Parametric Estimation in a Recurrent Competing Risks Model

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Abstract. A resource-efficient approach to making inferences about the distributional properties of the failure times in a competing risks setting is presented. Efficiency is gained by observing recurrences of the competing risks over a random monitoring period. The resulting model is called the recurrent competing risks model (RCRM) and is coupled with two repair strategies whenever the system fails. Maximum likelihood estimators of the parameters of the marginal distribution functions associated with each of the competing risks and also of the system lifetime distribution function are presented. Estimators are derived under perfect and partial repair strategies. Consistency and asymptotic properties of the estimators are obtained. The estimation methods are applied to a data set of failures for cars under warranty. Simulation studies are used to ascertain the small sample properties and the efficiency gains of the resulting estimators.

1 Introduction

Consider a series or a competing risks system with Q components. Denote by T_1, T_2, $\ldots$, T_Q the (possibly latent) times-to-failure of the Q

components. To avoid identifiability issues (cf., Tsiatis [17] and Heckman and Honoré [7]), these variables are assumed to be independent. Additionally, let $F_q(\cdot)$ be the distribution of T_q, which in this paper will be assumed continuous. The system life is denoted by S, the minimum of the T_q's. In terms of $F_1, F_2, \ldots, F_Q$, the system life distribution, F_S, is given by

$$F_S(s) = P(S \leq s) = 1 - \prod_{q=1}^{Q} [1 - F_q(s)]. \tag{1}$$

Let Λ_q be the cumulative hazard function associated with F_q so that

$$\Lambda_q(t) = -\log(1 - F_q(t)), q = 1, 2, \ldots, Q,$$

and let λ_q be its associated hazard rate function where $\lambda_q = f_q/(1 - F_q)$ with f_q the density function of F_q. These functions are related to the qth cumulative incidence function or sub-distribution function (cf., [10]), defined via $\breve{F}_q(t) = P\{S \leq t, S = T_q\}$, according to the relationship

$$\breve{F}_q(t) = \int_0^t \bar{F}_S(w) \Lambda_q(dw).$$

In the competing risks literature, there is also the notion of a cause-specific hazard rate function associated with the qth risk defined via

$$\breve{\lambda}_q(t) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{S \in [t, t+dt), S = T_q | S \geq t\}, q = 1, 2, \ldots, Q.$$

By virtue of the assumed independence of the T_q's, note that we have the identities $\lambda_q(\cdot) = \breve{\lambda}_q(\cdot)$ for $q = 1, 2, \ldots, Q$. Since the qth cause-specific cumulative hazard function is defined via $\breve{\Lambda}_q(t) = \int_0^t \breve{\lambda}_q(w)dw$, then this equals $\Lambda_q(t)$ under the independence assumption among the T_q's. Note, however, that it is *not* the case that $\breve{\Lambda}_q = -\log(1 - \breve{F}_q)$ when $Q > 1$.

For purposes of making inference about F_S, and of the F_q's or the $\breve{F}_q$'s, n independent systems or units will be monitored. The ith system will be under observation during the random period $[0, \tau_i]$. The monitoring times, $\tau_1, \tau_2, \ldots, \tau_n$, are assumed to be independent with common distribution function G, which is non-informative about the F_q's. The τ_i's are also assumed to be independent of the inter-failure times. In the usual competing risks model, the ith system is monitored either until its system failure S_i or until τ_i. It is assumed that when the system fails, the component that caused the failure could be determined. In this basic competing risks model, the random observables

are $\{(Z_i, \delta_i), i = 1, 2, \ldots, n\}$, where $Z_i = \min(S_i, \tau_i)$; $\delta_i = 0$ whenever $Z_i = \tau_i$, so the system life S_i is right-censored by τ_i; and $\delta_i = q$ whenever $Z_i = T_{iq}$ with T_{iq} the latent time-to-failure due to cause q, so the ith system failed due to the failure of component q which happened before τ_i. The δ_i's are the indicator variables of the component that caused system failure. Given these random observables, inference about the individual F_q's are made which may lead to a better inference about the system life, F_S, by exploiting equation (1). If interest is on the cause-specific sub-distribution functions $\breve{F}_q$'s, then estimators of these functions could be obtained from the estimators of $\bar{F}_S$ and the Λ_q's. This basic model is the so-called single-event competing risks model, which has been considered in Crowder [5] and will be hereon abbreviated SCRM.

Such studies may not, however, be resource-efficient. It is possible that a system will fail early during the monitoring period $[0, \tau_i]$ and, in the above scheme, this system will not anymore be monitored. The unfailed components in this system are in essence wasted. A more resource-efficient scheme may potentially be achieved by instituting a repair strategy after each system failure, similar in spirit to the idea of testing "with replacement" in [11]. Instead of discarding the system, a repair can be performed that may require either the entire system be replaced or the failed component that caused system failure be replaced by a new component. These types of repairs are, respectively, referred to as perfect repairs and partial repairs (cf., Bedford and Lindqvist [2]). Whichever repair strategy is adopted for the ith system is then continuously implemented for this system during the random monitoring period $[0, \tau_i]$. It will be assumed for simplicity, but clearly unrealistically, that the repair process of the system or the failed component can be performed instantaneously. The time of the jth failure for the ith system will be denoted by S_{ij}, while the associated event indicator will be denoted by δ_{ij}. For the ith system, the random number of system failures over $[0, \tau_i]$ caused by the qth risk or component will be denoted by K_{iq}. Observe that in this recurrent competing risks framework, in contrast to the basic competing risks model, the period $[S_{i1}, \tau_i]$ will still be utilized to continue the monitoring of system i, which may provide more information leading to improved inferences regarding the F_q's and consequently the system life distribution F_S.

We shall refer to this model as the recurrent competing risks model (RCRM). Note that a recurrent competing risks model has been mentioned in [4] in the context of analyzing data pertaining to recurrent failures of internal shunts described in [12, 18]. This RCRM offers a

more efficient use of the monitoring period to maximize the information obtained about the system life and the latent failure times. The primary goal of this paper is to obtain estimators of the marginal distribution functions of each of the competing risks failure times and of the system life distribution under a perfect repair RCRM and a partial repair RCRM. We restrict the coverage of this paper to parametric models without covariates and defer the development of nonparametric methods and models with covariates for future work.

The context of application of the methods presented in this paper are expected to be useful in reliability and engineering settings where a parametric assumption may be appropriate. However, further extensions of the model should be made for many realistic applications to biomedical data that incorporate dependencies across risks. In particular, a relaxation of the independence assumption of the T_q's may be needed to account for the deterioration of a system resulting from combined effects of other risks. Heckman and Honoré [7] present conditions and models that induce identifiability. Their results were extended recently by Abbring and van den Berg [1]. Other avenues for modeling competing risks have incorporated copulas. These methods have been explored by Zheng and Klein [19] and subsequently by Lo and Wilke [13].

We outline the contents of this paper. Section 2 presents a mathematical framework for the RCRM. Section 3 provides the maximum likelihood estimators of the model parameters. Gains in efficiency of the estimation procedure based on the RCRM relative to the non-recurrent SCRM are presented in Section 4. In Section 5, the estimation procedure is applied to a data set on failures of cars under warranty. To get an immediate idea of the type of data of interest, see Figure 5 on page 172, which is a graphical depiction of the car warranty data analyzed in Section 5. Small sample properties of the estimators are obtained through a simulation study in Section 6. Concluding remarks are provided in Section 7.

2   Recurrent Competing Risks Model

As mentioned in the preceding section, we will consider the situation where n systems or units are under consideration. For the ith system, the observable random vector will be

$$\mathbf{D}_i \equiv (K_i, \tau_i, S_{i1}, S_{i2}, \ldots, S_{i,K_i}, \tau_i - S_{i,K_i}, \delta_{i1}, \delta_{i2}, \ldots, \delta_{i,K_i}),$$

where $K_i = \max\{k : S_{ik} \leq \tau_i\}$ is the total number of events (failures) over $[0, \tau_i]$. Observe that K_i is random and will be informative about the F_q's and F_S. The random number of failures attributed to risk q is $K_{iq} = \sum_{j=1}^{K_i} I\{\delta_{ij} = q\}$. From $\mathbf{D}_i$, the inter-event times can be recovered via $T_{ij} = S_{ij} - S_{i,j-1}$ with $S_{i0} \equiv 0$. For the partial repair model, successive calendar times are additionally denoted for risk q by $S_{ijq}, j = 1, 2, \ldots, K_{iq}$, and the inter-event times for risk q are given by $T_{ijq} = S_{ijq} - S_{i,j-1,q}, j = 1, 2, \ldots, K_{iq}$, with $S_{i0q} \equiv 0$. [For our notation, we shall on occasion write $T_{i,j,q}$ for T_{ijq} and $S_{i,j,q}$ for S_{ijq} for clarity.] Under the perfect repair model, the jth inter-event time associated with risk q, denoted by T_{ijq}, will only be observed if the jth system failure is due to risk q. In such a case, the inter-event times $T_{ijv}, v \neq q$, will all be right-censored by T_{ijq}. In addition, for the perfect repair model, the inter-event times $T_{i,K_i+1,q}, q = 1, 2, \ldots, Q$, will all be right-censored by $\tau_i - S_{i,K_i}$; while for the partial repair model $\tau_i - S_{i,K_{iq},q}$ will be the right-censoring variable for $T_{i,K_{iq}+1,q}$ for each q.

Assume the T_{ijq}'s are independent, and for a fixed q, are iid from an unknown marginal distribution function F_q which belongs to a parametric family of distributions $\mathcal{C}_q = \{F_q(\cdot; \boldsymbol{\theta}_q) : \boldsymbol{\theta}_q \in \boldsymbol{\Theta}_q\}$ where $\boldsymbol{\Theta}_q$ is an open subset of $\Re^{p_q}$ where P_q denotes the number of parameters associate with F_q. The marginal hazard rate function is denoted by $\lambda_q(\cdot; \boldsymbol{\theta}_q)$, where $\boldsymbol{\theta}_q$ is a vector of distinct parameters associated with λ_q. As noted earlier, by virtue of the independence of the latent variables, this marginal hazard rate function is also the cause-specific hazard rate function. The parameter vector of interest is the $p_\bullet \times 1$ vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^\mathsf{T}, \boldsymbol{\theta}_2^\mathsf{T}, \ldots, \boldsymbol{\theta}_Q^\mathsf{T})^\mathsf{T}$, where $p_\bullet = \sum_{q=1}^{Q} p_q$. Observe that

$$\lambda_q(t; \boldsymbol{\theta}_q) = \frac{f_q(t; \boldsymbol{\theta}_q)}{S_q(t; \boldsymbol{\theta}_q)}, q = 1, 2, \ldots, Q,$$

where $f_q(t; \boldsymbol{\theta}_q)$ is the marginal density associated with the marginal distribution function, $F_q(t; \boldsymbol{\theta}_q) = P(T_{ijq} \leq t)$ and $S_q(t; \boldsymbol{\theta}_q) = 1 - F_q(t; \boldsymbol{\theta}_q)$. Therefore, it will suffice to estimate the parameters associated with the marginal hazard function in order to estimate the marginal distribution function. Figure 1 illustrates the observed data for two recurrent competing risks under a partial repair model. The calendar times of failures for the two competing causes are denoted by blue stars and green diamonds with the censoring time denoted by a red X. The observed system calendar times (S_{ij}) and system inter-event times (T_{ij}) are shown in black. The observed calendar times (S_{ijq}) and inter-event times (T_{ijq}) for each of the two causes are depicted in their associated colors as well.
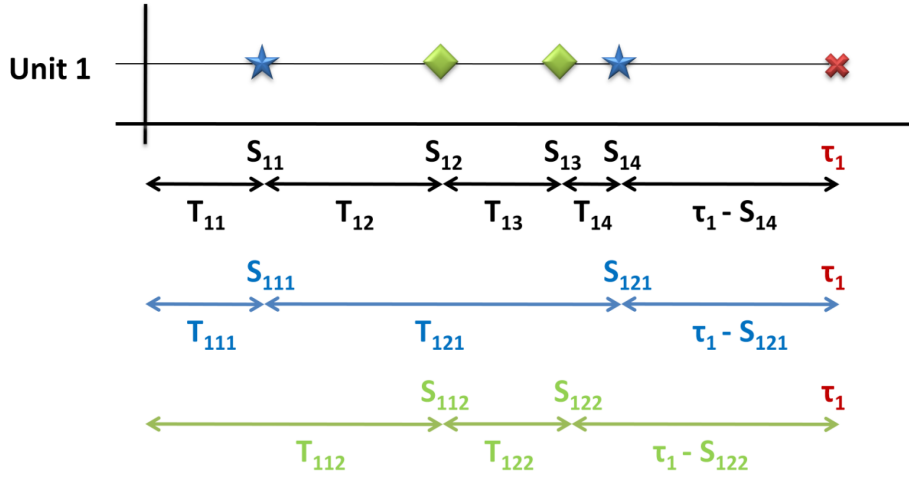
Figure 1: Observable quantities for data generated from two recurrent competing risks under a partial repair strategy for one system.

3 Maximum Likelihood Estimation

3.1 Likelihood and Estimators

The main focus of this paper is the estimation of the parameter vector $\boldsymbol{\theta}$ associated with the Q marginal distribution functions. To achieve more generality, we employ counting processes and martingales in the following. We shall define a generic counting process $N_{iq}^{\dagger} = \{N_{iq}^{\dagger}(s) : s \geq 0\}$, which counts the number of failures due to cause q for system i during calendar time period $[0, s]$, which will vary depending on whether we have perfect or partial repair. For the perfect repair model (E) and for the partial repair model (A) this counting process becomes, respectively,

$$N_{iq}^E(s) = \sum_{j=1}^{K_i} I\{S_{ij} \leq s \wedge \tau_i, \delta_{ij} = q\} \quad \text{and} \quad N_{iq}^A(s) = \sum_{j=1}^{K_{iq}} I\{S_{ijq} \leq s \wedge \tau_i\},$$

for $i = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, Q$. The at-risk process, $Y_i^{\dagger}(s) = I\{\tau_i \geq s\}$, indicates whether unit i is still under observation at time s. Additionally, define a generic backward recurrence process $E_{iq}^{\dagger} = \{E_{iq}^{\dagger}(s) : s \geq 0\}$, which for the perfect repair model (E) and the partial repair model (A) becomes, respectively,

$$E_{iq}^E(s) = s - S_{i,N_i^E(s-)} \quad \text{and} \quad E_{iq}^A(s) = s - S_{i,N_{iq}^A(s-),q},$$

where $N_i^E(s) = \sum_{q=1}^Q N_{iq}^E(s)$. Under the perfect repair model, $E_{iq}^E(s)$ is the time elapsed since the most recent failure due to any risk; while under the partial repair model, $E_{iq}^A(s)$ is the time elapsed since the most recent failure due to risk q only.

The filtration that is augmented to the basic probability space $(\Omega, \mathfrak{F}, P)$ on which all random entities are defined is $\mathbf{F} = \{\mathfrak{F}_s : s \geq 0\}$, where $\mathfrak{F}_s$ is the σ-field containing all information up to time s, defined via

$$\mathfrak{F}_s = \mathfrak{F}_0 \vee \left\{ \bigvee_{i=1}^n \sigma \left\{ \left(N_{iq}^\dagger(v), Y_i^\dagger(v+) \right) : q = 1, 2, \ldots, Q; \ 0 \leq v \leq s \right\} \right\},$$

where $\mathfrak{F}_0$ is the σ-field containing all the information available at time 0.

Proposition 3.1. *The intensity process of $N_{iq}^\dagger$ with respect to $\mathbf{F}$ is $Y_i^\dagger(s)\lambda_q(E_{iq}^\dagger(s); \boldsymbol{\theta}_q)$, that is, with $dN_{iq}^\dagger(s) = N_{iq}^\dagger((s+ds)-) - N_{iq}^\dagger(s-)$, $\lim_{0 < ds \to 0} (ds)^{-1} P\left\{ dN_{iq}^\dagger(s) = 1 | \mathfrak{F}_{s-} \right\} = Y_i^\dagger(s)\lambda_q(E_{iq}^\dagger(s); \boldsymbol{\theta}_q)$.*

Proof. For infinitesimal $ds > 0$, there will either be no failures or exactly one failure in $[s, s + ds)$. Now, $P(dN_{iq}^\dagger(s) = 1 | \mathfrak{F}_{s-})$ is the conditional probability, given $\mathfrak{F}_{s-}$, of exactly one failure for the ith system that is due to cause q in the interval of time $[s, s + ds)$. We have for such a ds that

$$P(dN_{iq}^\dagger(s) = 1 | \mathfrak{F}_{s-}) = P\{dN_{iq}^\dagger(s) = 1; \cap_{q' \neq q}[dN_{iq'}^\dagger(s) = 0] | \mathfrak{F}_{s-}\}$$

$$= (Y_i^\dagger(s)\lambda_q(E_{iq}^\dagger(s); \boldsymbol{\theta}_q)ds + o(ds))$$

$$\times \prod_{v=s}^{s+ds} \left\{ 1 - \sum_{q' \neq q} \lambda_{q'}(E_{iq'}^\dagger(v); \boldsymbol{\theta}_{q'})dv \right\}$$

$$= (Y_i^\dagger(s)\lambda_q(E_{iq}^\dagger(s); \boldsymbol{\theta}_q)ds + o(ds))$$

$$\times \exp\left\{ -\int_s^{s+ds} \sum_{q' \neq q} \lambda_{q'}(E_{iq'}^\dagger(v); \boldsymbol{\theta}_{q'})dv \right\}$$

$$= (Y_i^\dagger(s)\lambda_q(E_{iq}^\dagger(s); \boldsymbol{\theta}_q)ds + o(ds))$$

$$\times \exp\left\{ -\sum_{q' \neq q}[\Lambda_{q'}(E_{iq'}^\dagger((s+ds)-); \boldsymbol{\theta}_{q'}) - \Lambda_{q'}(E_{iq'}^\dagger(s-); \boldsymbol{\theta}_{q'})] \right\},$$

where $\prod$ denotes product integral. Dividing by ds and then taking the limit as $ds \to 0$, the result follows since $\Lambda_{q'}(E_{iq'}^{\dagger}(s); \boldsymbol{\theta}_{q'})$ is left-continuous in s. ∎

The intensity process of $N_{iq}^{\dagger}(\cdot)$ with respect to $\mathbf{F}$ is $Y_i^{\dagger}(\cdot)\lambda_q(E_{iq}^{\dagger}(\cdot); \boldsymbol{\theta}_q)$ so that the process $A_{iq}^{\dagger} = \{A_{iq}^{\dagger}(s; \boldsymbol{\theta}_q) : s \geq 0\}$, defined by

$$A_{iq}^{\dagger}(s; \boldsymbol{\theta}_q) = \int_0^s Y_i^{\dagger}(v)\lambda_q(E_{iq}^{\dagger}(v); \boldsymbol{\theta}_q)dv,$$

is therefore the compensator of $N_{iq}^{\dagger}(s)$. By the Doob-Meyer decomposition theorem,

$$\{M_{iq}^{\dagger}(s; \boldsymbol{\theta}_q) = N_{iq}^{\dagger}(s) - A_{iq}^{\dagger}(s; \boldsymbol{\theta}_q) : s \geq 0\}$$

is a zero-mean square-integrable $\mathbf{F}$-martingale for each i and q. The vector $\mathbf{M}_i^{\dagger}(v; \boldsymbol{\theta}) = (M_{i1}^{\dagger}(v; \boldsymbol{\theta}_1), \ldots, M_{iQ}^{\dagger}(v, \boldsymbol{\theta}_Q))^{\mathbf{T}}$ is a $Q \times 1$ vector of square-integrable zero-mean martingale processes. Its predictable quadratic variation process is a diagonal matrix process with diagonal elements

$$\langle M_{iq}^{\dagger} \rangle(s; \boldsymbol{\theta}_q) = A_{iq}^{\dagger}(s; \boldsymbol{\theta}_q), q = 1, 2, \ldots, Q.$$

By results of Jacod [9], the full likelihood process at time s is given by

$$\begin{aligned}
L(\boldsymbol{\theta}, s) &= \prod_{v=0}^{s}\prod_{i=1}^{n}\prod_{q=1}^{Q} \left\{ [Y_i^{\dagger}(v)\lambda_q(E_{iq}^{\dagger}(v); \boldsymbol{\theta}_q)]^{dN_{iq}^{\dagger}(v)} \right. \\
&\quad \times \left. \left[1 - Y_i^{\dagger}(v)\lambda_q(E_{iq}^{\dagger}(v); \boldsymbol{\theta}_q) \right]^{1-dN_{iq}^{\dagger}(v)} \right\} \\
&= \prod_{i=1}^{n}\prod_{q=1}^{Q} \left\{ \left(\prod_{v=0}^{s} [Y_i^{\dagger}(v)\lambda_q(E_{iq}^{\dagger}(v); \boldsymbol{\theta}_q)]^{dN_{iq}^{\dagger}(v)} \right) \right. \\
&\quad \times \left. \exp\left\{ -\int_0^s Y_i^{\dagger}(v)\lambda_q(E_{iq}^{\dagger}(v); \boldsymbol{\theta}_q)dv \right\} \right\}.
\end{aligned}$$

Consequently, the log-likelihood process is $\{\ell(\boldsymbol{\theta}, s) : s \geq 0\}$ with

$$\begin{aligned}
\ell(\boldsymbol{\theta}, s) &= \sum_{i=1}^{n} \left\{ \int_0^s \sum_{q=1}^{Q} \log[Y_i^{\dagger}(v)\lambda_q(E_{iq}^{\dagger}(v); \boldsymbol{\theta}_q)]dN_{iq}^{\dagger}(v) \right. \\
&\quad \left. -\int_0^s Y_i^{\dagger}(v) \sum_{q=1}^{Q} \lambda_q(E_{iq}^{\dagger}(v); \boldsymbol{\theta}_q)dv \right\}.
\end{aligned}$$

For purposes of obtaining the maximum likelihood estimators and their properties, we seek the $p_\bullet \times 1$ score vector process $\boldsymbol{U} = \{\boldsymbol{U}(\boldsymbol{\theta}, s) : s \geq 0\}$ and the $p_\bullet \times p_\bullet$ observed information matrix process $I(\boldsymbol{\theta}) = \{\boldsymbol{I}(\boldsymbol{\theta}, s) : s \geq 0\}$. For this purpose, for a vector $\boldsymbol{a}$, define the gradient operator $\nabla_{\boldsymbol{a}} \equiv \frac{\partial}{\partial \boldsymbol{a}}$ and, for $q = 1, 2, \ldots, Q$, let

$$\boldsymbol{\rho}_q(s; \boldsymbol{\theta}_q) = \nabla_{\boldsymbol{\theta}_q} \log \lambda_q(s; \boldsymbol{\theta}_q) \quad \text{and} \quad \boldsymbol{V}_q(s; \boldsymbol{\theta}_q) = \nabla_{\boldsymbol{\theta}_q^\mathsf{T}} \nabla_{\boldsymbol{\theta}_q} \log \lambda_q(s; \boldsymbol{\theta}_q).$$

Thus, $\boldsymbol{\rho}_q(s; \boldsymbol{\theta}_q)$ is a $p_q \times 1$ vector of functions, while $\boldsymbol{V}_q(s; \boldsymbol{\theta}_q)$ is a $p_q \times p_q$ matrix of functions.

Since the vector of score process is obtained via $\boldsymbol{U}(\boldsymbol{\theta}, s) = \nabla_{\boldsymbol{\theta}} \ell(\boldsymbol{\theta}, s)$, it is straightforward to obtain that the vector of score processes is $\boldsymbol{U}(\boldsymbol{\theta}, s) = (\boldsymbol{U}_1^\mathsf{T}(\boldsymbol{\theta}_1, s), \boldsymbol{U}_2^\mathsf{T}(\boldsymbol{\theta}_2, s), \ldots, \boldsymbol{U}_Q^\mathsf{T}(\boldsymbol{\theta}_Q, s))^\mathsf{T}$, where $\boldsymbol{U}_q(\boldsymbol{\theta}_q, s)$ is the $p_q \times 1$ vector given by

$$\boldsymbol{U}_q(\boldsymbol{\theta}_q, s) = \sum_{i=1}^{n} \int_0^s \boldsymbol{\rho}_q(E_{iq}^\dagger(v); \boldsymbol{\theta}_q) dM_{iq}^\dagger(v; \boldsymbol{\theta}_q).$$

On the other hand, the observed Fisher information matrix process is defined via $\boldsymbol{I}(\boldsymbol{\theta}, s) = -\nabla_{\boldsymbol{\theta}^\mathsf{T}} \boldsymbol{U}(\boldsymbol{\theta}, s)$. It is straightforward to verify that $\boldsymbol{I}(\boldsymbol{\theta}, s)$ is a block-diagonal matrix with the (q, q)th block matrix being the $p_q \times p_q$ matrix

$$\begin{aligned}
\boldsymbol{I}_{qq}(\boldsymbol{\theta}_q, s) &= -\nabla_{\boldsymbol{\theta}_q^\mathsf{T}} \boldsymbol{U}_q(\boldsymbol{\theta}_q, s) \\
&= \sum_{i=1}^{n} \int_0^s Y_i^\dagger(v) \boldsymbol{\rho}_q(E_{iq}^\dagger(v); \boldsymbol{\theta}_q)^{\otimes 2} \lambda_q(E_{iq}^\dagger(v); \boldsymbol{\theta}_q) dv \\
&\quad - \sum_{i=1}^{n} \int_0^s \boldsymbol{V}_q(E_{iq}^\dagger(v); \boldsymbol{\theta}_q) dM_{iq}^\dagger(v; \boldsymbol{\theta}_q),
\end{aligned}$$

where for a vector $\boldsymbol{a}$, we have $\boldsymbol{a}^{\otimes 2} = \boldsymbol{a}\boldsymbol{a}^\mathsf{T}$. The ML estimator of $\boldsymbol{\theta}_q$ based on the realization of the processes up to calendar time s^*, denoted $\hat{\boldsymbol{\theta}}_q(s^*)$, is obtained as a solution of the equation

$$\boldsymbol{U}_q(\boldsymbol{\theta}_q, s^*) = \boldsymbol{0}. \tag{2}$$

Numerical methods, such as the Newton-Raphson algorithm, will typically be needed to obtain the estimate $\hat{\boldsymbol{\theta}}_q(s^*)$ of $\boldsymbol{\theta}_q$ based on equation (2). The Newton-Raphson iteration is based on the updating

$$\hat{\boldsymbol{\theta}}_q^{new} \leftarrow \hat{\boldsymbol{\theta}}_q^{old} + \boldsymbol{I}_{qq}(\hat{\boldsymbol{\theta}}_q^{old}, s^*)^{-1} \boldsymbol{U}_q(\hat{\boldsymbol{\theta}}_q^{old}, s^*).$$

3.2 Asymptotics

Asymptotic properties of $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\theta}}_1^{\mathsf{T}}, \hat{\boldsymbol{\theta}}_2^{\mathsf{T}}, \ldots, \hat{\boldsymbol{\theta}}_Q^{\mathsf{T}})^{\mathsf{T}}$, such as consistency and asymptotic normality, as $n \to \infty$, follow from Borgan's [3] results regarding ML estimators from parametric counting process models. Define $\boldsymbol{\mathcal{I}}_{qq}(\boldsymbol{\theta}_q, s)$ to be the in-probability limit of $\frac{1}{n}\boldsymbol{I}_{qq}(\boldsymbol{\theta}_q, s)$, and also let $\boldsymbol{\mathcal{I}}(\boldsymbol{\theta}, s)$ be the $p_\bullet \times p_\bullet$ block-diagonal matrix with block-diagonal elements $\boldsymbol{\mathcal{I}}_{qq}(\boldsymbol{\theta}_q, s), q = 1, 2, \ldots, Q$. Recalling the standard 'asymptotic normality' notation (cf., [15]) that $\mathbf{T}_n \sim AN(\mu_n, \boldsymbol{\Sigma}_n)$ means that

$$\boldsymbol{\Sigma}_n^{-1/2}(\mathbf{T}_n - \mu_n) \xrightarrow{d} N(\mathbf{0}, \mathbf{I}),$$

we have that under certain regularity conditions,

$$\hat{\boldsymbol{\theta}}(s^*) \sim AN\left(\boldsymbol{\theta}, \frac{1}{n}\boldsymbol{\mathcal{I}}(\boldsymbol{\theta}, s^*)^{-1}\right).$$

Because of the block-diagonal structure of $\boldsymbol{\mathcal{I}}(\boldsymbol{\theta}, s^*)$, we can conclude that the $\hat{\boldsymbol{\theta}}_q$ are asymptotically independent and also that for each $q = 1, 2, \ldots, Q$, we have

$$\hat{\boldsymbol{\theta}}_q(s^*) \sim AN\left(\boldsymbol{\theta}_q, \frac{1}{n}\boldsymbol{\mathcal{I}}_{qq}(\boldsymbol{\theta}_q, s^*)^{-1}\right).$$

A consistent estimator of $\boldsymbol{\mathcal{I}}_{qq}(\boldsymbol{\theta}_q, s^*)$ obtained from the predictable quadratic variation process of the score process is given by

$$\frac{1}{n}\langle\mathbf{U}_q\rangle(\hat{\boldsymbol{\theta}}_q, s^*) = \frac{1}{n}\sum_{i=1}^{n}\int_0^{s^*} \boldsymbol{\rho}_q(E_{iq}^{\dagger}(v); \hat{\boldsymbol{\theta}}_q)^{\otimes 2} Y_i^{\dagger}(v)\lambda_q(E_{iq}^{\dagger}(v); \hat{\boldsymbol{\theta}}_q)dv. \quad (3)$$

The estimate could be computed on a case-by-case basis depending on the form of the hazard rate function λ_q.

For the purpose of getting exact efficiency expressions for some models, we now seek an expression of $\boldsymbol{\mathcal{I}}_{qq}(\boldsymbol{\theta}_q, s^*)$ for $q = 1, 2, \ldots, Q$. To achieve a unified notation for the perfect and partial repair schemes, for $i = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, Q$, let

$$S_{i0q}^* = 0 \quad \text{and} \quad S_{ijq}^* = \inf\{s > S_{i,j-1,q}^* : E_{iq}^{\dagger}(s) = 0\};$$
$$K_{iq}^*(s^*) = \max\{j : S_{ijq}^* \le (s^* \wedge \tau_i)\};$$
$$T_{ijq}^* = S_{ijq}^* - S_{i,j-1,q}^*, j = 1, 2, \ldots, K_{iq}^*(s^*).$$

A generalized at-risk process could now be defined via

$$Y_{iq}^*(s^*, w) = \sum_{j=1}^{K_{iq}^*(s^*-)} I\{T_{ijq}^* \ge w\} + I\{(s^* \wedge \tau_i) - S_{i,K_{iq}^*(s^*),q}^* \ge w\}.$$

Let us denote the expectation of this generalized at-risk process by

$$y_q^*(s^*, w; \boldsymbol{\theta}, G) \equiv y_q^*(s^*, w) = E\{Y_{iq}^*(s^*, w)\},$$

where G is the distribution of the censoring variable τ. Fortunately, not much work is needed since this expectation can be obtained from Proposition 1 of Peña, Strawderman, and Hollander [14]. It follows that

$$\frac{1}{n} \sum_{i=1}^{n} Y_{iq}^*(\cdot, w) \xrightarrow{up} y_q^*(\cdot, w),$$

where '$\xrightarrow{up}$' means uniform convergence in-probability on $[0, s^*]$. Analogously to the derivations in [14] using a change-of-variable in the integral, it follows that

$$\boldsymbol{\mathcal{I}}_{qq}(\boldsymbol{\theta}_q, s^*) = \int_0^\infty y_q^*(s^*, w; \boldsymbol{\theta}, G) \boldsymbol{\rho}_q(w; \boldsymbol{\theta}_q)^{\otimes 2} \lambda_q(w; \boldsymbol{\theta}_q) dw.$$

Specializing to the perfect repair scheme, we have in this case that $T_{ijq}^* = \min_{q'=1,2,\ldots,Q} T_{ijq'}$, so the distribution function is identical to the system life, which is

$$F_S(t; \boldsymbol{\theta}) = 1 - \prod_{q=1}^{Q} [1 - F_q(t; \boldsymbol{\theta}_q)].$$

Thus, from [14], with $G_s(t) = G(t)I\{t < s\} + I\{t \geq s\}$, it follows that for this perfect repair scheme, $y_q^*(s^*, t; \boldsymbol{\theta}, G)$ is equal to

$$y_q^E(s^*, t; \boldsymbol{\theta}, G) = \bar{F}_S(t; \boldsymbol{\theta}) \bar{G}_{s^*}(t) + \bar{F}_S(t; \boldsymbol{\theta}) \int_t^\infty \vartheta_S(w - t; \boldsymbol{\theta}) dG_{s^*}(w),$$

where $\vartheta_S(\cdot; \boldsymbol{\theta})$ is the renewal function of $F_S(\cdot; \boldsymbol{\theta})$.

On the other hand, for the partial repair scheme, $T_{ijq}^* = T_{ijq}$ whose common distribution is $F_q(\cdot; \boldsymbol{\theta}_q)$. Thus, again from [14], we obtain that for this partial repair scheme, $y_q^*(s^*, t; \boldsymbol{\theta}, G)$ is equal to

$$y_q^A(s^*, t; \boldsymbol{\theta}_q, G) = \bar{F}_q(t; \boldsymbol{\theta}_q) \bar{G}_{s^*}(t) + \bar{F}_q(t; \boldsymbol{\theta}_q) \int_t^\infty \vartheta_q(w - t; \boldsymbol{\theta}_q) dG_{s^*}(w),$$

where $\vartheta_q(\cdot; \boldsymbol{\theta}_q)$ is the renewal function of $F_q(\cdot; \boldsymbol{\theta}_q)$.

The estimation of the parameter vector of each of the component life distributions is perhaps more of secondary importance than the estimation of the system life distribution F_S. Upon obtaining the estimators of

the $\boldsymbol{\theta}_q$'s, we could now obtain an estimator of F_S by plugging-in $\hat{\boldsymbol{\theta}}_q(s^*)$ for $\boldsymbol{\theta}_q$ in the expression for $F_S(t; \boldsymbol{\theta})$ from equation (1) to obtain the estimator

$$\hat{F}_S(s^*; t) = F_S(t; \hat{\boldsymbol{\theta}}(s^*)) = 1 - \prod_{q=1}^{Q}[1 - F_q(t; \hat{\boldsymbol{\theta}}_q(s^*))].$$

By using the delta-method and noting that $\hat{\boldsymbol{\theta}}(s^*)$ is asymptotically multivariate normal with mean $\boldsymbol{\theta}$ and asymptotic covariance matrix $\frac{1}{n}\mathcal{I}(\boldsymbol{\theta}, s^*)^{-1}$, it follows that for each $t \leq s^*$,

$$\hat{\bar{F}}_S(s^*; t) \sim AN\left(\bar{F}_S(t; \boldsymbol{\theta}), \frac{1}{n}\sigma_S^2(s^*; t; \boldsymbol{\theta})\right),$$

where

$$\sigma_S^2(s^*; t; \boldsymbol{\theta}) = \bar{F}_S(t; \boldsymbol{\theta})^2 \sum_{q=1}^{Q} \overset{\bullet}{\Lambda}_q(t; \boldsymbol{\theta}_q)^{\mathrm{T}} \mathcal{I}_{qq}(\boldsymbol{\theta}_q, s^*)^{-1} \overset{\bullet}{\Lambda}_q(t; \boldsymbol{\theta}_q)$$

with $\overset{\bullet}{\Lambda}_q(t; \boldsymbol{\theta}_q) = \nabla_{\boldsymbol{\theta}_q}\Lambda_q(t; \boldsymbol{\theta}_q)$. We remark with regards to the notation that when we let $s^* \to \infty$, we simply drop the argument s^* in the above expressions. For example, $\hat{F}_S(s^* = \infty; t)$ will simply be written as $\hat{F}_S(t)$.

Since we would like to compare the gain in efficiency obtained by utilizing this RCRM relative to the SCRM, we recall that the associated ML estimator of $\boldsymbol{\theta}$ for the SCRM, denoted by $\tilde{\boldsymbol{\theta}}$, is asymptotically multivariate normal with mean $\boldsymbol{\theta}$ and asymptotic covariance matrix $\frac{1}{n}\tilde{\mathcal{I}}(\boldsymbol{\theta})^{-1}$, where $\tilde{\mathcal{I}}(\boldsymbol{\theta})$ is the $p_\bullet \times p_\bullet$ block-diagonal matrix with (q, q)th block matrix given by the $p_q \times p_q$ matrix

$$\tilde{\mathcal{I}}_{qq}(\boldsymbol{\theta}_q) = \int_0^\infty \bar{F}_S(w; \boldsymbol{\theta}_q)\bar{G}(w)\boldsymbol{\rho}_q(w; \boldsymbol{\theta}_q)^{\otimes 2}\lambda_q(w; \boldsymbol{\theta}_q)dw.$$

Thus, if one only had data from the SCRM, an estimator of $F_S(t; \boldsymbol{\theta})$ will be

$$\tilde{F}_S(t) = F_S(t; \tilde{\boldsymbol{\theta}}) = 1 - \prod_{q=1}^{Q}[1 - F_q(t; \tilde{\boldsymbol{\theta}}_q)].$$

Using again the delta-method, it follows that for each t,

$$\tilde{\bar{F}}_S(t) \sim AN\left(\bar{F}_S(t; \boldsymbol{\theta}), \frac{1}{n}\tilde{\sigma}_S^2(t; \boldsymbol{\theta})\right);$$

$$\tilde{\sigma}_S^2(t; \boldsymbol{\theta}) = \bar{F}_S(t; \boldsymbol{\theta})^2 \sum_{q=1}^{Q} \overset{\bullet}{\Lambda}_q(t; \boldsymbol{\theta}_q)^{\mathrm{T}} \tilde{\mathcal{I}}_{qq}(\boldsymbol{\theta}_q)^{-1} \overset{\bullet}{\Lambda}_q(t; \boldsymbol{\theta}_q).$$

A measure of asymptotic relative efficiency (ARE) of the RCRM-based estimator $\hat{\bar{F}}_S(t)$ relative to the SCRM-based estimator $\tilde{\bar{F}}_S(t)$ is given by

$$ARE(\hat{\bar{F}}_S(t) : \tilde{\bar{F}}_S(t)) = \frac{\tilde{\sigma}_S^2(t;\boldsymbol{\theta})}{\sigma_S^2(t;\boldsymbol{\theta})} = \frac{\sum_{q=1}^{Q} \overset{\bullet}{\Lambda}_q (t;\boldsymbol{\theta}_q)^{\mathrm{T}} \tilde{\mathcal{I}}_{qq}(\boldsymbol{\theta}_q)^{-1} \overset{\bullet}{\Lambda}_q (t;\boldsymbol{\theta}_q)}{\sum_{q=1}^{Q} \overset{\bullet}{\Lambda}_q (t;\boldsymbol{\theta}_q)^{\mathrm{T}} \mathcal{I}_{qq}(\boldsymbol{\theta}_q)^{-1} \overset{\bullet}{\Lambda}_q (t;\boldsymbol{\theta}_q)}.$$

(4)

Theorem 3.1. *The RCRM-based estimator $\hat{\bar{F}}_S(t)$, with either a perfect repair or a partial repair strategy, dominates the SCRM-based estimator $\tilde{\bar{F}}_S(t)$ in terms of asymptotic relative efficiency, that is, $ARE(\hat{\bar{F}}_S(t) : \tilde{\bar{F}}_S(t)) > 1.0$.*

Proof. For the perfect repair strategy, this result is immediate from the fact that for each $q \in \{1, 2, \ldots, Q\}$, we obtain from their respective expressions that $\mathcal{I}_{qq}^E(\boldsymbol{\theta}) > \tilde{\mathcal{I}}_{qq}(\boldsymbol{\theta})$. On the other hand, for the partial repair strategy, by first noting that for each $q' = 1, 2, \ldots, Q$, we have $\bar{F}_S(t) = \prod_{q=1}^{Q} \bar{F}_q(t) \leq \bar{F}_{q'}(t)$, it follows that $\mathcal{I}_{qq}^A(\boldsymbol{\theta}) > \tilde{\mathcal{I}}_{qq}(\boldsymbol{\theta})$, from which the assertion of the theorem for the partial repair strategy follows. ∎

4 Efficiency Comparisons

In this section we examine in concrete settings the magnitude of the gain in asymptotic efficiency when utilizing the RCRM versus the SCRM in estimating the system lifetime distribution F_S. From Theorem 3.1 we already know that the RCRM-based estimator will always be more efficient than the SCRM-based estimator, so getting an idea of the magnitude of this efficiency improvement will provide us more information regarding the merits of adopting the RCRM design in real studies. We will consider two concrete situations: (i) when the component failure distributions are exponential and (ii) when the component failure distributions are Weibull.

4.1 Exponential Component Lifetimes

Let us now assume that the component lifetime distributions are exponential, so that $F_q(t;\theta_q) = 1 - \exp(-\theta_q t)$ for $t \geq 0$. We also assume that G is exponential so that $G(t;\gamma) = 1 - \exp(-\gamma t)$ for $t \geq 0$. Solving for

θ_q in the ML estimating equation in (2) yields the estimator

$$\hat{\theta}_q(s^*) = \frac{\sum_{i=1}^n N_{iq}^\dagger(s^*)}{\sum_{i=1}^n \int_0^{s^*} Y_i^\dagger(v)dv},$$

which is the occurrence-exposure rate under risk q. With $\theta_\bullet = \sum_{q=1}^Q \theta_q$, from [14] (see also [8]) we are able to obtain that

$$y_q^E(w) = \exp\{-(\theta_\bullet + \gamma)w\}\left(1 + \frac{\theta_\bullet}{\gamma}\right);$$

$$y_q^A(w) = \exp\{-(\theta_q + \gamma)w\}\left(1 + \frac{\theta_q}{\gamma}\right).$$

Straightforward calculations then yield that

$$I_{qq}^E(\theta_q) = \int_0^\infty \left(1 + \frac{\theta_\bullet}{\gamma}\right) e^{-(\theta_\bullet + \gamma)w}\frac{1}{\theta_q^2}\theta_q dw = \frac{1}{\theta_q\gamma}.$$

Similarly, we find that

$$I_{qq}^A(\theta_q) = \int_0^\infty \left(1 + \frac{\theta_q}{\gamma}\right) e^{-(\theta_q + \gamma)w}\frac{1}{\theta_q^2}\theta_q dw = \frac{1}{\theta_q\gamma}.$$

It is not surprising that these information values under the perfect and partial repair schemes are identical owing to the memoryless property of the exponential distribution. It follows therefore that under the assumption of exponential component lifetime distributions and an exponentially-distributed monitoring period, the RCRM-based estimators of $\bar{F}_S(t; \boldsymbol{\theta})$ under either perfect or partial repair strategies satisfy

$$\hat{\bar{F}}_S(t) \sim AN\left(\bar{F}_S(t; \boldsymbol{\theta}), \frac{1}{n}t^2\theta_\bullet\gamma\exp\{-2t\theta_\bullet\}\right).$$

On the other hand, under the same distributional assumptions, the SCRM-based estimator of $\bar{F}_S(t; \boldsymbol{\theta})$ satisfies

$$\tilde{\bar{F}}_S(t) \sim AN\left(\bar{F}_S(t; \boldsymbol{\theta}), \frac{1}{n}t^2\theta_\bullet(\theta_\bullet + \gamma)\exp\{-2t\theta_\bullet\}\right).$$

As a consequence, the asymptotic efficiency of $\hat{\bar{F}}_S(t)$ relative to $\tilde{\bar{F}}_S(t)$ under the exponential distribution assumptions is

$$ARE(\hat{\bar{F}}_S(t) : \tilde{\bar{F}}_S(t)) = 1 + \frac{\theta_\bullet}{\gamma} = 1 + \frac{E(\tau)}{E(S)}.$$

That is, the gain in efficiency using the RCRM over the SCRM, which is always positive, is solely determined by the ratio of the mean length of the monitoring period and the mean lifetime of the competing risk system.

4.2 Weibull Component Lifetimes

We were able to obtain an exact expression of the asymptotic relative efficiency of $\hat{\bar{F}}_S(t)$ relative to $\tilde{\bar{F}}_S(t)$ under the exponential case because of the fact that the renewal function of the exponential distribution is in closed-form, which for an exponential distribution with rate θ is given by $\vartheta(t) = \theta t$. For many other distributions, such as the Weibull, no closed-form expressions of their renewal function are available, hence we are usually unable to obtain closed-form analytical expressions of asymptotic relative efficiencies. Thus, we resort to simulation studies to get an idea of the relative efficiencies under these non-exponentially distributed models.

Note also from equation (4) that the asymptotic relative efficiency of $\hat{\bar{F}}_S(t)$ relative to $\tilde{\bar{F}}_S(t)$ will, in general, depend on the time t. To examine the efficiency of the RCRM-based estimator $\hat{F}_S(t)$ relative to the SCRM-based estimator $\tilde{F}_S(t)$ under the Weibull distributional model we performed a simulation study. In this study, the component lifetime distributions for $Q = 2$ competing risks are Weibull distributions with respective shape and scale parameters $(\alpha_1 = 2, \beta_1 = 1/2)$ and $(\alpha_2 = 3, \beta_2 = 1/3)$. We used a sample of $n = 30$ systems, and the number of simulation replications was $N = 1000$. The distribution of the time to the end of monitoring period was an exponential distribution. We took several values of the parameter of this exponential distribution.

Figure 2 and Figure 3 show the simulated relative efficiency comparison between non-recurrent competing risks and recurrent competing risks under a *partial repair model* with two competing risks for the component lifetimes and the system lifetimes, respectively. For ease of comparison, the simulated relative efficiencies were plotted as SCRM-based estimator relative to the RCRM-based estimator so that efficiencies are bounded by 1. Data sets with 30 systems were generated ($N = 1000$ replications) in order to estimate the parameters of the Weibull distributions so that 1000 estimates of the component lifetimes parameters were obtained. The estimated components lifetime distributions were evaluated at the 1st to the 99th percentiles. For each method, the biases and variances of the component lifetime estimates were calculated for these percentiles and compared via their mean-squared error (MSE) to estimate the relative efficiencies. This process yielded the smoothed curves of Figure 2 and also of Figure 3. These curves are smooth since the same data sets were used to estimate the bias and variance, hence the MSE, at each of the percentiles. The system lifetime is plotted up to the 95th percentile associated with the Weibull(2,1/2) distribution of

Simulated Relative Efficiency for Cause 1   **Simulated Relative Efficiency for Cause 2**
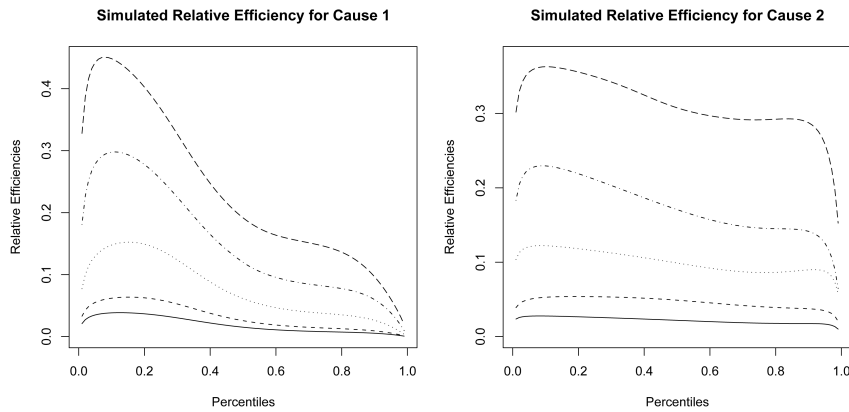
Figure 2: Simulated relative efficiencies of the SCRM-based estimators relative to the RCRM-based estimators for the two component lifetime distributions under Weibull models for 30 systems based on $N = 1000$ simulation replications. The curves correspond to rates of the exponential distribution pertaining to the length of the monitoring period. These exponential rates are $\gamma = 2, 1, 1/2, 1/5$, and $1/10$, from top to bottom of figures.

Cause 1. Individual curves represent a different rate $(2, 1, 1/2, 1/5,$ and $1/10$, respectively from top to bottom) associated with an exponential censoring distribution. As the rate increases (which results in a decrease in the mean of the monitoring times), the simulated efficiency remains less than 1. When the rate increases, the mean of the censoring times eventually becomes less than the mean of both of the component lifetime distributions for the two risks which ultimately results in data sets that contain mostly censored observations for each of the 30 systems. Therefore, the resulting relative efficiencies become closer to unity. However, it is also interesting to note what happens when the rate associated with the exponential censoring distribution decreases, which increases the mean length of the monitoring period. When this occurs, the simulated relative efficiency converges to 0, indicating that the increase in information gathered through repairing and recurring observations is highly beneficial. The interesting multi-peaked shapes of the curves can be attributed to the interplay between the variance curves of the estimates for the SCRM and RCRM scenarios as can be seen, for example, in Figure 4 which depicts these curves for the topmost curve of Figure 2 associated with an exponential censoring rate $\gamma = 2$. The simulated MSEs at the 10th percentile are 0.001170 and 0.00261, respectively for

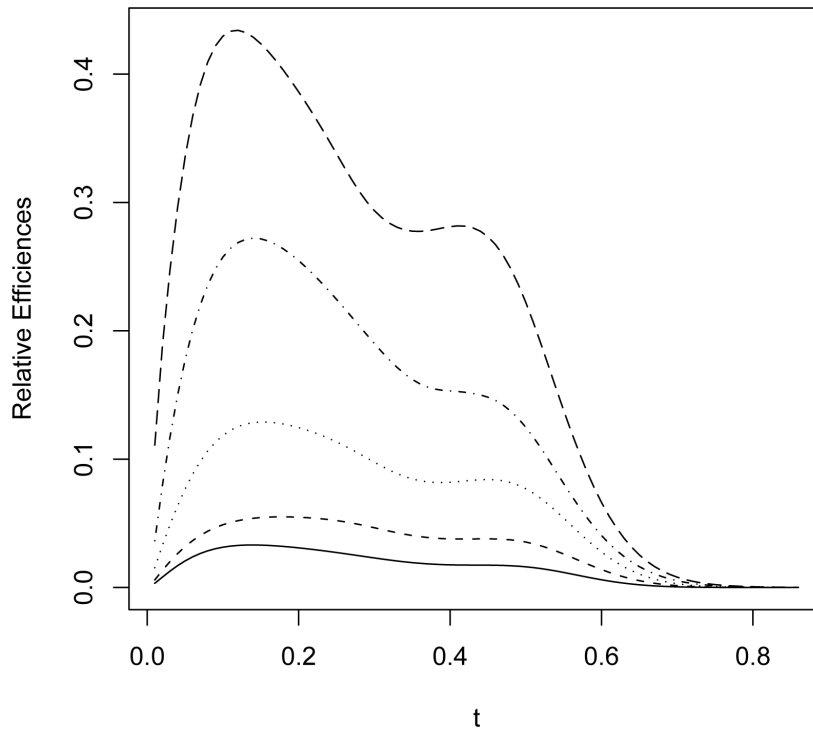**Simulated Relative Efficiences for System**

Figure 3: Simulated relative efficiency of the SCRM-based system lifetime estimator relative to the RCRM-based system lifetime estimator for the two component lifetime distributions under Weibull models for samples of size 30 based on $N = 1000$ simulation replications. The curves correspond to rates of the exponential distribution pertaining to the length of the monitoring period. These exponential rates are $\gamma = 2, 1, 1/2, 1/5$, and $1/10$, from top to bottom of figures.

the RCRM and SCRM models, which yields a simulated relative efficiency of approximately 0.4485. Comparatively, the simulated variances at the 10th percentile are 0.001168 and 0.00255, respectively. Visual inspection reveals that as the difference between the variances increases in Figure 4 (with the largest difference being at approximately the 70th percentile), a corresponding change can be seen in the simulated relative efficiency which changes concavely at approximately the 70th percentile. The biases of the SCRM and RCRM estimates are also shown in Figure

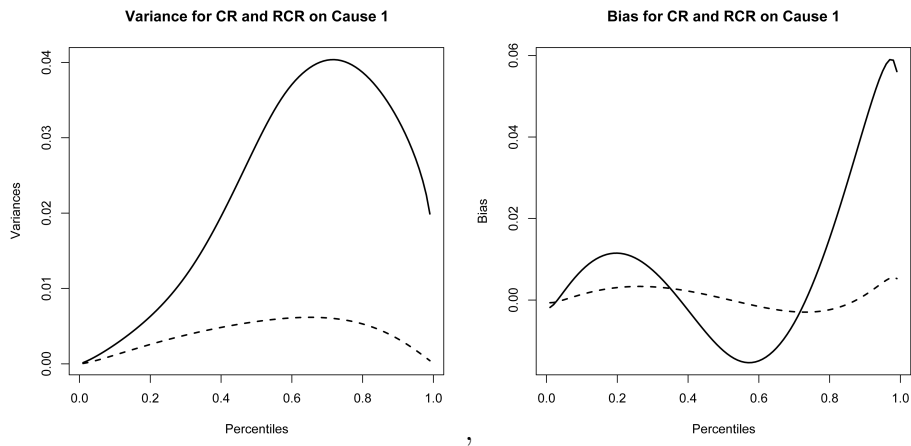4 but become essentially negligible in the MSEs since they get squared.

Figure 4: Variance and bias curves of the simulated estimates for the component lifetime distribution associated with $F_1(t)$ and with an exponential censoring rate of $\gamma = 2$. SCRM is depicted by a solid curve and RCRM by a dashed curve.

Overall, for the parameter values used in this simulation, there is a significant increase in efficiency of the RCRM-based estimator relative to the SCRM-based estimator. We expect that this efficiency behavior will also be the case for other parameter values, though more empirical investigations will be required especially for other distributional models.

5   Illustration Using Car Warranty Data

The time of failures (measured in mileage) for a sample of 189 cars under warranty were recorded together with the mode of failure (indicated by a 1 or 2). The circumstances of the data collection are not completely known to us, and it may be that this data suffers from selection bias, that is, only those cars that had warranty claims were in the sampled population. This will entail that the resulting estimates of the failure distributions based on this data set will tend to be stochastically smaller than the true failure time distributions of all the cars under warranty. This data, which was provided to us by Professor Ananda Sen of the University of Michigan, is pictured in Figure 5 with blue triangles representing failures attributed to mode 1 and red circles denoting failures attributed to mode 2. In some cases, there were multiple failures recorded

at the same time. When this occurred, only the first recorded failure mode was used in the analysis. The final observation for each of these cars is a failure event. Thus, note that the actual data accrual does not exactly coincide with our model where the time to the end of monitoring is noninformative about the failure distributions. Additionally, the data possess discrete inter-event times. As such, caution should be observed in interpreting the results of this data illustration.

There were 153 failures attributed to failure mode 1 and 158 attributed to failure mode 2. The average (standard deviation) of observed miles between mode 1 failures was 675.1 (812.7) miles. Comparatively, failure mode 2 had an average (standard deviation) of observed miles between failures of 880.5 (800.8) miles. The data is modeled as partial repairs since typically car mechanics, but as we all know not always, only repair the failed components. Weibull distributions are assumed for the inter-event times for each of the failure modes. The estimates of the parameters are $\hat{\alpha}_1 = 0.553$, $\hat{\beta}_1 = 1465.011$, $\hat{\alpha}_2 = 0.812$, and $\hat{\beta}_2 = 1351.176$. Utilizing equation (3), estimates of the standard errors of these parameter estimates are obtained to be $\hat{se}(\hat{\alpha}_1) = 0.040$, $\hat{se}(\hat{\beta}_1) = 192.609$, $\hat{se}(\hat{\alpha}_2) = 0.059$, and $\hat{se}(\hat{\beta}_2) = 121.099$. The associated marginal distribution curves and the system life distribution are depicted in Figure 6 for this data. Additionally, Figure 6 depicts the non-parametric estimator of the system life distribution based on the product limit estimators of the marginal distributions using *only* the first event (SCRM). Based on the discrepancies between the parametric estimator and the non-parametric estimator of the system life, we see that the Weibull assumption may not be an appropriate model for this data, *provided* that the partial repair assumption is valid.

6 Simulation Studies

In order to demonstrate the small sample properties of our estimators, simulations that are similar to those for the efficiency comparisons were performed. However, the intent of these simulations was to demonstrate the properties of the estimation procedures for small samples. In these simulations we consider a system with three competing risks operating under partial repair with Weibull inter-event time distributions with distinct parameters for each risk, (α_1, β_1), (α_2, β_2), and (α_3, β_3). For the first and second simulations, 1000 data sets were simulated with $n = 5$ and $n = 10$ units, respectively, each operating under three *partially repaired* Weibull causes of failure with $\boldsymbol{\theta} = (\alpha_1 = 2, \beta_1 = 2, \alpha_2 = 3, \beta_2 =$
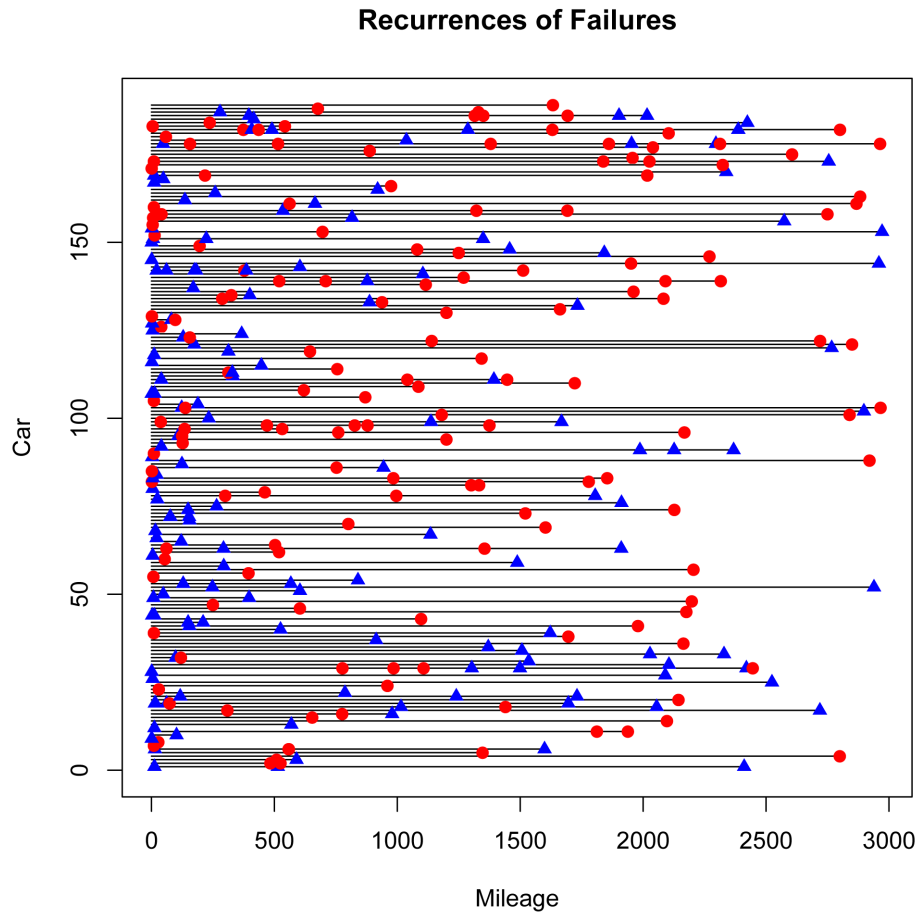
Recurrences of Failures

Figure 5: Recurrences of two types of failure modes for 189 cars under warranty. Failure mode 1 = blue triangle, Failure mode 2 = red circle.

$3, \alpha_3 = 4, \beta_3 = 4$). Censoring times were randomly generated from an exponential distribution with mean 4. Utilizing the `constrOptim` function in R to minimize the negative log-likelihood, the estimates are obtained. The average estimate and standard error for the 1000 estimates are reported in Table 1 for $n = 5$ units and Table 2 for $n = 10$ units. Histograms of the parameter estimates are given in Figure 7 for the five units case and Figure 9 for the ten units case. There were on average, approximately 43, 65, and 86 observations of each of the three Weibull causes, respectively, for the five units case. Even with these relatively large number of observations per simulation, the estimated sampling distributions of the estimators for each of the parameters ex-

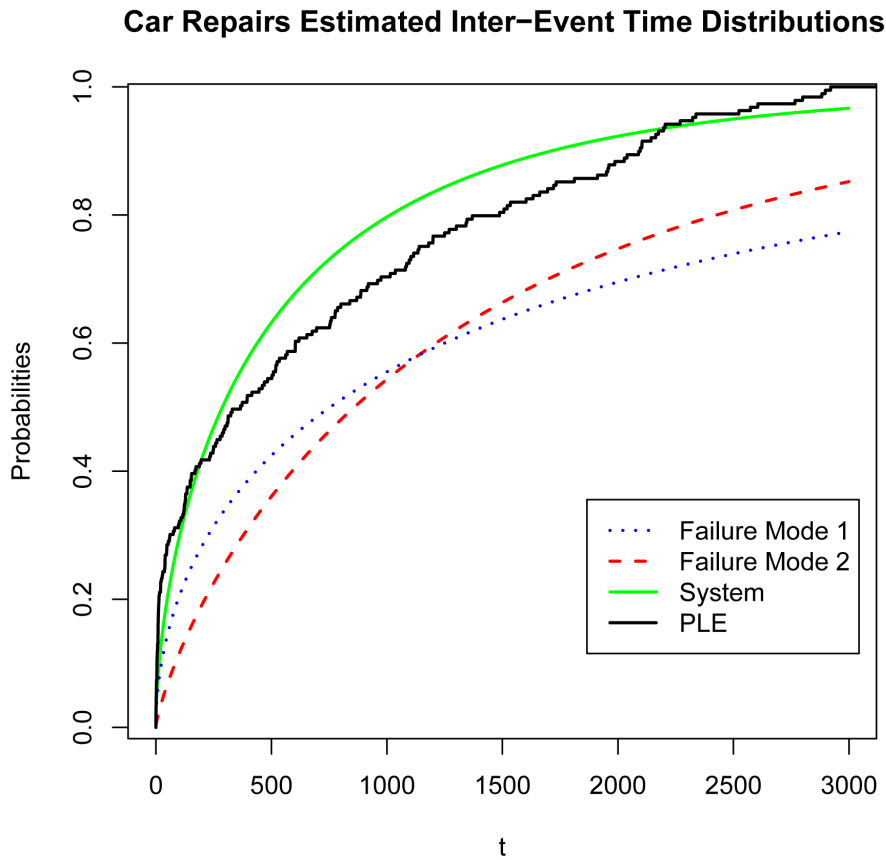**Car Repairs Estimated Inter−Event Time Distributions**

Figure 6: Estimated inter-event time distributions for two types of failure modes based on 189 cars under warranty.

hibit some skewness to the right for the shape parameters, α_1, α_2, and α_3. This feature of the sampling distributions is more pronounced for the shape parameters' estimates. With these sample sizes, the simulated sampling distributions of the scale parameters, β_1, β_2, and β_3, are approximately normal. A reviewer has suggested that a log-transformation of the estimates could improve the normal approximations to the sampling distributions. This suggestion is partly empirically validated by looking at Figure 8 which is the sampling distribution histogram of the logged estimates of α_1 from Figure 7. Utilizing the logged estimates may become beneficial, for instance, when constructing confidence intervals for the parameter values through the use of the normal approximation to the sampling distribution.

When we increased to $n = 10$ units, the average number of observations increased to approximately 88, 132, and 174 observations for each of the three Weibull causes, respectively. With this increase in the number of observations, the simulated sampling distributions for the shape parameters, as well as the scale parameters, show approximately normal distributions.

True	Average Number of Events per Replication	Estimated Parameter (Estimated Standard Error)	Estimated Bias
$\alpha_1 = 2$	43.184	2.082 (0.286)	0.082
$\beta_1 = 2$		1.996 (0.176)	-0.004
$\alpha_2 = 3$	64.943	3.096 (0.354)	0.096
$\beta_2 = 3$		2.997 (0.151)	-0.003
$\alpha_3 = 4$	85.984	4.072 (0.397)	0.072
$\beta_3 = 4$		4.002 (0.126)	0.002

Table 1: 1000 simulations of five units operating under three Weibull recurrent competing risks under a partial repair strategy and with a censoring mechanism generated by an exponential distribution with a mean of 4.

True	Average Number of Events per Replication	Estimated Parameter (Estimated Standard Error)	Estimated Bias
$\alpha_1 = 2$	87.746	2.037 (0.187)	0.037
$\beta_1 = 2$		2.001 (0.116)	0.001
$\alpha_2 = 3$	131.961	3.047 (0.213)	0.047
$\beta_2 = 3$		3.006 (0.097)	0.006
$\alpha_3 = 4$	174.411	4.033 (0.247)	0.033
$\beta_3 = 4$		4.004 (0.083)	0.004

Table 2: 1000 simulations of 10 units operating under three Weibull recurrent competing risks under a partial repair strategy and with a censoring mechanism generated by an exponential distribution with a mean of 4.

To increase the number of observations for each cause, the mean of the censoring mechanism's distribution and the number of units were increased to 6 and 50, respectively. The final simulation estimated the parameters for three partially repaired Weibull causes of failure with $\boldsymbol{\theta} = (\alpha_1 = 2, \beta_1 = 2, \alpha_2 = 2.1, \beta_2 = 2.1, \alpha_3 = 2.2, \beta_3 = 2.2)$ for 50 units under an exponentially distributed censoring mechanism with mean 6. Again, there were 1000 simulation replications. Results from this simulation are shown in Table 3 and Figure 10. The simulated sampling distributions for the shape and scale parameters exhibit approximate normality.
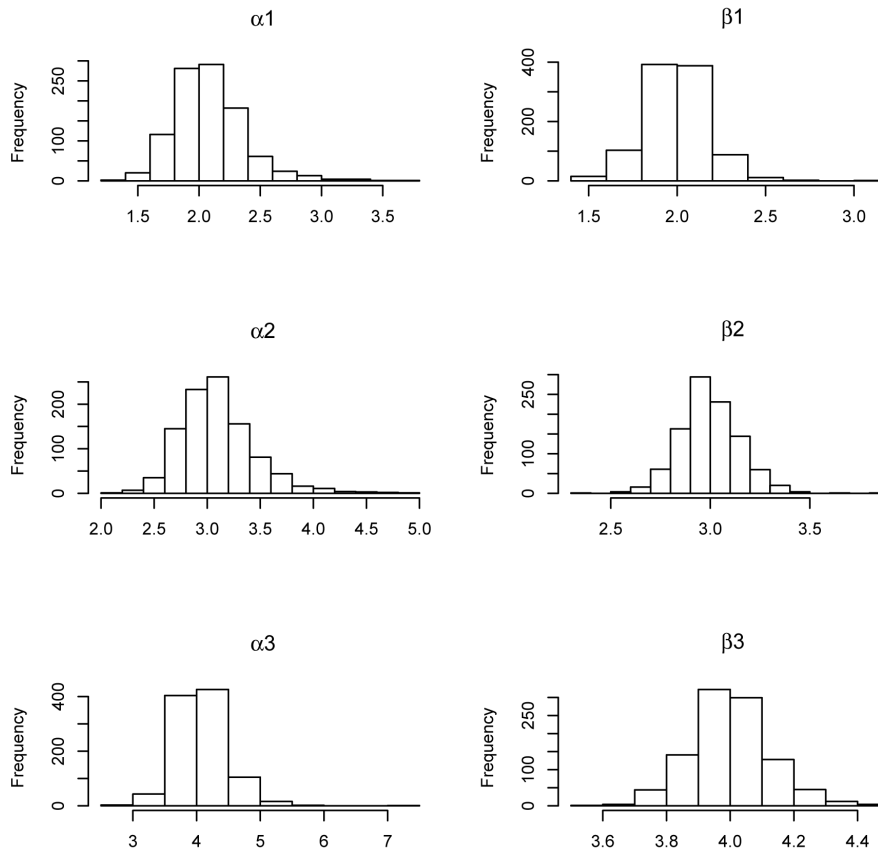
Figure 7: Histogram of parameter estimates using $\alpha_1 = 2, \beta_1 = 2, \alpha_2 = 3, \beta_2 = 3, \alpha_3 = 4$, and $\beta_3 = 4$ for 1000 simulations of 5 units.

7 Concluding Remarks

This paper proposes a resource-efficient method for analyzing recurrent competing risks data. In particular, recurrent competing risks allow for a better use of the monitoring period by utilizing a repair strategy. Perfect and partial repair strategies were considered here. Other repair strategies not utilized in this paper offer alternative realistic applications. For example, a minimal repair strategy would replace the failed component with another component of equivalent lifetime, which is an option when used parts are available. Given the resource-efficient nature of recurrent competing risks in which monitoring for more events continues after a failure, it is not surprising that the recurrent competing risks method leads to more efficient estimation of the marginal distribution
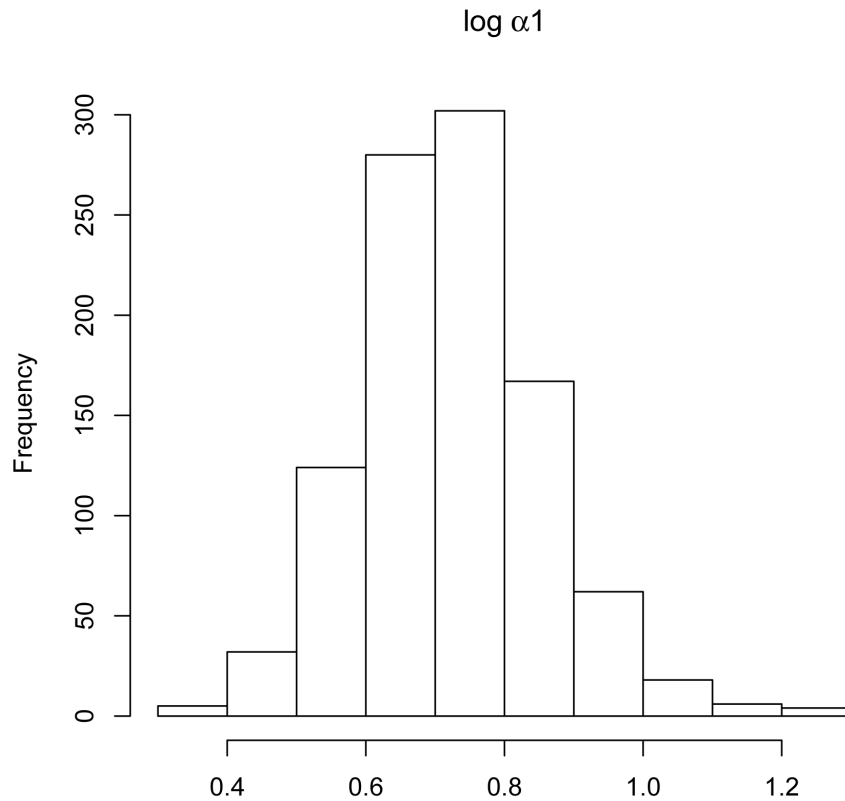
log α1

Figure 8: Histogram of the logged parameter estimates of α_1 depicted in Figure 7.

functions of the component lifetimes, and consequently of the system life distribution, when compared to the non-recurrent competing risks model. The sampling distribution of the estimators are shown to be approximately normal distributions for large sample sizes. Simulation studies demonstrated the small (to moderate) sample properties of the estimators.

The methods were applied to a car warranty data to estimate the inter-event time distributions for the latent failure times. For the car warranty data, it was assumed that the failures occurred under Weibull marginal distributions, and under a partial repair strategy, aside from the independence of the latent time-to-event variables. A possibly more robust approach to the analysis of such types of recurrent competing risks data sets are through nonparametric methods where parametric assumptions are not imposed. Recent work on this has been done in [6],
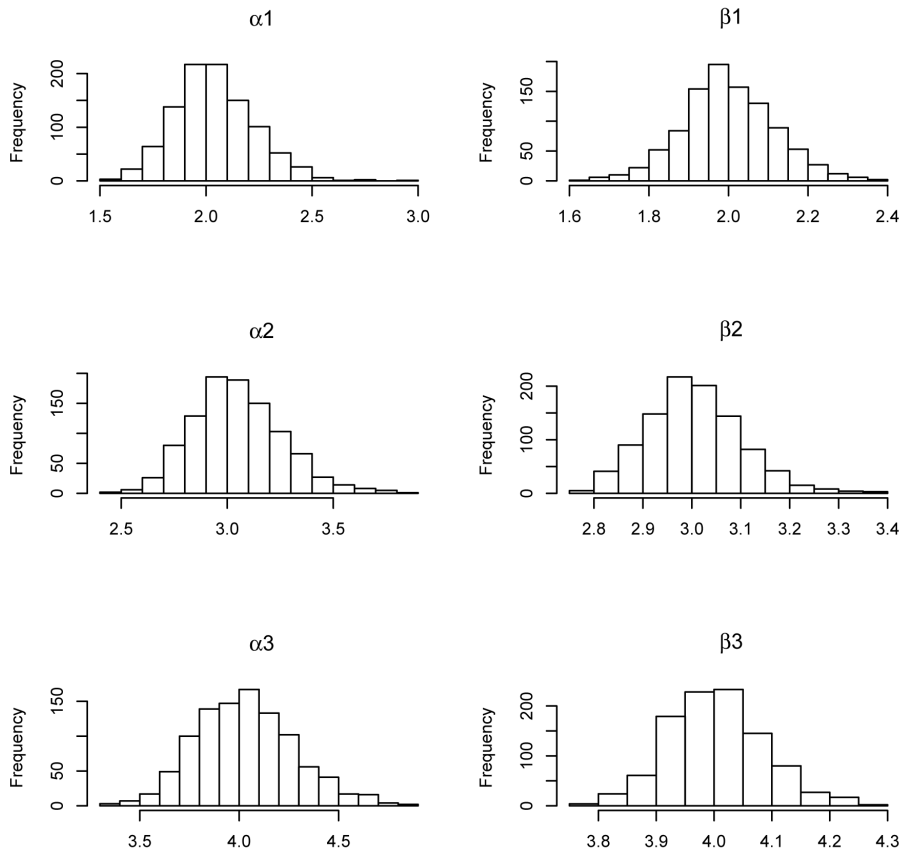
Figure 9: Histogram of parameter estimates using $\alpha_1 = 2, \beta_1 = 2, \alpha_2 = 3, \beta_2 = 3, \alpha_3 = 4$, and $\beta_3 = 4$ for 1000 simulations of 10 units.

as well as in [16]. We also mention that a latent variable approach to the modeling of competing risks has its limitation as has been pointed out in [7, 10, 11, 17] among others. An important limitation, for instance, is that a competing risks data is insufficient to empirically verify the independence assumption among the latent failure-time variables, which could be a serious limitation in biomedical applications, but possibly may not be so serious for engineering and reliability applications. It will therefore be of interest to develop methods for dealing with recurrent competing risks data without using a latent variable approach and with possible dependencies among the different risks.

We close by noting that in a World where Time is Money and Money is Time, it is important to be considerate of both. When systems can

True	Average Number of Events per Replication	Estimated Parameter (Estimated Standard Error)	Estimated Bias
$\alpha_1 = 2.0$	433.649	2.007 (0.077)	0.007
$\beta_1 = 2.0$		2.001 (0.049)	0.001
$\alpha_2 = 2.1$	455.463	2.108 (0.074)	0.008
$\beta_2 = 2.1$		2.100 (0.049)	0.000
$\alpha_3 = 2.2$	478.039	2.204 (0.079)	0.004
$\beta_3 = 2.2$		2.201 (0.050)	0.001

Table 3: 1000 simulations of 50 units operating under three Weibull recurrent competing risks under a partial repair strategy and with a censoring mechanism generated by an exponential distribution with a mean of 6.
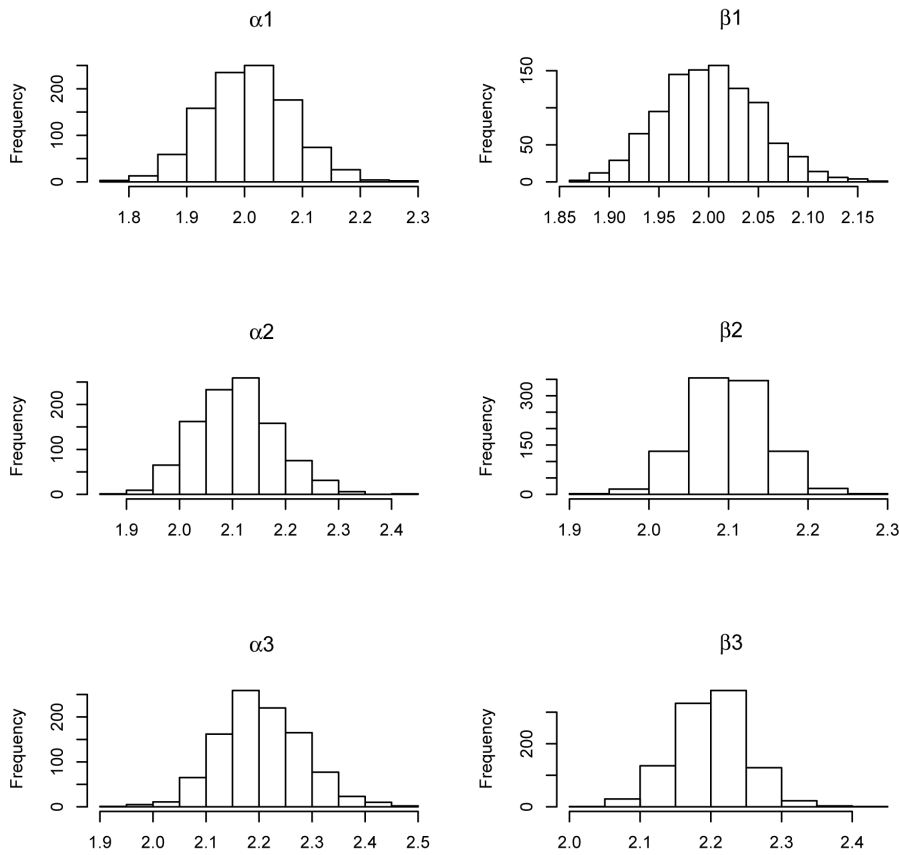

Figure 10: Histogram of parameter estimates using $\alpha_1 = 2, \beta_1 = 2, \alpha_2 = 2.1, \beta_2 = 2.1, \alpha_3 = 2.2, \beta_3 = 2.2$ for 1000 simulations of 50 units.

be monitored for periods that extend beyond the first failure time, it is beneficial to allow systems to be repaired and to observe the recur-

rences of failures over the monitoring period. So far, a limited amount of data has been collected in this manner so as to utilize information from recurrent events and competing risks. A data set that has been used in [12] concerns a recurrent competing risks setting under perfect repair by observing the failure of shunts due to multiple causes; see [4] for additional analysis of this data set. The analyses of this data set are first performed by ignoring the unique causes of failure and additionally using multiplicative regression models for each of the causes of failures to model three recurrences of shunt failures. However, we believe that it is now the right time that future studies and data collection should allow for recurrent competing risks to obtain more information, with a consequent beneficial result of producing more reliable conclusions, which will ultimately translate to better real-life decisions.

Acknowledgements

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