On the Monotone Behavior of Time Dependent Entropy of Order α

Mahdi Mahmoudi[1]**, Majid Asadi**[2]

[1]Faculty of Mathematics, University of Semnan, Iran.
[2]Department of Statistics, University of Isfahan, Iran. (m.asadi@stat.ui.ac.ir)

Abstract. In this paper we study some monotone behavior of the residual (past) entropy of order α. We prove that, under some relation between the hazard rates (reversed hazard rates) of two distributions functions F and G, when the residual (past) entropy of order α of F is decreasing (increasing) then the residual (past) entropy of G is decreasing (increasing). Using this, several conclusions regarding monotone behavior of residual (past) entropy of order α of $(n-k+1)$-out-of-n systems and record values are derived. Some results on the residual (past) entropy of order α of equilibrium distributions are also obtained.

1 Introduction

The entropy of order α (EO(α)) (also known as Rényi's entropy (Rényi (1961)) is a one parameter extension of Shannon entropy. It

has a wide range of applications in many fields from electrical engineering, physics, chemistry and computer sciences to economics, biology and medicine genetics. Let X be a non-negative absolutely continuous random variable with density function f. The EO(α) of X, which we denote by $H_\alpha(f)$, is defined as follows:

$$H_\alpha(f) = \frac{1}{1-\alpha} \log \int_0^\infty f^\alpha(x)dx,$$

where $\alpha > 0$, $\alpha \neq 1$. It is well known that when α tends to 1, $H_\alpha(f)$ tends to Shannon entropy (Shannon (1948)) which we denote by $H(f)$. That is

$$\lim_{\alpha \to 1} H_\alpha(f) = H(f) = -\int_0^\infty f(x) \log f(x)dx.$$

Let the random variable X denote a duration such as the lifetime of a system. Usually in reliability theory and survival analysis, when the system is still alive at time t, one is interested in studying the properties of the residual lifetime of the system. The residual lifetime of the system, which we denote by X_t, is $X_t = [X - t|X > t]$. Under the assumptions that the random variable X has distribution function F and survival function $\bar{F} = 1 - F$, the survival function of X_t is

$$\bar{F}_t(x) = \begin{cases} \frac{\bar{F}(x+t)}{\bar{F}(t)} & x > 0 \\ 1 & \text{otherwise,} \end{cases}$$

Ebrahimi (1996) has proposed a time dependent Shannon entropy which measures the information in the residual lifetime distribution as follows.

$$\begin{aligned} H(f;t) &= -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)}dx \\ &= \log \bar{F}(t) - \frac{1}{\bar{F}(t)} \int_t^\infty \log f(x)f(x)dx. \end{aligned}$$

$H(f;t)$ has the properties of $H(f)$ and clearly $H(f;0) = H(f)$. Several properties of this measure are derived by Ebrahimi (1996), Ebrahimi and Kirmani (1996), Asadi and Ebrahimi (2000) and Belzunce et al. (2004). If $\lambda_F(t) = \frac{f(t)}{\bar{F}(t)}$ denotes the hazard rate of X, then it can be shown, using simple algebra, that $H(f;t)$ is connected to $\lambda_F(t)$ as follows

$$H(f;t) = 1 - E(\log \lambda_F(X)|X > t). \tag{1}$$

Asadi et al. (2005) developed the concept of residual and past entropies of order α by giving some time dependent Rényi's entropy for measuring the information in residual and past lifetime distributions(see also Nanda and Paul (2006a)). The residual EO(α) (REO(α)), of the residual lifetime density is defined by

$$H_\alpha(f;t) = \frac{1}{1-\alpha} \log \int_t^\infty \frac{f^\alpha(x)}{\bar{F}^\alpha(t)} dx.$$

It is clear that $H_\alpha(f;0) = H_\alpha(f)$. The REO($\alpha$) can be represented in terms of hazard rate function $\lambda_F(t)$ as follows.

$$H_\alpha(f;t) = \frac{1}{1-\alpha} \log E\left[(\lambda_F(X_\alpha))^{\alpha-1}|X_\alpha > t\right] - \frac{1}{1-\alpha} \log \alpha, \quad (2)$$

where X_α is a random variable with survival function $\bar{F}^\alpha(t)$.

There are situations in which one is interested in past life time of a system. If again we assume that X denotes the lifetime of a system and the system is assumed to fail sometime before t, the past lifetime of the system is $X_{[t]} = [t - X|X < t]$. The Shannon entropy of the past lifetime distribution, which we denote by $H(f;[t])$, is defined by

$$H(f;[t]) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} dx.$$

Several properties of the past Shannon entropy are explored by Di Crescenzo and Longobardi (2002). The past entropy of order α (PEO(α)) of the past lifetime $X_{[t]}$ is also given by

$$H_\alpha(f;[t]) = \frac{1}{1-\alpha} \log \int_0^t \frac{f^\alpha(x)}{F^\alpha(t)} dx,$$

which is extensively studied by Nanda and Paul (2006b). The (PEO(α)) can be represented in terms of reversed hazard rate $r_F(t) = \frac{f(t)}{F(t)}$ as follows:

$$H_\alpha(f;[t]) = \frac{1}{1-\alpha} \log E[r_F^{\alpha-1}(X^*)|X^* < t] - \frac{1}{1-\alpha} \log \alpha.$$

The aim of the present paper is to explore the monotone behavior of $H_\alpha(f;t)$ and $H_\alpha(f;[t])$. In Section 2, we prove a theorem showing that under some relation between the hazard rates of two distribution functions F and G, when the distribution function F has an increasing $H_\alpha(f;t)$ then the REO(α) of G, $H_\alpha(g;t)$, is also increasing. Using this, we obtain several results regarding the monotone behavior

of REO(α) of $(n - k + 1)$-out-of-n systems and record values. It is shown that when the components of a parallel system has an increasing REO(α) then so is the REO(α) of the system. The monotonicity of REO(α) in the proportional odds family is also investigated in this section. In Section 3, we concentrate on PEO(α), $H_\alpha(f; [t])$. Analog results, as given in Section 2, are obtained for the PEO(α)'s. In this section the main result is proved based a relation between the reversed hazard rates of two distributions. Using that some results regarding $(n - k + 1)$-out-of-n systems are derived. Finally in Section 3, we obtain some results on the REO(α) of equilibrium distributions.

2   The Residual Entropy of Order α

In this section we focus on the REO(α), $\alpha > 0$, $\alpha \neq 1$. The following theorem is the key result to obtain the subsequent results of this section.

Theorem 2.1. *Let X and Y be two nonnegative absolutely continuous random variables with density functions f and g, hazard rates λ_F and λ_G, survival functions $\bar{F}$ and $\bar{G}$ and REO(α)'s $H_\alpha(f; t)$ and $H_\alpha(g; t)$, respectively. Let also $0 \leq \theta(t) \leq 1$ be a nonnegative increasing function such that $\lambda_G(t) = \theta(t)\lambda_F(t)$, $t \geq 0$. Further, let $\lim_{t \longrightarrow \infty} \frac{\bar{G}(t)}{\bar{F}(t)} < \infty$. Under these conditions if $H_\alpha(f; t)$ is decreasing then so is $H_\alpha(g; t)$.*

Proof. First, we assume that $\alpha > 1$. In this case, using equation (2), the assumption that $H_\alpha(f; t)$ is decreasing is equivalent to say that

$$\frac{\int_t^\infty \alpha \lambda_F^{\alpha-1}(x) f(x) \bar{F}^{\alpha-1}(x) dx}{\bar{F}^\alpha(t)}$$

is an increasing function of t. This, in turn, is equivalent to say that for $t > 0$

$$\frac{\int_t^\infty \alpha \lambda_F^{\alpha-1}(x) f(x) \bar{F}^{\alpha-1}(x) dx}{\bar{F}^\alpha(t)} \geq \lambda_F^{\alpha-1}(t). \qquad (3)$$

On the other hand since $\theta(t)$ is assumed to be increasing, we have

$$\frac{\int_t^\infty \alpha\theta^{\alpha-1}(x)\lambda_F^{\alpha-1}(x)f(x)\bar{F}^{\alpha-1}(x)dx}{\bar{F}^\alpha(t)}$$

$$\geq \theta^{\alpha-1}(t)\frac{\int_t^\infty \alpha\lambda_F^{\alpha-1}(x)f(x)\bar{F}^{\alpha-1}(x)dx}{\bar{F}^\alpha(t)}$$

$$\geq (\theta(t)\lambda_F(t))^{\alpha-1}, \tag{4}$$

where the last inequality is based on (3). Assuming that

$$m_1(t) = E[(\theta(X^*)\lambda_F(X^*))^{\alpha-1}|X^* > t],$$

where X^* is a random variable with survival function $\bar{F}^\alpha(t)$, $t > 0$, (4) implies that m_1 is an increasing function of t. We now show that

$$m_2(t) = E[(\theta(Y^*)\lambda_F(Y^*))^{\alpha-1}|Y^* > t]$$

is an increasing function of t, where Y^* is a random variable with survival function $\bar{G}^\alpha(t)$. Define $\beta(t)$ as follows

$$\beta(t) = \bar{G}^\alpha(t)[m_1(t) - m_2(t)].$$

Asadi and Ebrahimi (2000) showed that, under the same assumptions of the present theorem, $\beta(t)$ is an increasing function of t and that $\beta(t) < 0$ for all $t > 0$. This, in turn, implies that for $t > 0$, $m_1(t) \leq m_2(t)$. From this and inequality (8) we get

$$m_2(t) \geq (\theta(t)\lambda_F(t))^{\alpha-1} \qquad t > 0.$$

Hence, it is concluded that $m_2(t)$ is an increasing function of t. That is, $E[\lambda_G(Y^*)|Y^* > t]$ is increasing in t and thus $H_\alpha(g;t)$ is a decreasing function of t.

Now assume that $0 < \alpha < 1$. In this case $H_\alpha(f;t)$ is decreasing in t if and only if

$$E[\lambda_F^{\alpha-1}(X^*)|X^* > t] \leq \lambda_F^{\alpha-1}(t).$$

Under the assumption that $0 < \alpha < 1$ (on noting that $\theta^{\alpha-1}(t)$ is decreasing in t), we obtain

$$m_1(t) = E[(\theta(X^*)\lambda_F(X^*))^{\alpha-1}] \leq (\theta(t)\lambda_F(t))^{\alpha-1}. \tag{5}$$

Defining $\beta(t)$ as above and using the same arguments as used to prove the case $\alpha > 1$, we can show that $\beta(t) > 0$ and that $\beta(t)$ is increasing

in t. Hence, we conclude that, $m_1(t) \geq m_2(t)$. Using inequality in (5) we get that

$$m_2(t) = E[(\theta(Y^*)\lambda_F(Y^*))^{\alpha-1}] \leq (\theta(t)\lambda_F(t))^{\alpha-1}.$$

That is $E[\lambda_G^{\alpha-1}(Y^*)|Y^* > t]$ is decreasing and hence $H_\alpha(g;t)$ is decreasing. This completes the theorem.

Remark 2.1. Asadi and Ebrahimi (2000) have proved the same result for the case where $\alpha \to 1$. That is, under the assumption of Theorem 2.1 they showed that if the residual Shannon entropy of F is decreasing in t then so is the residual Shannon entropy of G. One can show that the result of Theorem 2.1 reduces to the result of Asadi and Ebrahimi (2000) if we take the limit when $\alpha \to 1$ in all steps of Theorem 2.1. Hence, using the result of Asadi and Ebrahimi and the result of Theorem 2.1, we conclude, under the assumptions of the theorem, if $H_\alpha(f;t)$ is decreasing then $H_\alpha(g;t)$ is decreasing in t for all $\alpha > 0$.

In order to see an immediate consequence of Theorem 2.1 we need to mention the concept of proportional hazards model. The concept of proportional hazards model, which plays an important role in reliability and survival analysis, is introduced by Cox (1972). Let X and Y be two continuous random variables with survival functions $\bar{F}$ and $\bar{G}$, respectively. The random variables X and Y are said to have proportional hazards if there exists a constant $c > 0$ such that for $t > 0$

$$\bar{G}(t) = (\bar{F}(t))^c. \tag{6}$$

In this case the hazard rates of X and Y are related as $\lambda_G(t) = c\lambda_F(t)$. Now the following corollary can easily be obtained from Theorem 2.1.

Corollary 2.1. *Let X and Y have proportional hazards as (6) with $c \in (0,1)$. If the $REO(\alpha)$ of X is decreasing in time then so is the $REO(\alpha)$ of Y.*

Order statistics play important role in many branches of applied probability and statistics. In particular, in reliability theory the lifetime of a $(n-k+1)$-out-of-n system is equivalent to the kth order statistics in a sample of size n. Let $X_1, X_2, ..., X_n$ be iid random variables from a distribution function F with density function f. Let also $X_{1:n}, X_{2:n}, ..., X_{n:n}$ denote the order statistics corresponding to

the sample. Then $X_{k:n}$ represents the lifetime of a $(n - k + 1)$-out-of-n system. Under the assumption that $F_{k:n}$, $f_{k:n}$ and $\lambda_{F_{k:n}}$ denote the distribution function, the density function and the hazard rate function of $X_{k:n}$, respectively, it can be shown that

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}(F(x))^{k-1}(\bar{F}(x))^{n-k}f(x),$$

$$F_{k:n}(x) = \sum_{i=k}^{n} \binom{n}{i}(F(x))^{i}(\bar{F}(x))^{n-i},$$

and

$$\lambda_{F_{k:n}}(x) = \frac{n!}{(k-1)!(n-k)!}\lambda_F(x)\frac{(\phi(x))^{k-1}}{\sum_{i=0}^{k-1}\binom{n}{i}(\phi(x))^{i}}, \qquad (7)$$

where $\phi(x) = \frac{F(x)}{\bar{F}(x)}$. For more details on order statistics, we refer to David and Nagaraja (2003). An important special case of $(n-k+1)$-out-of-n systems is parallel system which corresponds to the case of $k = n$. Now we have the following corollary to Theorem 2.1 regarding the parallel systems.

Corollary 2.2.   *If, $H_\alpha(f;t)$, the REO(α) of the components of a parallel system is decreasing in t then $H_\alpha(f_{n:n};t)$, the REO(α) of the system is also decreasing in t.*

Proof. It is easy to see from (7) that the hazard rate of the system is $\lambda_{F_{n:n}}(t) = \lambda_F(t)\theta(t)$ where

$$\begin{aligned}
\theta(t) &= \frac{n(F(t))^{n-1}}{\sum_{i=0}^{n-1}F(t)^{i}} \\
&= \frac{n}{\sum_{i=0}^{n-1}F(t)^{i-n+1}}
\end{aligned}$$

Now, since $F(t)$ is an increasing function of t, one can show that $\theta(t)$ is increasing in t. Also it is not difficult to show that $\theta(t) \in (0, 1)$. On the other hand it can be easily seen that $\lim_{t \longrightarrow \infty} \frac{\bar{F}_{n:n}(t)}{\bar{F}(t)} = n$. Therefore the assumptions of Theorem 2.1 hold and hence $H_\alpha(f_{n:n}, t)$ is a decreasing function of t.

Example 2.1.   Let $m_F(t)$ denote the mean residual life function of X. That is,

$$m_F(t) = E(X - t|X > t).$$

Nanda and Paul (2006a) proved that if X has a decreasing MRL then the REO(α) is a decreasing function of t. Some well known distributions with decreasing mean residual life are the Weibull distribution and the Gamma distribution with shape parameter greater than 1. Hence, as an application of Corollary 2.2, one can conclude that when the components of a parallel system have Weibull distribution with shape parameter greater than one (or Gamma distribution with shape parameter greater than one) then the REO(α) of the components are decreasing and hence, based on Corollary 2.2, the REO(α) of the system is also decreasing.

Consider now two sets of iid components of sizes n_1 and n_2 which are connected in systems with (n_1-k_1+1)-out-of-n_1 and (n_2-k_2+1)-out-of-n_2 structures, respectively. We assume that the components have a common distribution function F. If we denote the hazard rates of the systems by $\lambda_{F_{k_1:n_1}}$ and $\lambda_{F_{k_2:n_2}}$, respectively then it can be shown that $\lambda_{F_{k_2:n_2}}(x) = \theta(x)\lambda_{F_{k_1:n_1}}(x)$, where

$$\theta(x) = \frac{c(k_2, n_2)}{c(k_1, n_1)}\phi^{k_2-k_1}\frac{\sum_{i=0}^{k_1-1}\binom{n_1}{i}\phi^i}{\sum_{j=1}^{k_2-1}\binom{n_2}{j}\phi^j},$$

in which $c(k, n) = \frac{n!}{(n-k)!(k-1)!}$ and $\phi = \phi(x) = F(x)/\bar{F}(x)$ is increasing in x. It is not difficult to verify, in the following cases, that $\theta(x)$ is increasing in x and its range is a subset of $(0, 1)$ (see, Nagaraja (1990)).

- $n_1 = n_2 = n$, $k_1 = k$, $k_2 = k + 1$

- $n_1 = n$, $n_2 = n - 1$, $k_1 = k_2 = k$

- $n_1 = n$, $n_2 = n + 1$, $k_1 = k$, $k_2 = k + 1$

Also it can be easily shown that other conditions of Theorem 2.1 hold. Hence, we have the following corollary to Theorem 2.1.

Corollary 2.3. *Let $H_\alpha(f_{k:n}; t)$ denote the REO(α) of a $(n-k+1)$-out-of-n system which is decreasing in t. Then*

(a) *$H_\alpha(f_{k+1:n}; t)$, the REO(α) of a $(n - k)$-out-of-n system, is also decreasing.*

(b) *$H_\alpha(f_{k:n-1}; t)$, the REO(α) of a $(n - k)$-out-of-$(n - 1)$ system, is also decreasing.*

(c) $H_\alpha(f_{k+1:n+1}; t)$, *the REO(α) of a $(n-k+1)$-out-of-$(n+1)$ system, is also decreasing.*

   The next corollary gives an application of Theorem 2.1 associated to record values. The record values appear in many branches of applied sciences. The upper record values can be defined as a model for successive extremes in a sequence of iid random variables and hence they may be helpful to model successively largest insurance claims in non-life insurance, the highest water levels or highest temperatures etc. For more details about records and applications, one may refer to Ahsanullah (2004). Let $X_1, X_2, ...,$ be a sequence of iid random variables with a common absolutely continuous distribution function (cdf) F, probability density function (pdf) f, and survival function $\overline{F} = 1 - F$. Suppose that $X_{i:n}$ stands for the ith order statistic obtained from the first n observations. The sequence of upper records can be defined as

$$X_{U(n)} = X_{U_n:U_n}, \quad n = 0, 1, ...,$$

where

$$
\begin{aligned}
U_0 &= 1, \\
U_n &= \min\left\{ j : j > U_{n-1},\ X_j > X_{U_{n-1}:U_{n-1}} \right\}, \quad n \geq 1.
\end{aligned}
$$

Records can be viewed as maximum of a sample whose size is determined by the values of occurrence of the observations. It can be shown that under mild conditions, the structure of record values is the same as that of the occurrence times of some corresponding non-homogeneous Poisson process of some minimal repair and of the relevation transform (see Gupta and Kirmani, 1988).

   The marginal pdf of $X_{U(n)}$ is

$$f_{U(n)}(x) = \frac{[-\log \overline{F}(x)]^n}{n!} f(x), \quad x > 0, \quad n \geq 0,$$

and the survival function of $X_{U(n)}$, at value $v > 0$, is

$$\overline{F}_{U(n)}(v) = \sum_{j=0}^{n} \frac{[-\log \overline{F}(v)]^j}{j!} \overline{F}(v).$$

Hence the hazard rate of $X_{U(n)}$ can be represented as

$$\lambda_{U_n}(t) = \theta(x)\lambda_F(t),$$

where

$$\theta(t) = \frac{\frac{[\Lambda_F(t)]^n}{n!}}{\sum_{k=0}^{n} \frac{1}{k!}[\Lambda_F(t)]^k}$$

and $\Lambda_F(t) = -\log \bar{F}(t)$. Since $\theta(t)$ is increasing in t with range $(0, 1)$, then for the case where $\lim_{t \to \infty} \beta(t) = 0$ in Theorem 2.1, we get the following corollary.

Corollary 2.4. *Let $\{X_n; n \geq 1\}$ be a sequence of iid random variables having the distribution function F, density function f, hazard function λ_F and decreasing residual uncertainty $H(f; x)$. If f_{U_n} denotes the density function of nth upper record values then $H(f_{U_n}; x)$, the residual uncertainty of f_{U_n}, is decreasing.*

As an application of this corollary one can show that when the underlying distribution is Weibull with shape parameter greater than one, then all conditions of Theorem 2.1 (in particular the condition $\lim_{t \to \infty} \beta(t) = 0$) hold. Hence upper record values (and epoch times of the non-homogeneous Poisson process) generated by Weibull model has decreasing residual lifetime entropy of order α.

Corollary 2.5. Proportional odds family (also known as tilt parameter family) is a well known family in the literature (see e.g., Kirmani and Gupta (2001) and Marshall and Olkin (2007)). Let $F(x)$ be a distribution function and assume that $F(x|\eta)$ is defined in terms of F as follows

$$\frac{F(x|\eta)}{\bar{F}(x|\eta)} = \frac{1}{\eta} \frac{F(x)}{\bar{F}(x)}, \quad x > 0, \ \eta > 0,$$

where $\bar{F}(x)$ and $\bar{F}(x|\eta)$ denote the survival functions corresponding to F and $F(x|\eta)$, respectively. The parameter η is called a tilt parameter and $F(x|\eta)$ is said to be the *proportional odds family.* It is easily seen that

$$\bar{F}(x|\eta) = \frac{\eta \bar{F}(x)}{F(x) + \eta \bar{F}(x)}.$$

If $\lambda(x|\eta)$ and $\lambda(x)$ denote the hazard rates of $F(x|\eta)$ and $F(x)$, respectively, then we have

$$\lambda(x|\eta) = \theta(x)\lambda(x)$$

where
$$\theta(x) = \frac{1}{F(x) + \eta \bar{F}(x)}.$$

It is seen that for $\eta > 1$, $\theta(x)$ is an increasing function of x and also $\theta(x) \in (0, 1)$. Hence based on Theorem 2.1 if REO(α) of $F(x)$ is decreasing in time, then REO(α) of $F(x|\eta)$ is also a decreasing function of time. It should be pointed out that for $\eta \in (0, 1)$, using the same argument, it can be concluded that when REO(α) of $F(x|\eta)$ is decreasing in time, then REO(α) of $F(x)$ is also an decreasing function of time.

3   The Past Entropy of Order α

In this section we deal with PEO(α). The following theorem is the main result of the section.

Theorem 3.1.   *Let X and Y be two absolutely continuous non-negative random variables with density functions f and g, reversed hazard rates r_F and r_G, and PEO(α)'s $H_\alpha(f; [t])$ and $H_\alpha(g; [t])$, respectively. Let also $0 \le \theta \le 1$ be a nonnegative decreasing function such that $r_G(t) = \theta(x) r_F(t)$, $t \ge 0$. Further, let $\lim_{t \to 0} \frac{G(t)}{F(t)} < 0$. If $H_\alpha(f; [t])$ is increasing in t then so is $H_\alpha(g; [t])$.*

Proof. First note that the PEO(α) can be represented in terms of the reversed hazard rate $r_F(t)$ as follows

$$H_\alpha(f; [t]) = \frac{1}{1-\alpha} \log E[r_F^{\alpha-1}(X^*)|X^* < t] - \frac{1}{1-\alpha} \log \alpha \qquad (8)$$

where X^* is a random variable with distribution function $F^\alpha(t)$. Assume now that $\alpha > 1$. Then, using (8), $H_\alpha(f; [t])$ is increasing in t if and only if $E[r_F^{\alpha-1}(X^*)|X^* < t]$ is a decreasing function of t. Under the assumptions of the theorem we can conclude that

$$
\begin{aligned}
k_1(t) &= E\left[(\theta(X^*)r_F(X^*))^{\alpha-1} |X^* < t\right] \\
&\ge (\theta(t)r_F(t))^{\alpha-1}.
\end{aligned}
$$

We show that

$$k_2(t) = E\left[(\theta(Y^*)r_F(Y^*))^{\alpha-1} |Y^* < t\right]$$

is a decreasing function of t where Y^* has distribution $G^\alpha(t)$. If we define

$$\gamma(t) = G^{\alpha-1}(t)[k_1(t) - k_2(t)]$$

and use the same arguments as used to prove Theorem 2.1, we conclude under the conditions of the theorem that $\gamma(t)$ is deceasing in t and that $\gamma(t)$ is non-positive. This implies that $k_1(t) \le k_2(t)$ and hence

$$
\begin{aligned}
k_2(t) &\ge (\theta(t)r_F(t))^{\alpha-1} \\
&= (r_G(t))^{\alpha-1}.
\end{aligned}
$$

That is $k_2(t)$ is a decreasing function of t. Hence we get that $E[r_G^{\alpha-1}(Y^*)| Y^* < t]$ is decreasing in t and therefore $H_\alpha(g; [t])$ is increasing in t. For $0 < \alpha < 1$ the result follows similarly. Hence we have the theorem.

Remark 3.1. In Zohrevand and Asadi (2004) it is shown that the result of Theorem 3.1 is also true for $\alpha = 1$, i.e. for the past Shannon entropy. Hence it can be concluded that under the assumptions of Theorem 3.1 if PEO(α) of F is increasing then so is the PEO(α) of G for all values of $\alpha > 0$.

Corollary 3.1. *Let $X_1, \ldots, X_n$ denote the lifetimes of n independent components which are connected in a series system. If the PEO(α)'s of the components, $H_\alpha(f; [t])$, is increasing, then so is, $H_\alpha(f_{1:n}; [t])$, the PEO(α) of the system.*

Proof. One can easily see that the reversed hazard rate of X_i's, $r_F(t)$, and the reversed hazard rate of the system, $r_{F_{1:n}}(t)$, have the following relation.

$$r_{F_{1:n}}(t) = \theta(t)r_F(t),$$

where

$$\theta(t) = \frac{n(\bar{F}(t))^{n-1}}{\sum_{i=0}^{n-1}(\bar{F}(t))^i}.$$

It is easy to verify, in this case, that $\theta(t) \in (0, 1)$ and that $\theta(t)$ is a decreasing function of t. Hence from Theorem 3.1 we have the result.

Remark 3.2. The representation (8) shows that if the reversed hazard rate $r_F(t)$ is a decreasing function of t then the PEO(α) of F is an increasing function of t. Hence, as an application of Corollary

3.1, assume that the components of a series system have exponential distribution with mean η. Then the reversed hazard rate of the components is

$$r_F(t) = \frac{1/\eta e^{-t/\eta}}{1 - e^{-t/\eta}},$$

which is a decreasing function of t. Thus, the PEO(α) of the components is increasing in t and hence, based on Corollary 3.1, the PEO(α) of the system is increasing in t.

Consider again two sets of *iid* components of sizes n_1 and n_2 which are connected in systems with (n_1-k_1+1)-out-of-n_1 and (n_2-k_2+1)-out-of-n_2 structures, respectively. If we denote the reversed hazard rates of the systems by $r_{F_{k_1:n_1}}$ and $r_{F_{k_2:n_2}}$, respectively, then it can be seen that

$$r_{F_{k_2:n_2}}(x) = \theta(x) r_{F_{k_1:n_1}}(x),$$

where

$$\theta(t) = \frac{c(k_2, n_2)}{c(k_1, n_1)} \phi^{k_2-k_1} \frac{\sum_{i=k_1}^{n_1} \binom{n_1}{i} \phi^i}{\sum_{j=k_2}^{n_2} \binom{n_2}{j} \phi^j},$$

in which $c(k,n) = \frac{n!}{(n-k)!(k-1)!}$ and $\phi = \phi(x) = F(x)/\bar{F}(x)$ is increasing in x. It is not difficult to prove that in the following cases $\theta(x)$ is decreasing in x and its range is a subset of $(0,1)$.

- $n_1 = n_2 = n$, $k_1 = k$, $k_2 = k - 1$

- $n_1 = n$, $n_2 = n + 1$, $k_1 = k_2 = k$

- $n_1 = n$, $n_2 = n + 1$, $k_1 = k$, $k_2 = k + 1$

The discussion above leads to the following corollary.

Corollary 3.2.   *Let $H_\alpha(f_{k:n}; t)$ denote the PEO(α) of a $(n-k+1)$-out-of-n system which is increasing in t. Then*

(a) *$H_\alpha(f_{k-1:n}; t)$, the PEO(α) of a $(n - k + 2)$-out-of-n system, is also increasing.*

(b) *$H_\alpha(f_{k:n+1}; t)$, the PEO(α)'s of a $(n-k+2)$-out-of-n-1 system, is also increasing .*

(c) *$H_\alpha(f_{k+1:n+1}; t)$ system, the PEO(α)'s of a $(n - k + 1)$-out-of-n $+ 1$, is also increasing.*

4 Equilibrium Distributions

Let a renewal process be generated by random variables $X_i \geq 0$, $i = 1, 2, ...$ in which X_i's have a common distribution F with a finite mean μ. The X_i's may, for example, be the lifetimes of a device which is replaced upon failure. The process is observed at some given time t. For a device operating at time t assume that Y denotes its residual (excess) lifetime . It can be shown that when the process is stationary or when $t \to \infty$, Y has an asymptotic distribution, which is known as the equilibrium distribution,

$$P(Y \leq t) = \frac{\int_0^t \bar{F}(x)dx}{\mu},$$

with density function

$$f^*(t) = \frac{\bar{F}(t)}{\mu}$$

where $\bar{F}$ denotes the survival function. The REO(α) corresponding to f^* is therefore

$$H_\alpha(f^*; t) = \frac{1}{1 - \alpha} \log \frac{\int_t^\infty \bar{F}^\alpha(x)dx}{\left(\int_t^\infty \bar{F}(x)dx\right)^\alpha}.$$

The following theorem gives a lower bound for $H_\alpha(f^*; t)$ in terms of $m_F(t)$, the MRL of the parent distribution F.

Theorem 4.1. *For all values of $t > 0$ and $\alpha > 0$, we have*

$$H_\alpha(f^*; t) \geq \log m_F(t), \tag{9}$$

where $m_F(t)$ denotes the MRL function of F.

Proof. First assume that $\alpha > 1$. Note that since $\bar{F}(t)$ is a decreasing function of t, we have

$$
\begin{aligned}
H_\alpha(f^*; t) &= \frac{1}{1 - \alpha} \log \frac{\int_t^\infty \bar{F}^\alpha(x)dx}{\left(\int_t^\infty \bar{F}(x)dx\right)^\alpha} \\
&\geq \frac{1}{1 - \alpha} \log \frac{\bar{F}^{\alpha-1}(t) \int_t^\infty \bar{F}(x)dx}{\left(\int_t^\infty \bar{F}(x)dx\right)^\alpha} \\
&= \frac{1}{1 - \alpha} \log \frac{1}{m_F^{\alpha-1}(t)} \\
&= \log m_F(t)
\end{aligned}
$$

In the case of $0 < \alpha < 1$ the result follows similarly. Now we show that the result is also true for $\alpha = 1$, i.e, the result is true for residual Shannon entropy. This is so because the residual Shannon entropy, in this case, can be written as

$$
\begin{aligned}
H(f^*;t) &= -\int_t^\infty \frac{\bar{F}(x)}{\int_t^\infty \bar{F}(x)dx} \log \frac{\bar{F}(x)}{\int_t^\infty \bar{F}(x)dx} dx \\
&= -\frac{\int_t^\infty \bar{F}(x) \log \bar{F}(x)dx}{\int_t^\infty \bar{F}(x)dx} + \log \int_t^\infty \bar{F}(x)dx \\
&\geq -\log \bar{F}(t) + \log \int_t^\infty \bar{F}(x)dx \\
&= \log m_F(t)
\end{aligned}
$$

That is, for all $\alpha > 0$, the REO(α) of the equilibrium distribution is bounded from below by $\log m_F(t)$. This completes the proof of the theorem.

Classification of distributions with respect to ageing properties is a popular subject in reliability theory. Two classes of distributions which arise in the study of replacement and maintenance policies are the class of decreasing (increasing) MRL and the class of new better (worse) than used in expectation (NBUE) (NWUE) distributions. Let X be the lifetime of a system with a continuous distribution function F and the MRL function m_F.

- F is said to be a decreasing (increasing) MRL distribution, if $m_F(t)$ is a decreasing (increasing) function of t, $t > 0$.

- F is said to be a NBUE (NWUE) distribution if

$$
m_F(t) \leq (\geq)m(0) = \mu \quad t \geq 0.
$$

For the details of these concepts and some other concepts of ageing properties, we refer the reader to Barlow and Proschan (1981).

Based on the lower bound for $H_\alpha(f^*;t)$ and under the assumption that F is NWUE with mean μ we get

$$
H_\alpha(f^*;t) \geq \log \mu
$$

Example 4.1. The mixture of distributions arises in many branches of statistics and applied probability. Let X be distributed as the mixture of two exponential distributions with mean λ_1 and λ_2, respectively. Then the survival function of X is given by

$$
\bar{F}(x) = pe^{-x/\lambda_1} + (1 - p)e^{-x/\lambda_2},
$$

where $p \in (0, 1)$. It can be shown that this distribution has a decreasing hazard rate which, in turn, implies that F belongs to the class of NWUE distributions, (see, Barlow and Proschan (1975, p. 101)). Thus, based on above result and noting that the mean of X is $p\lambda_1 + (1-p)\lambda_2$, we can conclude that the REO(α) of the equilibrium distribution corresponding to X is bounded below as follows

$$H_\alpha(f^*; t) \geq \log(p\lambda_1 + (1-p)\lambda_2).$$

The following theorem gives some results on REO(α) of the equilibrium distributions in the class of decreasing (increasing) MRL distributions.

Theorem 4.2. *If the distribution function F belongs to the class of decreasing (increasing) MRL distributions then*

(a) *For all $\alpha > 0$ and all $t > 0$ $H_\alpha(f^*; t)$ is decreasing (increasing) in t.*

(b) *For all $t > 0$*

$$H_\alpha(f^*; t) \leq (\geq) \begin{cases} \log m_F(t) - \frac{1}{1-\alpha} \log \alpha, & \alpha \neq 1 \\ \log m_F(t) + 1 & \alpha = 1, \end{cases}$$

(c) *The equality in part (b) holds if and only if the distribution is exponential.*

Proof.

(a) To prove the result first assume that $\alpha \neq 1$. Note, in this case, that $H_\alpha(f^*; t)$ can be represented as

$$H_\alpha(f^*; t) = \frac{1}{1-\alpha} E(m_F^{1-\alpha}(Y^*)|Y^* \geq t) - \frac{1}{1-\alpha} \log \alpha, \quad (10)$$

where Y^* is a random variable with density function

$$g^*(t) = \frac{\alpha \bar{F}(t)(\int_t^\infty \bar{F}(x)dx)^{\alpha-1}}{\mu}.$$

This implies that if $m_F(t)$ is a decreasing (increasing) function of t then $H_\alpha(f^*; t)$ is also a decreasing (increasing) function of t. Now assume that $\alpha \to 1$. In this case, using the relation between the residual Shannon entropy and the hazard rate of

F given in (1) and the fact that the hazard rate of equilibrium distribution is the reciprocal of the MRL function of F we have

$$H_\alpha(f^*; t) = E(\log m_F(Y)|Y > t) + 1, \qquad (11)$$

where Y has the equilibrium distribution $\frac{\bar{F}(t)}{\mu}$. This implies that for a distribution function with decreasing (increasing) MRL the REO(α), $H_\alpha(f^*; t)$ is a decreasing (increasing) function of t for all values of $\alpha > 0$. This completes part (a).

(b) Again first consider the case of $\alpha \neq 1$. Under the assumption that F is decreasing (increasing) MRL we have from (10)

$$H_\alpha(f^*; t) \leq (\geq) \log m_F(t) - \frac{1}{1 - \alpha} \log \alpha. \qquad (12)$$

For the case of $\alpha \to 1$ the result follows from (11).

(c) The 'if' part of the theorem is easy to prove. To prove the 'only if' part assume that we have the equality in (12). This is equivalent to say that for $t > 0$,

$$\int_t^\infty \bar{F}^\alpha(x)dx = \frac{m_F(t)}{\alpha} \bar{F}^\alpha(t).$$

Differentiation of both sides of this equation in terms of t implies that $m'_F(t) = 0$. That is the underlying distribution F is exponential. This completes part (c) and hence the theorem.

Remark 4.1.   For the class of decreasing (increasing) MRL distributions the following upper(lower) bound which does not depend on t holds.

$$H_\alpha(f^*; t) \leq (\geq) \log \mu - \frac{1}{1 - \alpha} \log \alpha.$$

The result follows from the fact that the class of decreasing (increasing) MRL distributions is a subclass of NBUE (NWUE) distributions.

Remark 4.2.   Based on the upper bound for REO(α) in the class of decreasing MRL distributions in Theorem 4.2 and the lower bound in (9) one can conclude that

$$0 \leq H_\alpha(f^*; t) - \log m_F(t) \leq \frac{1}{\alpha - 1} \log \alpha.$$

Acknowledgment

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