377	-2.071699	-2.136362	-1.145558
GAR	9.8128884	9.565402	9.898201	9.203474	9.717438	9.565402	9.848449
ANI	-	-	-	-	-	-	-
$tr(\hat{Cov})$	267.1337	126.7402	191.6829	100.9167	144.1346	126.7402	253.3436
$\hat{R}$	267.1337	242.2252	245.7824	224.9466	247.3418	242.2252	255.7899
k Coefficients	$\hat{k}_{KS}$	$\hat{k}_{AKS1}$	$\hat{k}_{AKS2}$	$\hat{k}_{AKS3}$	$\hat{k}_{MK1}$	$\hat{k}_{MK2}$	$\hat{k}_{MK3}$
intercept	-	-	-	-	-	-	-
LT	4.780614	4.790988	4.777431	4.795170	4.794695	4.774013	4.759604
SFH	39.884659	44.159313	38.342581	45.651162	45.486682	36.522379	27.766056
FP	7.331019	6.553419	7.581736	6.256864	6.290140	7.854406	8.721178
DHW	-1.618666	-1.278818	-1.724321	-1.145435	-1.160489	-1.836274	-2.148912
GAR	9.907375	9.874836	9.901165	9.848422	9.851679	9.879711	9.501758
ANI	-	-	-	-	-	-	-
$tr(\hat{Cov})$	206.2219	240.7151	194.3838	253.3552	251.9459	180.8068	121.0748
$\hat{R}$	244.4944	249.2271	245.4815	255.7974	254.8962	247.0628	239.5009
k Coefficients	$\hat{k}_{MK4}$	$\hat{k}_{MK5}$	$\hat{k}_{MK6}$	$\hat{k}_{MK7}$	$\hat{k}_{RAK1}$	$\hat{k}_{RAK2}$	$\hat{k}_{RAK3}$
intercept	-	-	-	-	-	-	-
LT	4.786836	4.773337	4.788108	4.771497	4.796517	4.788593	4.776190
SFH	42.562553	36.140046	43.065291	35.061722	46.110587	43.253807	37.703216
FP	6.857065	7.908254	6.763065	8.053317	6.163182	6.727432	7.680499
DHW	-1.413407	-1.857947	-1.371975	-1.915481	-1.102939	-1.356212	-1.765264
GAR	9.894896	9.873116	9.889539	9.850349	9.838886	9.887299	9.895445
ANI	-	-	-	-	-	-	-
$tr(\hat{Cov})$	227.5356	178.0083	231.6466	170.2133	257.3123	233.1973	189.5664
$\hat{R}$	245.5463	247.3743	246.3921	248.1039	258.5732	246.7786	246.0281

In the real example, a near dependency among the column of $\boldsymbol{X}'\boldsymbol{X}$ identified from the condition number $\sqrt{\lambda_7/\lambda_1} = \sqrt{141750.3}$, that is, the design matrix may be considered as being quite ill-conditioned and we had to consider the ridge form of proposed estimators in our study. As it can be seen from Table 14 as well as Figure 4, a nonlinear relation between sale price and average neighborhood income (ANI) can be detected. From Table 15 and Figure 5, it can be found that pure parametric model does not fit to the data well and a semiparametric regression model fits more significantly. Further, from Table 16 and Figure 6, it can be deduced that the proposed estimators are efficient in the sense that they have significant goodness of fit values.

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