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Some New Results on Stochastic Orderings between Generalized Order Statistics

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Abstract. In this paper we specify the conditions on the parameters of pairs of g**OS**'s under which the corresponding generalized order statistics are ordered according to usual stochastic ordering, hazard rate ordering, likelihood ratio ordering and dispersive ordering. We consider this problem in one-sample as well as two-sample problems. We show that some of the results obtained by Franco et al. [*Probab. Engrg. Inform. Sci.* 2002, **16**, 471-484] and Belzunce et al. [*Probab. Engrg. Inform. Sci.* (2004), *to appear*] for stochastic orderings of g**OS**'s are contained in our new results.

1 Introduction

Order statistics and record values play an important role in statistics, in general, and in Reliability Theory and Life Testing, in particular. Their distributional and stochastic properties have been studied extensively but separately in the literature. However, they can be

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considered as special cases of Generalized order statistics (g**OS**'s) (cf. Kamps, 1995) which in addition cover sequential order statistics, kth record values, Pfeifer's record model, k_n record from nonidentical distributions, and ordered random variables which arise from truncated distributions. It is well known that a sequence of record values can be viewed as a sequence of the occurrence times of a certain non-homogeneous Poisson process. It is also connected to the failure times of a minimal repair process. There is a close connection between Pfeifer's records and the occurrence times of a pure birth process (cf. Pfeifer, 1982b).

Many interesting stochastic orderings results for order statistics on one hand; and for record values on the other hand, have been obtained separately by many investigators without realizing that analogous properties can also be found for g**OS**'s .

In this paper we study the connection between various types of stochastic orderings between two probability distributions and their corresponding g**OS**'s . A consequence of these results will be applicable to all those models which are covered under g**OS**'s .

This problem has also been studied by Franco et al. (2002) and Belzunce et al. (2004).

There are several notions of stochastic orderings of varying degree of strength and they have been discussed in details in Shaked and Shanthikumar (1994). We first briefly review some of these here.

Throughout this paper *increasing* means *nondecreasing* and *decreasing* means *nonincreasing*; and we shall be assuming that all distributions under study are absolutely continuous.

Let X and Y be univariate random variables with distribution functions F and G, survival functions $\overline{F}$ and $\overline{G}$, density functions f and g; and hazard rates r_F $(= f/\overline{F})$ and r_G $(= g/\overline{G})$, respectively. Let $l_X\,(l_Y)$ and $u_X\,(u_Y)$ be the left and the right endpoints of the support of $X\,(Y)$. X is said to be *stochastically* smaller than Y (denoted by $X \leq_{st} Y$) if $\overline{F}(x) \leq \overline{G}(x)$ for all x. This is equivalent to saying that $Eg(X) \leq Eg(Y)$ for any increasing function g for which expectations exist. X is said to be smaller than Y in *hazard rate* ordering (denoted by $X \leq_{hr} Y$) if $\overline{G}(x)/\overline{F}(x)$ is increasing in $x \in (-\infty, max(u_X, u_Y))$. In case the hazard rates exist, it is easy to see that $X \leq_{hr} Y$, if and only if, $r_G(x) \leq r_F(x)$ for every x. X is said to be *smaller than* Y *in the likelihood ratio order* (and written as $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in $x \in (-\infty, max(u_X, u_Y))$. Note that likelihood ratio ordering implies hazard rate ordering which in turn implies stochastic ordering.

Let F^{-1} and G^{-1} be the right continuous inverses (quantile functions) of F and G, respectively. We say that X is less *dispersed* than Y (denoted by $X \leq_{disp} Y$) if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$, for all $0 \leq \alpha \leq \beta \leq 1$. A consequence of $X \leq_{disp} Y$ is that $|X_1 - X_2| \leq_{st} |Y_1 - Y_2|$ and which in turn implies $var(X) \leq var(Y)$ as well as $E[|X_1 - X_2|] \leq E[|Y_1 - Y_2|]$, where X_1, X_2 (Y_1, Y_2) are two independent copies of X (Y). For more details on stochastic orderings, see Chapters 1 and 4 of Shaked and Shanthikumar (1994).

One of the basic tools in establishing various inequalities in statistics and probability is the notion of majorization. Let $\{x_{(1)} \leq \ldots \leq x_{(n)}\}$ denote the increasing arrangement of the components of a vector $\mathbf{x} = (x_1, \ldots, x_n)$. A vector $\mathbf{x}$ is said to majorize another vector $\mathbf{y}$ (written $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$. Marshall and Olkin (1979) provides extensive and comprehensive details on the theory of majorization and its applications in statistics. A vector $\mathbf{x}$ in $I\!R^{+^n}$ is said to be *p-larger* than another vector $\mathbf{y}$ also in $I\!R^{+^n}$ (written $\mathbf{x} \overset{p}{\succeq} \mathbf{y}$) if $\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)}, j = 1, \ldots, n$. It is known that when $\mathbf{x}, \mathbf{y} \in I\!R^{+^n}$, $\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{p}{\succeq} \mathbf{y}$. The converse is, however, not true. (cf. Khaledi and Kochar, 2002).

The organization of the paper is as follows. In Section 2, we introduce g**OS**'s and state the main theorem which describes the conditions under which various kinds of stochastic orderings between g**OS**'s are established. It is seen that the results of stochastic orderings among ordered random variables as order statistics from i.i.d random variables, classic record values, k-records, Pfeifer's records among others (which are particular cases of g**OS**'s) follow from the new results obtained in this paper. In Section 3 we establish likelihood ratio ordering, hazard rate ordering, stochastic ordering and dispersive ordering among g**OS**'s in one-sample problems and then we generalize these results to two-sample problems.

2 Main Results

First we give the definition of the joint distribution of n generalized order statistics (cf. Kamps, 1995, p. 49).

Definition 2.1. *Let $n \in N$, $k \geq 1$, $m_1, \ldots, m_{n-1} \in I\!R$, $M_r = \sum_{j=r}^{n-1} m_r$, $1 \leq r \leq n-1$ be parameters such that*

$$\gamma_r = k + n - r + M_r \geq 1 \text{ for all } r \in \{1, \ldots, n-1\}$$

and let $\tilde{m} = (m_1, \ldots, m_{n-1})$, if $n \geq 2$ ($\tilde{m} \in I\!R$ arbitrary, if $n = 1$). If the random variables $U(r, n, \tilde{m}, k)$, $r = 1, \ldots, n$, possess a joint density function of the form

$$f^{U(1,n,\tilde{m},\ k),\ldots,U(n,n,\tilde{m},\ k)}(u_1, \ldots, u_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} (1-u_i)^{m_i} \right) (1-u_n)^{k-1}$$

on the cone $0 \leq u_1 \leq \ldots \leq u_n < 1$ of $I\!R^n$, then they are called uniform generalized order statistics. Generalized order statistics based on some distribution function F are now defined by means of the quantile transformation

$$X(r, n, \tilde{m},\ k) = F^{-1}(U(r, n, \tilde{m},\ k)),\ r = 1, \ldots, n$$

*and they are denoted by g***OS***'s.*

As discussed in Kamps (1995), for suitable choices of the parameters these reduce to the joint distributions of order statistics from a continuous distribution, record values, Pfeifer's record values and so on.

Throughout this paper for $r = 1, \ldots, n$

$$X(r, n, \tilde{m}, k) \text{ and } Y(r, n, \tilde{m}, k)$$

stand for rth generalized order statistics based on continuous distribution function F and G respectively.

Kamps (1995, Section 5.2) for a particular set of parameters proved a likelihood ratio ordering between pairs of g**OS**'s based on distribution F. Then, Franco et al. (2002, Theorem 3.5 and Theorem 3.6) and Belzunce et al. (2004, Corollary 3.2, Theorem 3.5 and

Theorem 3.12) further studied this problem in one sample problem as well as two sample problem and proved somewhat general results about preservation of stochastic ordering, hazard rate ordering and dispersive ordering under the formation of g**OS**'s.

In the next Theorem whose proof is given in Section 3, we show that the above results of Franco et al. (2002) and Belzunce et al. (2004) can be extended for more general set of parameters.

Theorem 2.1. *For $i \geq i'$,*

(a) $G \leq_{st} F \Rightarrow Y(i', n', \tilde{m}', k') \leq_{st} X(i, n, \tilde{m}, k)$,

(b) $G \leq_{hr} F \Rightarrow Y(i', n', \tilde{m}', k') \leq_{hr} X(i, n, \tilde{m}, k)$, *and*

(c) if either F or G is DFR, $G \leq_{disp} F \Rightarrow Y(i', n', \tilde{m}', k') \leq_{disp} X(i, n, \tilde{m}, k)$;

provided

$$(\gamma_{\ell_1}, \ldots, \gamma_{\ell_{i'}}) \overset{p}{\succeq} (\gamma_1', \ldots, \gamma_{i'}') \text{ for some set } \{\ell_1, \ldots, \ell_{i'}\} \subset \{1, \ldots, i\}, \tag{2.1}$$

where $\gamma_r = k + n - r + \sum_{h=r}^{n-1} m_h$ *and* $\gamma_r' = k' + n' - r + \sum_{h=r}^{n'-1} m_h'$.

It is easy to see that the conditions $m \geq m' \geq -1$, $i \geq i'$ and $\gamma_i \leq \gamma_{i'}'$ of Theorem 3.5 and Theorem 3.6 in Franco et al. (2002) implies that $\gamma_{i-j+1} \leq \gamma_{i'-j+1}'$, $j = 1, \ldots, i'$, from which it follows that condition (2.1) is satisfied. That is, these Theorems are particular cases of Theorem 2.1. The cases when $i = i'$, $n = n'$, $\tilde{m} = \tilde{m}'$ and $k = k'$ from which it follows that $\gamma_i = \gamma_i'$ has been considered in Belzunce et al. (2004). They are also particular cases of Theorem 2.1.

It is well known that for specific sets of parameters, n, k and m_i, $i = 1, \ldots, n-1$, the g**OS**'s reduce to the well known ordered random variables. Below we characterize the required index sets for which Theorem 2.1 holds.

(A) Order Statistics from i.i.d random variables. For $n \geq 1$, let $X_{i:n}$ denote the *ith* order statistic based on a random sample $X_1, \ldots, X_n$ from a continuous distribution with cdf F and let $Y_{i':n'}$ denote the i'th order statistic based on a random sample $Y_1, \ldots, Y_{n'}$ from a continuous distribution with cdf G. These are respectively, special cases of g**OS**'s with $m_1 = \ldots = m_{n-1} = 0$, $k = 1$ and $m'_1 = \ldots = m'_{n'-1} = 0$, $k' = 1$. In this case $\gamma_r = n - r + 1$, $r = 1, \ldots, n-1$ and $\gamma'_r = n' - r + 1$, $r = 1, \ldots, n' - 1$. With these settings we see that (2.1) is satisfied when $n - i \leq n^{'} - i^{'}$. That is for $i \geq i^{'}$ and $n - i \leq n^{'} - i^{'}$ it follows from Theorem 2.1 that

(a) $G \leq_{st} F \Rightarrow Y_{i^{'}:n'} \leq_{st} X_{i:n}$,

(b) $G \leq_{hr} F \Rightarrow Y_{i^{'}:n'} \leq_{hr} X_{i:n}$, and

(c) if either F or G is DFR, $G \leq_{disp} F \Rightarrow Y_{i^{'}:n'} \leq_{disp} X_{i:n}$,

as part (a) and (b) can be found in Boland et al. (2002) and (c) proved by Khaledi and Kochar (2000).

(B) k**-Records.** Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d random variables from a continuous distribution F and let k be a positive integer. The random variables $L^{(k)}(n)$ given by $L^{(k)}(1) = 1$,

$$L^{(k)}(n+1) \;=\; min\{j \in N; X_{j:j+k-1} > X_{L^{(k)}(n):L^{(k)}((n)+k-1)}\}, \quad n \geq 1,$$

are called the nth k-th record times and the quantities

$$X_{L^{(k)}(n):L^{(k)}((n)+k-1)}$$

which we denote by $R^X_{n:k}$ are termed the nth k-records (cf. Kamps, 1995, p.34). Let $R^Y_{n:k'}$, $n \geq 1$, be another sequence of k'-records corresponding to continuous distribution G.

In the case $m_1 = \ldots = m_{n-1} = -1$ and $k \in N$, the density function of the first n g**OS**'s based on distribution F reduces to the joint density function of the first n k-records corresponding to a sequence of independent random variables from a continuous distribution F. In this case $\gamma_r = k$, $r = 1, \ldots, n-1$, $\gamma'_r = k'$, $r = 1, \ldots, n-1$ and $m_i = m_i^{'} = -1$, $i = 1, \ldots, n-1$. Let $k \leq k^{'}$. Using the above setting it follows that the conditions (2.1) of Theorem 2.1 is satisfied. Therefore, for $i \geq i^{'}$, we have

(a) $G \leq_{st} F \Rightarrow R^Y_{i^{'}:k'} \leq_{st} R^X_{i:k}$,

(b) $G \leq_{hr} F \Rightarrow R^Y_{i^{'}:k'} \leq_{hr} R^X_{i:k}$, and

(c) if either F or G is DFR, $G \leq_{disp} F \Rightarrow R^Y_{i^{'}:k'} \leq_{disp} R^X_{i:k}$.

As for the case when $X =_{st} Y$, (b) was proved by Raqab and Amin (1996). For the case when $k = k' = 1$, classic record model, (c) was proved by Kochar (1996) and (b) has been shown by Ahmadi and Arghami (2001) and Belzunce et al. (2001).

(C) Pfeifer Model For $k = 1$ the k-records model reduces to the well know classic record model and for this model it is known that successive record values follows the conditional distribution given by

$$P(R_n > x | R_{n-1} = x) = \frac{1 - F(y)}{1 - F(x)}, \quad \text{for } y > x. \tag{2.2}$$

Pfeifer (1982a) generalized the above model and consider a model in which the successive (upper) records values constitute a Markov chain with nonstationary transition distribution given by

$$P(R_n > x | R_{n-1} = x) = \frac{1 - F_n(y)}{1 - F_n(x)}, \quad \text{for } y > x.$$

Such a dependence structure for the record value sequence can be produced as follows. Suppose we have a double array of independent random variables $\{X_{01}, X_{nj}; n, j \geq 1\}$ such that X_{nj} distribution function F_n, $n \geq 0$. Now take $R_0 = X_{01}$ and define $\delta_n = min\{j : X_{nj} > R_{n-1}\}$ and $R_n = X_{n,\delta_n}$ for $n \geq 1$. This setting is called *Pfeifer record model* (cf. Arnold, Balakrishnan and Nagaraja, 1998, p.198). Pfeifer (1982b) showed that the sequence of jump-time generated by a pure birth process is identically distributed with records from Pfeifer models. Therefore the new results obtained here can be applied to this kind of Process.

For given positive real numbers $\beta_1, \ldots \beta_n$, the model of g**OS**'s based on distribution F with parameters $m_i = \beta_i - \beta_{i+1} - 1$, $i = 1, \ldots, n-1$ and $k = \beta_n$ is reduced to Pfeifer's record model based on distribution

$$F_r(t) = 1 - (1 - F(t))^{\beta_r}.$$

Let $\{R_n^X, n \geq 0\}$ and $\{R_n^Y, n \geq 0\}$ be two independent sequences of Pfeifer's records based on distributions $F_r(t) = 1 - (1 - F(t))^{\beta_r}$ and $G_r(t) = 1 - (1 - G(t))^{\beta_r}$, respectively, where F and G both are continuous distributions. Then for $i \geq i'$

(a) $G \leq_{st} F \Rightarrow R_{i'}^{Y} \leq_{st} R_i^X$,

(b) $G \leq_{hr} F \Rightarrow R_{i'}^{Y} \leq_{hr} R_i^X$, and

(c) if either F or G is DFR, $G \leq_{disp} F \Rightarrow R_{i'}^{Y} \leq_{disp} R_i^X$,

provided $(\beta_{\ell_1}, \ldots, \beta_{\ell_{i'}}) \stackrel{p}{\succeq} (\beta_1', \ldots, \beta_{i'}')$ for some set $\{\ell_1, \ldots, \ell_{i'}\} \subset \{1, \ldots, i\}$ is satisfied.

In particular let $i = i'$. Then it follows from the above result that $R_i^{\tilde{X}} \leq_{hr} R_i^X$ and $R_i^{\tilde{X}} \leq_{disp} R_i^X$, where $R_i^{\tilde{X}}$ is the ith record corresponding to classic record model based on exponential distribution with hazard rate $\tilde{\beta}$, the geometric mean of $\beta_1, \ldots, \beta_i$.

As discussed in Kamps (1995), there are many other models like sequential order statistics, order statistics with non-integral sample size etc which can also be expressed as special cases of g**OS**'s.

3 Auxiliary results and proofs

We shall be using the following known results to prove the main results in this paper. The following lemma can be found in Shaked and Shanthikumar (1994).

Lemma 3.1. *The random variable X satisfies*

$$(a)\ X \leq_{disp} X + Y \qquad \text{and} \qquad (b)\ X \leq_{lr} X + Y$$

for any random variable Y independent of X if and only if X has a logconcave density.

Bagai and Kochar (1986) established the following connections between hazard rate ordering and dispersive ordering under some restrictions on the shapes of the distributions.

Theorem 3.1. *Let X and Y be two nonnegative random variables.*
(a) If $Y \leq_{hr} X$ and either F or G is DFR (decreasing failure rate), then $Y \leq_{disp} X$,
(b) if $Y \leq_{disp} X$ and either F or G is IFR (increasing failure rate), then $Y \leq_{hr} X$.

we need the following result due to Rojo and He (1991).

Theorem 3.2. *Let X and Y be two random variables such that $X \leq_{st} Y$. Then $X \leq_{disp} Y$ implies that $\gamma(X) \leq_{disp} \gamma(Y)$ where γ is a nondecreasing convex function.*

Theorem 3.3. *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, for $i = 1, \ldots, n$. Then,*

(a) $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}'$ *implies* $\sum_{k=1}^{n} X_{\lambda_k} \geq_{disp} \sum_{k=1}^{n} X_{\lambda'_k}$,

(b) $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}'$ *implies* $\sum_{k=1}^{n} X_{\lambda_k} \geq_{hr} \sum_{k=1}^{n} X_{\lambda'_k}$ *and,*

(c) $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\lambda}'$ *implies* $\sum_{k=1}^{n} X_{\lambda_k} \geq_{lr} \sum_{k=1}^{n} X_{\lambda'_k}$.

Parts (a) and (b) proved by Khaledi and Kochar (2004) and part (c) proved by Korwar (2002).

Theorem 3.4. *(cf. Kamps, 1995, p.81) Let $X(r, n, \tilde{m}, k)$, $r = 1, \ldots, n$ be Generalized order statistics based on the distribution F with $F(x) = 1 - e^{-x}$, $x \geq 0$. Then the random variables*

$$Y_1 = \gamma_1 X(1, n, \tilde{m}, k),$$

$$Y_j = \gamma_j (X(j, n, \tilde{m}, k) - X(j-1, n, \tilde{m}, k)),\ j = 2, \ldots, n,$$

$$\text{with } \gamma_j = k + n - j + \sum_{i=j}^{n-1} m_i$$

are stochastically independent and identically distributed according to F.

Moreover, for $r = 2, \ldots, n$ we have the representation

$$X(r, n, \tilde{m}, k) =_{st} \sum_{j=1}^{r} X_{\gamma_j}, \tag{3.1}$$

where X_{γ_j} has exponential distribution with hazard rate γ_j, $j = 1, \ldots, r$.

Khaledi and Kochar (2000) proved that for $i \leq j$ and $n-i \geq m-j$, $X_{i:n} \leq_{disp} X_{j:m}$, where $X_{i:n}$, $i = 1, \ldots, n$ is the ith order statistics of a random sample of size n from a DFR distribution. In the next theorem we prove this result for the generalized order statistics.

In the following let $Z(i,n,\tilde{m},k)$, $i = 1,\ldots,n$, denotes the ith generalized order statistic based on standard exponential distribution (denoted by $E(x)$).

Theorem 3.5. *Under the condition (2.1), for $i \geq i^{'}$,*
(a) $X(i^{'},n^{'},\tilde{m}^{'},k^{'}) \leq_{hr} X(i,n,\tilde{m},k)$, and

(b) if F is DFR, then $X(i^{'},n^{'},\tilde{m}^{'},k^{'}) \leq_{disp} X(i,n,\tilde{m},k)$.

Proof: First we prove (b).

It follows from Theorem 3.4 that $Z(i,n,\tilde{m},k) = \sum_{h=1}^{i} X_{\gamma_h}$, where X_{γ_h} has exponential distribution with hazard rate γ_h, $h = 1,\ldots,i$. For $i \geq i'$, we have

$$\begin{aligned}
\sum_{\nu=1}^{i} X_{\gamma_\nu} &= \sum_{\nu=1}^{i'} X_{\gamma_{\ell_\nu}} + \sum_{\nu\notin\{\ell_1,\ldots,\ell_{i'}\}} X_{\gamma_\nu} \\
&\geq_{disp} \sum_{\nu=1}^{i'} X_{\gamma_{\ell_\nu}} \\
&\geq_{disp} \sum_{\nu=1}^{i'} X_{\gamma'_\nu},
\end{aligned}$$

since the density function of a gamma random variable with shape parameter $a \geq 1$ is logconcave and a convolutions of independent random variables with logconcave densities is logconcave, the first inequality follows from Lemma 3.1(a). Under the condition (2.1) The second inequality follows from Theorem 3.3 (a). That is we have shown that

$$Z(i^{'},n^{'},\tilde{m}^{'},k^{'}) \leq_{disp} Z(i,n,\tilde{m},k). \tag{3.2}$$

By definition of generalized order statistics, we have that for $i = 1,\ldots,n$

$$\begin{aligned}
X(i,n,\tilde{m},k) &= F^{-1}\left(1 - e^{-Z(i,n,\tilde{m},k)}\right) \qquad (3.3)\\
&= F^{-1}oE(Z(i,n,\tilde{m},k)).
\end{aligned}$$

Function $F^{-1}oE$ is increasing and convex, since F is DFR. Using this and (3.2), the required result follows from Theorem 3.2. This completes the proof of (b).

It is known that the convolutions of independent exponential distribution is IFR. Using this and (3.2), it follows from Theorem 3.1 (b) that $Z(i^{'}, n^{'}, \tilde{m}^{'}, k^{'}) \leq_{hr} Z(i, n, \tilde{m}, k)$. Now (a) follows from this and the fact that hazard rate ordering is preserved by increasing transformation. ■

The relation (3.3) was used by Cramer and Kamps (2003) to prove that *increasing failure rate* property is preserved by $g\mathbf{OS's}$.

Next we establish the likelihood ratio ordering between generalized order statistics based on distribution F.

Theorem 3.6. *For $i \geq i^{'}$, $X(i^{'}, n^{'}, \tilde{m}^{'}, k^{'}) \leq_{lr} X(i, n, \tilde{m}, k)$ provided,*

$$(\gamma_{\ell_1}, \ldots, \gamma_{\ell_{i'}}) \stackrel{m}{\succeq} (\gamma_1^{'}, \ldots, \gamma_{i'}^{'}) \text{ for some set } \{\ell_1, \ldots, \ell_{i'}\} \subset \{1, \ldots, i\}. \tag{3.4}$$

Proof: Using the similar kind of arguments as used to prove (3.2), it follows from Theorem 3.3 (c) and Lemma 3.1(b) that

$$Z(i^{'}, n^{'}, \tilde{m}^{'}, k^{'}) \leq_{lr} Z(i, n, \tilde{m}, k). \tag{3.5}$$

It is also known that the likelihood ratio ordering is closed under increasing transformation. Since the transformation (3.3) is increasing, now the required result follows from (3.5). ■

Using similar idea of stochastic comparisons of convolutions of independent random variables (cf. Shanthikumar and Yao, 1991), Korwar (2003) proved likelihood ratio orders for particular censored order statistics which are in some cases special cases of $g\mathbf{OS's}$.

In the next theorem we prove that without loss of generality the condition $m_i = m$, $i = 1, \ldots, n-1$ of Theorem 3.2 and Theorem 3.3 in Franco et al. (2002) can be removed from the statement of the above theorems. The same result was proved in Belzunce et al. (2004) by using different kind of arguments.

Theorem 3.7.

(a) $G \leq_{st} F \Rightarrow Y(i,n,\tilde{m},k) \leq_{st} X(i,n,\tilde{m},k)$.

(b) $G \leq_{hr} F \Rightarrow Y(i,n,\tilde{m},k) \leq_{hr} X(i,n,\tilde{m},k)$.

Proof: (a) From (3.3) we have

$$\begin{aligned} \overline{F}_{X(i,n,\tilde{m},k)}(x) &= \overline{F}_{Z(i,n,\tilde{m},k)}\Big(-\log(\overline{F}(x))\Big) \\ &\geq \overline{F}_{Z(i,n,\tilde{m},k)}\Big(-\log(\overline{G}(x))\Big) \\ &= \overline{G}_{Y(i,n,\tilde{m},k)}(x). \end{aligned} \tag{3.6}$$

It follows from $\overline{F}(x) \geq \overline{G}(x)$ that $-\log(\overline{F}(x)) \leq -\log(\overline{G}(x))$. The above inequality follows from the fact that $\overline{F}_{Z(i,n,\tilde{m},k)}(x)$ is decreasing function of x. This proves (a).

(b) Using (3.6) the hazard rate of $X(i,n,\tilde{m},k)$ can be written as

$$r_{X(i,n,\tilde{m},k)}(x) = r_F(x)\frac{f_{Z(i,n,\tilde{m},k)}\Big(-\log(\overline{F}(x))\Big)}{\overline{F}_{Z(i,n,\tilde{m},k)}\Big(-\log(\overline{F}(x))\Big)}.$$

By assumption $r_F(x) \leq r_G(x)$. $Z(i,n,\tilde{m},k)$ is a convolutions of independent exponential random variables, hence is IFR. On the other hand it follows from $G \leq_{hr} F$ that $\overline{F}(x) \geq \overline{G}(x)$ which in turn implies that $-\log(\overline{F}(x)) \leq -\log(\overline{G}(x))$. Combining these observations it follows that

$$\begin{aligned} r_{X(i,n,\tilde{m},k)}(x) &\leq r_G(x)\frac{f_{Z(i,n,\tilde{m},k)}\Big(-\log(\overline{G}(x))\Big)}{\overline{F}_{Z(i,n,\tilde{m},k)}\Big(-\log(\overline{G}(x))\Big)} \\ &= r_{Y(i,n,\tilde{m},k)}(x). \end{aligned}$$

This completes the proof of (b). ■

The following theorem due to Belzunce et al. (2004) establishes dispersive ordering between generalized order statistics based on different distributions.

Theorem 3.8. *For $i = 1, \ldots, n$,*

$$G \leq_{disp} F \Rightarrow Y(i,n,\tilde{m},k) \leq_{disp} X(i,n,\tilde{m},k).$$

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1 From Theorem 3.7 (a) we have that

$$Y(i, n, \tilde{m}, k) \leq_{st} X(i, n, \tilde{m}, k). \tag{3.7}$$

It also follows from Theorem 3.5 (a) that

$$Y(i^{'}, n^{'}, \tilde{m}^{'}, k^{'}) \leq_{hr} Y(i, n, \tilde{m}, k),$$

which in turn implies that

$$Y(i^{'}, n^{'}, \tilde{m}^{'}, k^{'}) \leq_{st} Y(i, n, \tilde{m}, k).$$

Now the required results of (a) follows from this and (3.7).

Under the same line as used to prove (a), part (b) follows from Theorem 3.7 (b) and Theorem 3.5 (a).

Let G be DFR, then it follows from Theorem 3.5 (b) that

$$Y(i^{'}, n^{'}, \tilde{m}^{'}, k^{'}) \leq_{disp} Y(i, n, \tilde{m}, k). \tag{3.8}$$

It follows from Theorem 3.8 that

$$Y(i, n, \tilde{m}, k) \leq_{disp} X(i, n, \tilde{m}, k).$$

This and (3.8) proves (c). ■

The last result of this paper is about establishing likelihood ratio ordering between generalized order statistics based on different distributions. The proof is based on Theorem 3.4 in Franco et al. (2002) and Theorem 3.6.

Theorem 3.9. *Under the condition (3.4) for $i \geq i'$, either $m_i = m \geq 0$, $i = 1, \ldots, n$ or $m_i' = m \geq 0$, $i = 1, \ldots, n'$ we have that* $G \leq_{lr} F \Rightarrow Y(i^{'}, n^{'}, \tilde{m}^{'}, k^{'}) \leq_{lr} X(i, n, \tilde{m}, k)$.

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