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# Record Range of Uniform Distribution

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**Abstract.** We consider a sequence of independent and identicaly distributed (iid) random variables with absolutely continuous distribution function $F(x)$ and probability density function (pdf) $f(x)$. Let $R_{nl}$ be the largest observation after observing $n$th record and $R_{(ns)}$ be the smallest observation after observing the $n$th record. Then we say $W_{nr} = R_{nl} - R_{(ns)},\ n > 1$, as the $n$th record range. We will consider some distributional properties of $W_{nr}$ when $f(x) = 1$ , $0 \leq x \leq 1$.

## 1 Introduction

Let $\{X_i, i = 1, 2, ..\}$ be a sequence of independent and identically distributed random varibles with an absolutely continuous( with respect to Lebegue measure ) ditribution function F(x) with pdf f(x).Let $R_{U(1)} = X_1$, $R_{U(2)}, ...$,be the upper records and $R_{L(1)}, R_{L(2)}, ...$be lower records of $\{X_i, i = 1, 2, ..\}$.For various properties of record values see Ahsanullah( 2005) Arnold et. al.(1998).

Suppose $R_{nl}$ is the largest observation after observing nth record and $R_{(ns)}$is the smallest observation after observing the nth record. Then we say $W_{nr} = R_{nl} - R_{(ns)}, n > 1$,as the nth record range.The

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joint pdf of $f_{(nl,(ns))}$ of $R_{nl}$ and $R_{(ns)}$ is given by ( see Arnold et. al. 1998, p. 275) as

$$f_{(nl,(ns))}(x,y) = \frac{2^{n-1}}{(n-2)!}\ [-\ln(\overline{F}(y)+F(x))]^{n-2} f(x)\ f(y), \quad (1.1)$$

$$-\infty < x < y < \infty$$

The pdf of $f_{W_{nr}}$ of $W_{nr}$ is given by

$$f_{W_{nr}}(w) \ = \int_{-\infty}^{\infty} \frac{2^{n-1}}{(n-2)!}\ [-\ln(\overline{F}(w+u)+F(u))]^{n-2} f(w+u)\ f(u)\ du \quad (1.2)$$

Suppose $X_i's$ are distributed as uniform with

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (1.3)$$

Using (1.3) in (1.2), we obtain

$$f_{W_{nr}}(w) = \begin{cases} \frac{2^{n-1}(1-w)}{\Gamma(n-1)}[-\ln\{1-w)]^{n-2}, & 0 < w < 1, n \geq 2 \\ 0, & \text{otherwise} \end{cases} \quad (1.4)$$

Figure 1.1 gives the pdf of $W_{nr}$ for $n = 10$ when $X_i'$ are distributed as uniform.

In this paper we will consider distributional properties of $W_{nr}$ for the case $X_i's$ are distributed as uniform distribution.

## 2 Main Results

**Lemma 2.1.** *For $n \geq 2$ and $0 < x < 1$,*

$$F_{W_{nr}}(x) = \Gamma_{-2\ln(1-x)}(n-1),$$

*where*

$$\Gamma_x(r) = \int_0^x \frac{1}{\Gamma(r)} u^{r-1} e^{-u} du$$

**Proof.**

$$\begin{aligned} F_{W_{nr}}(x) &= \int_0^x \frac{2^{n-1}(1-u)}{\Gamma(n-1)}[-\ln(1-u)]^{n-2} du \\ &= \int_0^{-2\ln(1-x)} \frac{1}{\Gamma(n-1)} e^{-t} t^{n-2} dt \\ &= \Gamma_{-2\ln(1-x)}(n-1). \ \square \end{aligned}$$

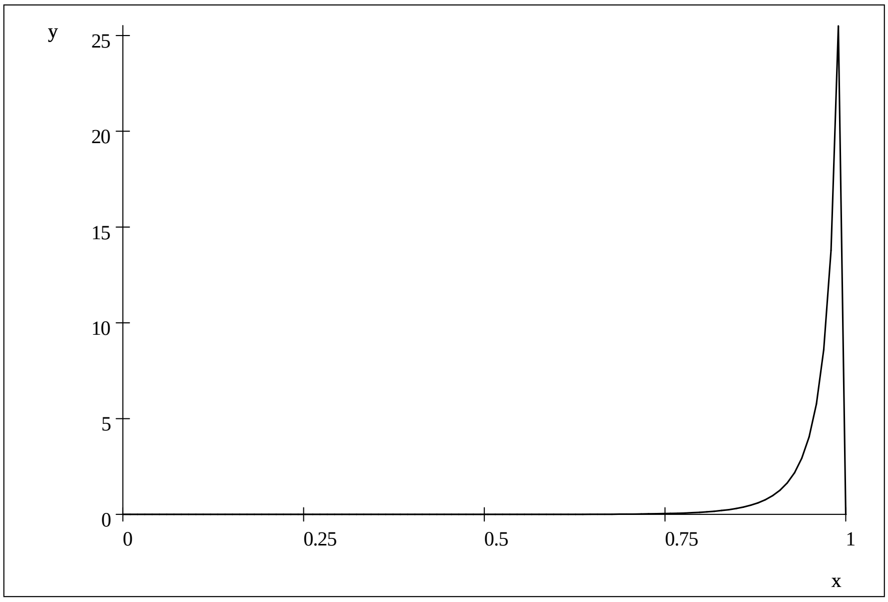


Figure 1.1: $f(x) = \frac{2^9(1-x)}{\Gamma(9)}(-\ln(1-x))^8$

**Remark 2.1.**

$$1 - F_{W_{nr}}(x) = (1-x)^2 \sum_{j=0}^{n-2} \frac{(-2\ln(1-x))^j}{j!}, \quad 0 < x < 1, \quad n \geq 2$$

**Lemma 2.2.**

$$F_{W_{nr}}(x) - F_{W_{n+1r}}(x) = \frac{f_{W_{n+1r}}(x)}{2(1-x)}$$

**Proof.**

$$\begin{aligned} F_{W_{nr}}(x) - F_{W_{n+1r}}(x) &= \Gamma_{-2\ln(1-x)}(n-1) - \Gamma_{-2\ln(1-x)}(n) \\ &= \frac{1}{\Gamma(n)}(-2\ln(1-x))^{n-1}e^{-2\ln(1-x)} \\ &= \frac{(1-x)^2}{\Gamma(n)}(-2\ln(1-x))^{n-1} \\ &= \frac{f_{W_{n+1r}}(x)}{2(1-x)}. \end{aligned}$$

$$\begin{aligned}
\mu_{nr}^{p} &= E(W_{nr})^{p} = \frac{2^{n-1}}{(n-2)!}\int_0^1 w^p(1-w)\ [-\ln(1-u)]^{n-2}dw \\
&= \frac{2^{n-1}(n-2)!}{(n-2)!}\int_0^1 (1-w)^p w\ [-\ln(w)]^{n-2}dw \\
&= 2^{n-1}(\sum_{k=0}^{p}\binom{p}{k}\frac{(-1)^k}{(2+k)^{n-1}} \qquad (2.1)
\end{aligned}$$

Using $p = 1$ and $p = 2$, we can get the mean and variance of $W_{nr}$ as

$$E(W_{nr}) \ = 1 - (\frac{2}{3})^{n-1}$$

and

$$Var(W_{nr}) \ = (\frac{1}{2})^{n-1} - (\frac{4}{9})^{n-1}. \ \square$$

**Theorem 2.1.** *Let $\mu_n^r = E(W_{nr}^r)$, then for $n \geq 2$ and $r = 1, 2, \ldots$*

$$(r+2)\mu_n^r - r\mu_n^{r-1} = 2\mu_{n-1}^r. \qquad (2.2)$$

**Proof.**

$$\begin{aligned}
&r(\mu_n^{r-1} - \mu_n^r) \\
&= \frac{2^{n-1}}{\Gamma(n-1)\theta^2}\int_0^1 [(1-w)^2 r w^{r-1}[-\ln(1-w)]^{n-2}dw \\
&= \frac{2^{n-1}}{\Gamma(n-1)}\int_0^1 w^r[2(1-w)[-\ln(1-w)]^{n-2}dw \\
&-\frac{2^{n-1}(n-2)}{\Gamma(n-1)}\int_0^1 (1-w)^2 w^r[-\ln(1-w)]^{n-3}\frac{1}{1-w}dw \\
&= 2\mu_n^r - 2\mu_{n-1}^r.
\end{aligned}$$

On simplification we get the result. $\square$

**Theorem 2.2.** *For $n \geq 2, p > 0$,*

$$\mu_{nr}^{p} - \mu_{nr}^{p+1} = 2^{n-1}\sum_{k=0}^{p}\binom{p}{k}\frac{(-1)^k}{(3+k)^{n-1}}$$

**Proof.**

$$\begin{aligned}
\mu_{nr}^{p} - \mu_{nr}^{p+1} &= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (w^p - w^{p+1})(1-w)\ [-\ln(1-w)]^{n-2} dw \\
&= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^p (1-w)^2\ [-\ln(1-w)]^{n-2} dw \\
&= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (1-w)^p\ w^2 [-\ln(w)]^{n-2} dw \\
&= 2^{n-1} \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{(3+k)^{n-1}} \qquad (2.3)
\end{aligned}$$

**Theorem 2.3.** *For* $n \geq 1$,

$$1 - W_{n+1r} \overset{d}{=} \Pi_{j=1}^{n} V_j, \qquad (2.4)$$

*where* $V_1, V_2, ..., V_{n-1}$ *are i.i.d. with* $F(v) = v^2, \quad 0 < v < 1.$

**Proof.** We will show first that

$$1 - W_{n+1r} \overset{d}{=} (1 - W_{nr}) V_n,$$

where $V_n$, is indendent of $1 - W_{nr}$ and is distributed with pdf as $f_V(v). = 2v$, $0 < v < 1$.

Let $Y_{n-1} = (1 - W_{nr})V_n, \quad n \geq 2$, and $f_n$ be the pdf of $1 - W_{nr}$, then

$$f_n = \frac{2^{n-1} w}{\Gamma(n-1)} [-\ln(1-w)]^{n-2}, \quad 0 < w < 1,$$

$$\begin{aligned}
F(y) &= P(Y_{n+1} \leq y) = P(1 - W_{nr}) V_n \leq y) \\
&= y^2 + \int_y^1 F_n(\frac{y}{v}) 2v dv, \text{ where } F_{n-1} \text{ is the df of } Y_{(n)} \\
&= y^2 + y^2 \int_y^1 F_n(t) \frac{2}{t^3} dt \\
&= y^2 + y^2 [F_n(t) \frac{-1}{t^2}]_y^1 + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \\
&= y^2 - y^3 + F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \\
&= F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \qquad (2.5)
\end{aligned}$$

Differentiang both sides of (2.5), with respect to y, we obtain

$$\begin{aligned} f(y) &= f_n(y) - f_n(y) + 2y\int_y^1 f_n(t)\frac{1}{t^2}dt \\ &= 2y\int_y^1 f_n(t)\frac{1}{t^2}dt \end{aligned}$$

i.e.

$$\begin{aligned} \frac{f(y)}{y} &= 2\int_y^1 f_n(t)\frac{1}{t^2}dt \\ &= 2\int_y^1 \frac{2^{n-1}t}{\Gamma(n-1)}[-\ln(1-t]^{n-2}\frac{1}{t^2}dt \\ &= [\frac{2^n}{\Gamma(n-1)}[-\ln t]^{n-1}\frac{-1}{n-1}]_y^1 \\ &= \frac{2^n}{\Gamma(n)}[-\ln y]^{n-1} \qquad (2.6) \end{aligned}$$

Hence

$$f(y) = \frac{2^n}{\Gamma(n-1)}y[-\ln y]^{n-1} \qquad (2.7)$$

which is the pdf of $Y = 1 - W_{n+1r}$.

Note that the sequence $Y_2, Y_3, ...$forms a Markov chain. □

Using (2.6), we have the following representation of $W_{nr}$ for

$$1 - W_{n+1r} \stackrel{d}{=} \Pi_{j=1}^n V_j, \quad n \geq 1 \qquad (2.8)$$

where $V_1, V_2, ..., V_{n-1}$ are i.i.d. with $F(v) = v^2, \quad 0 < v < 1$.

The conditional expectation of

$$1 - W_{nr}|1 - W_{mr} = x,\ 2 \leq m < n-1,\ 0 < x < 1,$$

is

$$E(1 - W_{nr}|1 - W_{mr} = x) = x(\frac{2}{3})^{n-m}.$$

Thus

$$\begin{aligned} Cov(W_{nr}W_{mr}) &= (\frac{2}{3})^{n-m}Var(W_{mr}) \\ &= (\frac{2}{3})^{n-m}\ [(\frac{1}{2})^{m-1} - (\frac{4}{9})^{m-1}]. \end{aligned}$$

The correlation coefficient $\rho_{m,n}$ between $W_{nr}W_{mr}$ is given by

$$\rho_{m,n} = \frac{(\frac{2}{3})^{n-m}\sqrt{[(\frac{1}{2})^{m-1} - (\frac{4}{9})^{m-1}]}}{\sqrt{[(\frac{1}{2})^{n-1} - (\frac{4}{9})^{n-1}]}} = \frac{\sqrt{[(\frac{9}{8})^{m-1} - 1]}}{\sqrt{[(\frac{9}{8})^{n-1} - 1]}} \longrightarrow 0$$

for any fixed m as $n \longrightarrow \infty$.

The following table gives the variances and covariances of $W_{nr}$ and $W_{mr}$ for $2 \le m \le n = 5$.

Table 2.1. variances and covariances of $W_{nr}$ and $W_{mr}$

| | | m | | | |
|---|---|---|---|---|---|
| | | 2 | 3 | 4 | 5 |
| n | 2 | $\frac{1}{18}$ | $\frac{1}{27}$ | $\frac{2}{81}$ | $\frac{4}{243}$ |
| | 3 | $\frac{1}{27}$ | $\frac{17}{324}$ | $\frac{17}{486}$ | $\frac{17}{729}$ |
| | 4 | $\frac{2}{81}$ | $\frac{17}{486}$ | $\frac{217}{5862}$ | $\frac{217}{8748}$ |
| | 5 | $\frac{4}{243}$ | $\frac{17}{729}$ | $\frac{217}{8748}$ | $\frac{2465}{104976}$ |

**Theorem 2.4.** *The joint pdf $f^*_{m,n}$ of $W_{mr}$ and $W_{nr}$, $2 \le m \le n$, is given by*

$$\begin{aligned} f^*_{m,n}(x,y) &= \frac{1}{\Gamma(m-1)\Gamma(n-m)} 2^{n-1}(-\ln(1-x))^{m-2} \\ &\times [-\ln(1-x) + \ln(1-y)]^{n-m-1} \frac{1-y}{(1-x)}, \\ & \quad 0 < x < y < 1. \end{aligned}$$

**Proof.** Let $U_1 = \Pi_{j=1}^{m} V_j$ and $U_2 = \Pi_{j=1}^{n-m} V_{m+j}$, then the joint pdf of $U_1$ and $U_2$ is given by

$$f_{U_1U_2}(u_1, u_2) = \frac{2u_1}{\Gamma(m-1)}(-2\ln u_1)^{m-2} \frac{2u_2}{\Gamma(n-m)}(-2\ln u_2)^{n-m-1}.$$

Let $T_1 = U_1$ and $T_2 = U_1U_2$, then the joint pdf of $T_1$ and $T_2$ is

$$\begin{aligned} & f_{T_1T_2}(t_1, t_2) \\ &= \frac{2t_1}{\Gamma(m-1)\Gamma(n-m)}(-2\ln t_1)^{m-2} 2\frac{t_2}{t_1^2}\ (-2\ln(\frac{t_2}{t_1}))^{n-m-1} \\ &= \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)}(-\ln t_1)^{m-2} \frac{t_2}{t_1}\ (-(\ln t_2 - \ln t_1))^{n-m-1} \end{aligned}$$

Sustituting $T_1 = 1 - W_{mr}$ and $T_2 = 1 - W_{nr}$, We obtain the joint pdf of $W_{mr}$ and $W_{nr}$, $2 \leq m < n$ as

$$\begin{aligned} f^*_{m,n}(x,y) &= \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)}(-\ln(1-x))^{m-2} \\ &\times [-\ln(1-x)+\ln(1-y)]^{n-m-1}\frac{1-y}{(1-x)}, \\ & 0 < x < y < 1. \quad \square \end{aligned}$$

**Theorem 2.5.** *For $m \geq 2$, $p \geq 0$ and $q \geq 0$,*

$$\begin{aligned} & E(W_{mr})^p(W_{m+1r})^{q+1} \\ & \quad = \frac{q+1}{q+3}E[(W_{mr})^p(W_{m+1r})^q + \frac{2}{q+3}E(W_{mr})^{p+q+1} \end{aligned}$$

**Proof.**

$$\begin{aligned} & E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}] \\ &= \int_o^1 \int_x^1 [(x)^p\{(y)^q(1-y)\}]f^*_{m,m+1}(x,y)dydx \\ &= \int_o^\theta [(x)^p \frac{1}{\Gamma(m-1)}2^{n-1}(-\ln(1-x))^{m-2}\frac{1}{(1-x)}H(x)dx, \quad (2.9) \end{aligned}$$

where

$$\begin{aligned} H(x) &= \int_x^1 (y)^q(1-y)\}(1-y)dy \\ &= \int_x^1 (y)^q(1-y)^2dy \\ &= \frac{y^{q+1}}{(q+1)}(1-y)^2|_x^1 + \int_x^1 \frac{2y^{q+1}}{\theta^{q+1}(q+1)}(1-y) \\ &= -\frac{x^{q+1}}{(q+1)} + 2\int_x^1 \frac{y^{q+1}}{(q+1)}(1-y)dy \end{aligned}$$

Substituting in (2.9), we obtain

$$\begin{aligned} & E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}] \\ &= \int_o^1 [(x)^p(1-y)\frac{1}{\Gamma(m-1)}2^m(-\ln(1-x))^{m-2}\frac{1}{(1-x)} \\ & \quad .[-\frac{2x^{q+1}}{(q+1)} + 2\int_x^1 \frac{y^{q+1}}{(q+1)}(1-y)dy]dx \\ &= -\frac{2}{q+1}E((W_{mr})^{p+q+1}) + \frac{2}{q+1}E(W_{mr})^p(W_{m+1r})^{q+1} \end{aligned}$$

Thus

$$E(W_{mr})^p(W_{m+1r})^{q+1}$$
$$= \frac{q+1}{q+3}E[(W_{mr})^p(W_{m+1r})^q + \frac{2}{q+3}E(W_{mr})^{p+q+1}. \quad \square$$

**Theorem 2.6.** *For $m \geq 2$, $n > m \geq 2$, $p \geq 0$ and $q \geq 0$,*

$$E(W_{mr})^p(W_{nr})^{q+1}$$
$$= \frac{q+1}{q+3}E((W_{mr})^p(W_{nr})^q) + \frac{2}{q+3}E(W_{mr})^{p+q+1}$$
$$E((W_{mr})^p(W_{nr})^{q+1})$$
$$= \frac{q+1}{q+3}E[(W_{mr})^p(W_{nr})^{q+1}] + \frac{2}{q+3}E((W_{mr})^p(W_{n-1r})^q)$$

**Proof.**

$$E[(W_{mr})^p(W_{nr})^q - (W_{mr})^p(W_{n+1r})^{q+1}]$$
$$= \int_o^1 \int_x^1 [(x)^p\{(y)^q(1-y)\}]f^*_{m,n}(x,y)dydx$$
$$= \int_o^1 [(x)^p \frac{1}{\Gamma(m-1)\Gamma(n-m)} 2^{m+k-1}$$
$$(-\ln(1-x))^{m-2}\frac{1}{(1-x)}H(x)dx, \qquad (2.10)$$

where

$$\frac{1}{\Gamma(n-m)}H(x)$$
$$= \int_x^1 \{(y)^q(1-y)\}[-\ln(1-x)+\ln(1-y)]^{n-m-1}$$
$$\times(1-y)dy$$
$$= \int_x^1 (\frac{y}{\theta})^q[-\ln(1-x)+\ln(1-y)]^{n-m-1}(1-y)^2dy$$
$$= \frac{y^{q+1}}{(q+1)}[-\ln(1-x)+\ln(1-y)]^{n-m-1}(1-y)^2]_x^1$$
$$- \int_x^1 \frac{y^{q+1}}{q+1}\frac{d}{dy}[-\ln(1-x)+\ln(1-y)]^{n-m-1}$$
$$\times(1-y)^2dy$$

$$= -\int_x^1 \frac{y^{q+1}}{q+1}\frac{d}{dy}[-\ln(1-x)+\ln(1-y)]^{n-m-1}$$

$$\times(1-y)^2]dy$$

$$= 2\int_x^1 \frac{y^{q+1}}{q+1}[-\ln(1-x)+\ln(1-y)]^{n-m-1}(1-y)]$$

$$+\int_x^1 \frac{y^{q+1}}{\theta^q(q+1)}(n-m-1)[-\ln(1-x)+\ln(1-y)]^{n-m-2}$$

$$\times(1-\frac{y}{\theta})dy$$

Substituting H(x) in (2.10), we obtain

$$E[(W_{mr})^p(W_{nr})^q-(W_{mr})^p(W_{nr})^{q+1}]$$
$$=\frac{2}{q+1}E((W_{mr})^p(W_{nr})^{q+1})-\frac{1}{q+1}E((W_{mr})^p(W_{n-1r})^{q+1})$$

On simplification, we obtain

$$E((W_{mr})^p(W_{nr})^{q+1})$$
$$=\frac{q+1}{q+3}E[(W_{mr})^p(W_{nr})^{q+1}]+\frac{2}{q+3}E((W_{mr})^p(W_{n-1r})^{q+1})\ \square$$

**Entropy of $W_{nr}$.**
The entropy of $W_{nr}$ is given in the following theorem.

**Theorem 2.7.** *The entropy, $I_n$ of $W_{nr}$, $n \geq 2$, is given by*

$$I_n = \ln\Gamma(n-1)+\frac{n-1}{2}-(n-2)\Psi(n-1)-\ln 2,$$

*where* $\Psi(n-1)=\frac{\Gamma'(n-1)}{\Gamma(n-1)}$.
**Proof.**

$$\begin{aligned} I_n &= E(-\ln f_{W_{nr}}) \\ &= \int_0^1 [\ln\Gamma(n-1)-(n-1)\ln 2-\ln(1-u)-(n-2) \\ &\quad \ln(-\ln(1-u))]\frac{2^{n-1}(1-u)}{\Gamma(n-1)}[-\ln\{1-u)]^{n-2}du \\ &= [\ln\Gamma(n-1)-(n-1)\ln 2-H_1-H_2, \end{aligned} \quad (2.11)$$

where

$$H_1 = \int_0^1 \ln(1-u)\frac{2^{n-1}(1-u)}{\Gamma(n-1)}[-\ln\{1-u)]^{n-2} = -\frac{n-1}{2},$$
$$H_2 = \int_0^1 (n-2)ln(-ln(1-u))]\frac{2^{n-1}(1-u)}{\Gamma(n-1)}[-\ln\{1-u)]^{n-2}du$$

Substituing $-\ln\{1-u) = t$, we obtain

$$\begin{aligned} H_2 &= \frac{2^{n-1}(n-2)}{\Gamma(n-1)}\int_0^\infty t^{n-2}\ln te^{-2t}dt \\ &= \frac{n-2}{\Gamma(n-1)}\int_0^\infty t^{n-2}\ln te^{-t}dt - (n-2)\ln 2 \\ &= (n-2)[\Psi(n-1) - \ln 2]. \end{aligned}$$

Substituting $H_1$ and $H_2$ in (2.11), we obtain

$$\begin{aligned} I_n &= \ln\Gamma(n-1) - (n-1)\ln 2 + \frac{n-1}{2} \\ &\quad -(n-2)[\Psi(n-1) - \ln 2]. \\ &= \ln\Gamma(n-1) + \frac{n-1}{2} - (n-2)\Psi(n-1) - \ln 2. \quad \Box(2.12) \end{aligned}$$

The following table gives $-I_n$ for $n = 3$ to 10.

Table 2.2. Values of $-I_n$ for $2 \leq n \leq 10$.

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| -I$_n$ | 0.1159 | 0.3456 | 0.6698 | 1.0396 | 1, 4363 | 1.8506 | 2.2775 | 2.7141 |

We will consider the estimation $\theta$ based on the record range. when $X_{1,}X_2, \ldots$ are i.i.d with $f(x) = \frac{1}{\theta},\ \ 0 < x < \theta$.

**Theorem 2.8.** *The minimum varaiance linear unbiased estimator of $\hat{\theta}$ of $\theta$ is*

$$\hat{\theta} = \frac{1}{3(2^n-1)}[3.2^{n-1}W_{n+1r} - 2^{n-2}W_{nr} - 2^{n-3}W_{n-1r} - ... - W_{2r}]$$

*and*

$$Var(\hat{\theta}) = \frac{\theta^2}{2(2^n-1)}.$$

**Proof.** Let

$$\begin{aligned} Z_1 &= d_1 W_{2r}, d_1 = 3.2^{\frac{1}{2}} \\ Z_2 &= d_2(W_{3r} - \frac{2}{3}W_{2r}), d_2 = 3.2 \\ &\vdots \\ Z_n &= d_n(W_{n+1r} - \frac{2}{3}W_{nr}), d_n = 3.2^{\frac{n}{2}}. \\ Z' &= (Z_1, Z_2, )..., Z_n), \end{aligned}$$

then $E(Z') = A\theta$, where

$$A' = (2^{\frac{1}{2}}, 2, ..., 2^{\frac{n}{2}}).A'A = 2(2^n - 1).$$

Then the minimum variance linear unbiased estimator (MVLUE) $\hat{\theta}$ of $\theta$ ( see Ahsanullah and Nevzorov (2005), Nagaraja and David (2003)) is

$$\begin{aligned} \hat{\theta} &= (A'A)^{-1}A'Z \\ &= \frac{1}{2(2^n - 1)}[2^{\frac{1}{2}}Z_1 + 2Z_2 + ... + 2^{\frac{n}{2}}Z_n] \\ &= \frac{1}{2(2^n - 1)}[3.2^n W_{n+1r} - 2^{n-1}W_{nr} - 2^{n-2}W_{n-1r} - 2W_{2r}]. \quad (2.13) \end{aligned}$$

$$Var(\hat{\theta}) = \theta^2(A'A)^{-1} = \frac{\theta^2}{2(2^n - 1)}. \quad \square$$

For example, if $n = 4$, then

$$\hat{\theta} - \frac{1}{15}[24W_{5r} - 4W_{4r} - 2W_{3r} - W_{2r}]$$

and

$$Var(\hat{\theta}) = \frac{\theta^2}{30}.$$

Table 2.3. Coefficient of $W_{nr}$ in MVLUE of $\theta$.

| $n$ | $W_{2r}$ | $W_{3r}$ | $W_{4r}$ | $W_{5r}$ | $W_{6r}$ | $W_{7r}$ | $W_{8r}$ | $W_{9r}$ | $W_{10r}$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | $-\frac{1}{3}$ | 2 | | | | | | | |
| 3 | $-\frac{1}{7}$ | $-\frac{2}{7}$ | $\frac{12}{7}$ | | | | | | |
| 4 | $-\frac{1}{15}$ | $-\frac{2}{15}$ | $-\frac{4}{15}$ | $\frac{24}{15}$ | | | | | |
| 5 | $-\frac{1}{31}$ | $-\frac{2}{31}$ | $-\frac{4}{31}$ | $-\frac{8}{31}$ | $\frac{48}{31}$ | | | | |
| 6 | $-\frac{1}{63}$ | $-\frac{2}{63}$ | $-\frac{4}{63}$ | $-\frac{8}{63}$ | $-\frac{16}{63}$ | $\frac{96}{63}$ | | | |
| 7 | $-\frac{1}{127}$ | $-\frac{2}{127}$ | $-\frac{4}{127}$ | $-\frac{8}{127}$ | $-\frac{16}{127}$ | $-\frac{32}{127}$ | $\frac{192}{127}$ | | |
| 8 | $-\frac{1}{255}$ | $-\frac{2}{299}$ | $-\frac{4}{255}$ | $-\frac{8}{255}$ | $-\frac{16}{255}$ | $-\frac{32}{255}$ | $-\frac{64}{255}$ | $\frac{384}{255}$ | |
| 9 | $-\frac{1}{511}$ | $-\frac{2}{511}$ | $-\frac{4}{511}$ | $-\frac{8}{311}$ | $-\frac{16}{511}$ | $-\frac{32}{511}$ | $-\frac{64}{511}$ | $-\frac{128}{511}$ | $\frac{768}{511}$ |

Let $\tilde{\theta} = c\hat{\theta}$, then bias of $\tilde{\theta}$ is $(c-1)\theta$ and mean squared error (MSE) of $\tilde{\theta}$ is $\text{MSE}(\tilde{\theta}) = c^2 \frac{\theta^2}{2(2^n-1)} + (c-1)^2\theta^2$. The MSE of $\tilde{\theta}$ will be minimum if $c = \frac{2^{n+1}-2}{2^{n+1}-1}$.

The bias of $\tilde{\theta} = (c-1)\theta = \frac{-1}{2^{n+1}-1}$ and $\text{MSE}(\tilde{\theta}) = \frac{1}{2^{n+1}-1}$.

**Prediction of $W_{n+sr}$.**
We consider the prediction of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$.

**Theorem 2.5.** *The best linear least squares predictor, $W^*_{n+sr}$ of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$ is $\theta[1-(\frac{2}{3})^s] + (\frac{2}{3})^s W_{nr}$.*

**Proof.** The best linear least squares predictor, $W^*_{n+sr}$ of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$ is

$$\begin{aligned} W^*_{n+sr} &= E(W_{n+sr}|W_{2r} = x_2, W_{3r} = x_3, ..., W_{nr} = x_n) \\ &= E(W_{n+sr}|W_{nr} = x_n), \text{by Markov property of } W_{2r}, W_{3r}, \ldots \\ &= \theta[1-(\frac{2}{3})^s] + x_n(\frac{2}{3})^s. \end{aligned}$$

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