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Second Order Moment Asymptotic Expansions for a Randomly Stopped and Standardized Sum

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Abstract. This paper establishes the first four moment expansions to the order $o(a^{-1})$ of $S'_{t_a}/\sqrt{t_a}$, where $S'_n = \sum_{i=1}^{n} Y_i$ is a simple random walk with $E(Y_i) = 0$, and t_a is a stopping time given by

$$t_a = \inf\{n \geq 1 : n + S_n + \zeta_n > a\}$$

where $S_n = \sum_{i=1}^{n} X_i$ is another simple random walk with $E(X_i) = 0$, and $\{\zeta_n, n \geq 1\}$ is a sequence of random variables satifying certain assumptions. These moment expansions complement the classical central limit theorem for a random number of i.i.d. random variables when the random number has the form t_a, which arises from

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many sequential statistical procedures. They can be used to correct higher order bias and/or skewness in $S'_{t_a}/\sqrt{t_a}$ to make asymptotic approximation more accurate for small and moderate sample sizes.

1 Introduction

Let $\{Y_i, i \geq 1\}$ be a sequence of i.i.d. random variables with zero mean and unit variance, and denote $S'_n = \sum_{i=1}^n Y_i$ and $\bar{Y}_n = S'_n/n$. It is well known (see e.g. Anscombe, 1952 and Renyi, 1957) that $\sqrt{t}\bar{Y}_t$ has an asymptotic normal distribution when t is a random variable and satisfies certain assumptions. However asymptotic normality alone is often not accurate enough for many applications, see e.g. Ghosh, Mukhopadhyay and Sen (1997), pp336. In this paper we derive rigorously the first four moment expansions of $\sqrt{t}\bar{Y}_t$ when t is a stopping time given by

$$t = t_a = \inf\{n \geq 1 : n + S_n + \zeta_n > a\}, \tag{1}$$

where $S_n = \sum_{i=1}^n X_i$ with $\{X_i, i \geq 1\}$ being i.i.d. zero-mean random variables, and $\{\zeta_n, n \geq 1\}$ is a sequence of random variables such that ζ_n is independent of $\{(X_{n+k}, Y_{n+k}), k \geq 1\}$ for each n. Stopping time t_a arises from many sequential statistical procedures, see for example, Lai and Siegmund (1977, 1979), Woodroofe (1977, 1982), and Ghosh, Mukhopadhyay and Sen (1997). The moment expansions of $\sqrt{t}\bar{Y}_t$ to the order $o(a^{-1})$ given here can be used directly to correct higher order bias and/or skewness in $\sqrt{t}\bar{Y}_t$ and therefore make large sample approximation more accurate for small or moderate sample sizes; see e.g. Liu, Wang and Wang (2002) and Liu (2002). Aras and Woodroofe (1993) give the first four moment expansions of $\sqrt{a}\bar{Y}_{t_a}$.

In establishing the expansions, one important result used is the nonlinear renewal theorem (see e.g. Lai and Siegmund, 1977) which states that, under some suitable conditions, the overshoot at the crossing time t_a, $t_a + S_{t_a} + \zeta_{t_a} - a$, has the same limit distribution as the overshoot $\tau_a + S_{\tau_a} - a$ at stopping time

$$\tau_a = \inf\{n \geq 1 : n + S_n > a\}$$

as $a \to \infty$. This limit distribution has the form

$$H(dx) = \frac{1}{E(\tau_0 + S_{\tau_0})} P\{\tau_0 + S_{\tau_0} > x\} dx,$$

where $\tau_0 = \inf\{n \geq 1 : n + S_n > 0\}$ is the first ladder epoch. The conditions imposed on $\{\zeta_n, n \geq 1\}$ are "slowly changing". As stated in Woodroofe (1982), they are

(i) $n^{-1} \max\{|\zeta_1|, |\zeta_2|, \cdots, |\zeta_2|\} \to 0$ in probability,

(ii) there is a $\theta > 0$ such that $P\{\max_{0 \leq k \leq \theta n} |\zeta_{n+k} - \zeta_n| > \varepsilon\} < \varepsilon$ for all $n \geq 1$, given a $\varepsilon > 0$.

Certain uniform integrabilities are also necessary in proving the moment expansions. Although these uniform integrabilities can be deduced from the general results of Zhang (1988) as in Aras and Woodroofe (1993), we establish the required uniform integrabilities directly. The reason for this is that Zhang's, 1988 discussion is for the more general "curved boundary" situation, where the upper boundary 'a' in (1) is replaced by '$A(n, a)$'. So the conditions for uniform integrabilities deduced from Zhang's results seem unnecessarily complex. By tackling stopping time (1) directly, we can establish the uniform integrabilities under relatively simpler conditions. As all the four moment expansions are of order $o(a^{-1})$, Wald's lemmas of high order, which are not readily available in the literature, are required. They are provided in this paper.

The layout of the paper is as follows. Section 2 deals with the uniform integrabilities. Section 3 establishes the moment expansions. The required higher order Wald's lemmas are provided in the Appendix.

2 Uniform integrabilities in nonlinear renewal theory

Let $(X_n, Y_n), n \geq 1$ be i.i.d. zero-mean random vectors on a probability space $\{\Omega, \mathcal{F}, P\}$ and $\zeta_n, n \geq 1$ be random variables on $\{\Omega, \mathcal{F}, P\}$ such that ζ_n is independent of $\{(X_{n+k}, Y_{n+k}),\ k \geq 1\}$ for each n. Let $\{\mathcal{F}_n, n \geq 1\}$ be the natural σ-algebras associated with $\{(X_n, Y_n, \zeta_n), n \geq 1\}$, i.e. $\mathcal{F}_n = \sigma(X_1, Y_1, \zeta_1, \cdots, X_n, Y_n, \zeta_n)$. It is clear that (X_n, Y_n) is independent of $\mathcal{F}_{n-1}$ for all $n \geq 2$, and $\{t_a, a > 0\}$ is a family of increasing stopping times with respect to $\{\mathcal{F}_n, n \geq 1\}$. The overshoot at time t_a will be denoted as $R_{t_a} = S_{t_a} + \zeta_{t_a} + t_a - a$.

In this section we establish the uniform integrabilities for $\{(t_a - a)/\sqrt{a}, a \geq 1\}$ and $\{R_{t_a} - \zeta_{t_a}, a > 0\}$. The following four conditions are required:

(C1) $E\left|X^{p+1}\right|$ exists for some $p \geq 1$;
(C2) $a^p P\{t_a \leq \delta a\} \to 0$ as $a \to \infty$, for some positive $\delta < 1$;
(C3) $\sum_{n=1}^{\infty} n^{p-1} P\{\zeta_n \leq -n\epsilon\} < \infty$ for some $0 < \epsilon < 1$;
(C4) $\{\max_{1 \leq j \leq n} \mid \zeta_{n+j} \mid^p,\ n \geq 1\}$ is uniformly integrable.

Baum-Katz Inequality (Baum and Katz, 1965) and a lemma from Chow and Yu (1981) are important in this discussion, so we state them here for easy reference.

Baum-Katz Inequality asserts that, for constants $s > 1$ and $r > 1$ such that $1/2 < r/s \leq 1$, the following three statements are equivalent:
(a) $E \mid X_1 \mid^s < \infty$;
(b) $\sum_{n=1}^{\infty} n^{r-2} P\{\mid S_n \mid > n^{r/s}\epsilon\} < \infty$ for all $\epsilon > 0$;
(c) $\sum_{n=1}^{\infty} n^{r-2} P\{\sup_{k \geq n} \mid S_k/k^{r/s} \mid > \epsilon\} < \infty$ for all $\epsilon > 0$.

Lemma 3.2 of Chow and Yu (1981) states that if, for some $s \geq 2$, $E \mid X_1 \mid^s < \infty$ and, for some $\mathcal{F}_n$-stopping time family $\{\eta_d, d \in \Lambda\}$,

$\{(\eta_d/d)^{s/2}, d \in \Lambda\}$ is uniformly integrable, then $\{| S_{\eta_d}/\sqrt{d} |^s, d \in \Lambda\}$ is uniformly integrable. Here $\Lambda \subset (0, \infty)$.

Another useful fact is that, for any $\mathcal{F}_n$-stopping time η satisfying $P\{\eta < \infty\} = 1$ (called proper), $X_{\eta+n}, n \geq 1$ are still i.i.d. random variables and each has the same distribution as that of X_1. See, for example, Chow and Teicher (1988), pp138.

The following lemma is an extension of Theorem 4.4 of Woodroofe (1982).

Lemma 2.1. *Under (C1) and (C3), we have* $E(t_a^p) < \infty$ *for any given* $a > 0$. *Moreover,* $\lim_{a\to\infty} \int_{t_a > ha} t_a^p dP = 0$ *for any* $h > (1-\epsilon)^{-1}$, *and consequently* $\{(a^{-1}t_a)^p, a \geq 1\}$ *is uniformly integrable.*

Proof. Let $\epsilon_1 > 0$ and $\epsilon_1 + \epsilon < 1$, where ϵ is as given in (C3). Denote $K_a = [a/(1-\epsilon_1-\epsilon)]+1$. Then, for $n > K_a$, one has $a - n < -n(\epsilon_1+\epsilon)$ and

$$n^{p-1}P\{t_a > n\} \leq n^{p-1}P\{S_n + \zeta_n \leq a - n\}$$
$$\leq n^{p-1}P\{S_n \leq -n\epsilon_1\} + n^{p-1}P\{\zeta_n \leq -n\epsilon\}.$$

Summing up over n, we get

$$\sum_{n=K_a}^{\infty} n^{p-1}P\{t_a > n\} <$$
$$\sum_{n=K_a}^{\infty} n^{p-1}P\{S_n \leq -n\epsilon_1\} + \sum_{n=K_a}^{\infty} n^{p-1}P\{\zeta_n \leq -n\epsilon\}$$

and hence, by Baum-Katz Inequality, (C1) and (C3), $E(t_a^p) < \infty$ for any fixed $a > 0$. By the same arguments, we also have

$$\sum_{n=K_a}^{\infty} n^{p-1}P(t_a > n) \to 0 \quad as\ a \to \infty,$$

and this implies $(K_a)^p P(t_a > K_a) \to 0$. Combining this result with the following integral by parts formula

$$\int_{t_a > y} t_a^p dP = \int_y^{\infty} px^{p-1}P(t_a > x)dx + (y)^p P(t_a > y),$$

we obtain $\lim_{a\to\infty}\int_{t_a>K_a} t_a^p dP = 0$. Noting that $\epsilon_1 > 0$ is arbitrarily fixed as long as $\epsilon_1+\epsilon < 1$, so we conclude that $\lim_{a\to\infty}\int_{t_a>ha} t_a^p dP = 0$ for any $h > 1/(1-\epsilon)$. The uniform integrability of $\{(a^{-1}t_a)^p,\ a \geq 1\}$ follows from the above result and the fact that t_a is monotone in a.
□

Lemma 2.2. *Suppose $\{T_a, a > 0\}$ be a family of positive and increasing random variables satisfying $\lim_{a\to\infty}\int_{T_a>ha} T_a^r dP = 0$ for some $h > 1$, $r > 0$, and B_a be an event in $\mathcal{F}$ such that $a^r P(B_a^c) \to 0$ as $a \to \infty$. Then $\{(T_a I_{B_a^c})^r, a > 0\}$ is uniformly integrable.*

The proof is trival, thus omitted.

For $h > (1-\epsilon)^{-1}$, define

$$\nu_n = \begin{cases} \zeta_n & \text{when} \quad n \leq ha \\ 0 & \text{when} \quad n > ha, \end{cases}$$

$$\tilde{t}_a = \inf\{n \geq \delta a : n + S_n + \nu_n > a\}, \quad a \geq 1,$$

and

$$\tau_a = \inf\{n \geq \delta a : n + S_n > a\}, \quad a \geq 1.$$

Observe that $P\{\nu_n \leq -n\epsilon\} \leq P\{\zeta_n \leq -n\epsilon\}$. So it is not difficult to verify that Lemma 2.1 holds for $\tilde{t}_a$. Lemma 2.1 holds also for both t_a^+ and t_a^- defined below in the proof of Lemma 2.3.

Lemma 2.3. *Under (C1)-(C4), $\left\{\left|\tilde{t}_a - \tau_a\right|^p, a > 0\right\}$ is uniformly integrable.*

Proof. From the definition of ν_n, we have

$$\max_{\delta a \leq n < \infty} |\nu_n| = \max_{\delta a \leq n \leq ha} |\nu_n|$$

which is uniformly integrable (about $a \in (0,\infty)$) to the power p by (C4). Let

$$\begin{aligned} t_a^+ &= \inf\{n \geq \delta a : n + S_n + \nu_n^+ > a\} \\ t_a^- &= \inf\{n \geq \delta a : n + S_n - \nu_n^- > a\} \end{aligned}$$

where $\nu_n^+ = \max\{0, \nu_n\}$, $\nu_n^- = -\min\{0, \nu_n\}$. It is apparent that $t_a^+ \le \tau_a \le t_a^-$ and $t_a^+ \le \tilde{t}_a \le t_a^-$, so

$$| \tilde{t}_a - \tau_a | \le \max\{\tau_a - t_a^+,\ t_a^- - \tau_a\}.$$

Note that

$$\begin{aligned}
\tau_a - t_a^+ &\le \inf\{n \ge 1 : n + X_{t_a^+ +1} + \cdots + X_{t_a^+ +n} > \nu^+_{t_a^+}\} \\
&\le \inf\{n \ge 1 : n + X_{t_a^+ +1} + \cdots + X_{t_a^+ +n} > \max_{\delta a \le n < \infty} |\nu_n|\} \triangleq d_1, \\
t_a^- - \tau_a &\le \inf\{n \ge 1 : n + X_{\tau_a +1} + \cdots + X_{\tau_a +n} > \nu^-_{\tau_a + n}\} \\
&\le \inf\{n \ge 1 : n + X_{\tau_a +1} + \cdots + X_{\tau_a +n} > \max_{\delta a \le n < \infty} |\nu_n|\} \triangleq d_2,
\end{aligned}$$

Therefore, if both d_1^p and d_2^p can be proved to be uniformly integrable, $| \tilde{t}_a - \tau_a |^p$ is then uniformly integrable. The proofs for these two cases are similar, so we give the proof for d_2 only.

It is clear that

$$P\{d_2 > n\} \le P\{n + X_{\tau_a+1} + \cdots + X_{\tau_a+n} - \max_{\delta a \le n < \infty} |\nu_n| \le 0\}$$

$$\le P\{X_{\tau_a+1} + \cdots + X_{\tau_a+n} \le -\theta_1 n\} + P\{-\max_{\delta a \le n < \infty} |\nu_n| \le -\theta_2 n\}$$

$$\le P\{|X_{\tau_a+1} + \cdots + X_{\tau_a+n}| \ge \theta_1 n\} + P\{\max_{\delta a \le n < \infty} |\nu_n| \ge \theta_2 n\}$$

where $\theta_1, \theta_2 > 0$ and $\theta_1 + \theta_2 = 1$. Summing up over n from an integer N gives

$$\sum_N^\infty n^{p-1} P\{d_2 > n\} \le \sum_N^\infty n^{p-1} P\{|X_{\tau_a+1} + \cdots + X_{\tau_a+n}| \ge \theta_1 n\}$$

$$+ \sum_N^\infty n^{p-1} P\{\max_{\delta a \le n < \infty} |\nu_n| \ge \theta_2 n\}$$

$$= \sum_N^\infty n^{p-1} P\{|X_1 + \cdots + X_n| \ge \theta_1 n\} + \sum_N^\infty n^{p-1} P\{\max_{\delta a \le n < \infty} |\nu_n| \ge \theta_2 n\}.$$

The last equality is due to the fact that $\{X_{\tau_a+n}, n \ge 1\}$ are i.i.d. sequence with each element having the same distribution as X_1 (since

τ_a is a proper $\mathcal{F}_n$-stopping times). For the first term, by Baum-Katz inequality and (C1), we have

$$\sum_N^\infty n^{p-1}P\{|X_1+\cdots+X_n|\geq\theta_1 n\}\to 0 \ \text{ as } N\to\infty$$

and it is free from a. For the second term, note that $\{\max_{\delta a\leq n<\infty}|\nu_n|^p\,,$ $a>0\}$ is uniformly integrable, so from the integral by parts formula we have

$$\sum_N^\infty n^{p-1}P\{\max_{\delta a\leq n<\infty}|\nu_n|>\theta_2 n\}\to 0 \ \text{ uniformly in } \ a \ \text{ as } \ N\to\infty.$$

Thus $\sum_N^\infty n^{p-1}P\{d_2>n\}\to 0$ uniformly in a as $N\to\infty$, and hence d_2^p is uniformly integrable. □

Theorem 2.1. *Under (C1)-(C4),* $\{|t_a-\tau_a|^p\,,a>0\}$ *is uniformly integrable and, as a consequence,* $\{|R_{t_a}-\zeta_{t_a}|^p\,,a>0\}$ *is uniformly integrable.*

Proof. $\mid t_a-\tau_a\mid^p$ is different from $\mid\tilde{t}_a-\tau_a\mid^p$ only on $\{t_a<\delta a\}\cup\{t_a> ha\}$. Note that

$$\mid t_a-\tau_a\mid^p<2^p(t_a^p+\tau_a^p)$$

and

$$a^pP\{(t_a<\delta a)\cup(t_a>ha)\}\to 0 \ \text{ as } \ a\to\infty$$

by (C2) and Lemma 2.1. Thus the uniform integrability of $\mid t_a-\tau_a\mid^p$ follows from Lemma 2.2 and Lemma 2.3. To prove the second part, let $R_{\tau_a}=\tau_a+S_{\tau_a}-a$. Comparing it with $R_{t_a}=t_a+S_{t_a}+\zeta_{t_a}-a$, we get

$$R_{t_a}-\zeta_{t_a}=t_a-\tau_a+S_{t_a}-S_{\tau_a}+R_{\tau_a}$$

and

$$|R_{t_a}-\zeta_{t_a}|\leq|t_a-\tau_a|+\left|\sum_{i=t_a\wedge\tau_a+1}^{t_a\vee\tau_a}X_i\right|+|R_{\tau_a}|\,.$$

According to a result of Gut, 1988, Theorem 5.1 of Chapter 1,

$$E\left|\sum_{i=t_a\wedge\tau_a+1}^{t_a\vee\tau_a} X_i\right|^{p+1} \leq CE\left|t_a-\tau_a\right|^{(p+1)/2} \leq CE\left|t_a-\tau_a\right|^p$$

for some constant $C,$ which depends only on the distribution of X_1 and parameter p. So, $E\left|\sum_{i=t_a\wedge\tau_a+1}^{t_a\vee\tau_a} X_i\right|^{p+1}$ is uniformly bounded and hence $\left|\sum_{i=t_a\wedge\tau_a+1}^{t_a\vee\tau_a} X_i\right|^p$ is uniformly integrable. The uniform integrability of $|R_{\tau_a}|^p$ is a result in renewal theory (see, for instance, the proof of Theorem 3.1 of Woodroofe 1982, and note that the delay δa doesn't affect the proof). Therefore $|R_{t_a}-\zeta_{t_a}|^p$ is uniformly integrable. $\square$

Remark. Gut, 1988 states that, if $E\left|X_1^r\right| < \infty$, then $E\left|S_\eta\right|^r \leq CE(\eta)^{r/2}$ for any proper $\mathcal{F}_n$-stopping time η where constant C is free from the choice of η. For two proper stopping times η and s, let $\mathcal{B}_n = \mathcal{F}_{(\eta\wedge s)+n}$, the σ-algebra of events before stopping time $(\eta\wedge s)+n$, and let $Z_n = X_{(\eta\wedge s)+n}$. Then $\{Z_n, \mathcal{B}_n, n\geq 1\}$ is still an adapted i.i.d. process and $(\eta\vee s)-(\eta\wedge s)$ is a $\mathcal{B}_n$-stopping time. Thus, by Gut's result,

$$E\left|\sum_{i=1}^{(\eta\vee s)-(\eta\wedge s)} Z_i\right|^r \leq CE[(\eta\vee s)-(\eta\wedge s)]^{r/2}$$

which is

$$E\left|\sum_{i=\eta\wedge s+1}^{\eta\vee s} X_i\right|^r \leq CE\left|\eta - s\right|^{r/2}.$$

Corollary 2.1. *Denote $t_a^* = (t_a-a)/\sqrt{a}$. Under (C1)-(C4), $\{(t_a^*)^p, a\geq 1\}$ is uniformly integrable.*

Proof. The uniform integrability of $(t_a^*)^p$ follows clearly from the equality

$$\frac{t_a-a}{\sqrt{a}} = \frac{R_{t_a}-\zeta_{t_a}}{\sqrt{a}} - \frac{S_{t_a}}{\sqrt{a}},$$

Theorem 2.1 and Lemma 3.2 of Chow and Yu (1981).

Theorem 2.1 and Corollary 2.1 are still true if condition (C4) is replaced by a weaker condition as follows

(C4') There are events $A_n \in \mathcal{F}_n, n \geq 1$ such that $\sum_{n=1}^{\infty} n^{p-1} P\{\cup_{k=n}^{\infty} A_k^c\}$ $< \infty$ and $\{\max_{1\leq j\leq n} \mid \zeta_{n+j} I_{A_{n+j}} \mid^p,\ n \geq 1\}$ is uniformly integrable.

Here A_n^c is the complement of A_n and I_{A_n} is an indicator function. The proof is not difficult by making use of Lemma 2.2.

3 Moment expansions

Now we are ready to discuss the expansion for $[\sqrt{t_a}\bar{Y}_{t_a}]^k, k = 1, 2, 3, 4$. Note that

$$[\sqrt{t_a}\bar{Y}_{t_a}]^k = \frac{1}{\sqrt{a^k}}\left[(S'_{t_a})^k + (S'_{t_a})^k\left(\sqrt{\frac{a^k}{t_a^k}} - 1\right)\right].$$

Applying Taylor expansion to $(t_a/a)^{-k/2} - 1$, we have

$$\begin{aligned}[\sqrt{t_a}\bar{Y}_{t_a}]^k &= \frac{1}{\sqrt{a^k}}(S'_{t_a})^k - \frac{k}{2\sqrt{a^k}}(S'_{t_a})^k\left(\frac{t_a - a}{a}\right) \\ &+ \frac{k(k+2)}{8u_a^{(k+4)/2}\sqrt{a^k}}(S'_{t_a})^k\left(\frac{t_a - a}{a}\right)^2 \\ &= \frac{1}{\sqrt{a^k}}(S'_{t_a})^k + \frac{k}{2\sqrt{a^{k+2}}}[(S'_{t_a})^k S_{t_a}] + \frac{1}{a}T_1 + \frac{1}{a}T_2 \quad (2)\end{aligned}$$

where u_a is between 1 and t_a/a,

$$T_1 = -\frac{k}{2}\left(\frac{S'_{t_a}}{\sqrt{a}}\right)^k (R_{t_a} - \zeta_{t_a}), \quad \text{and} \quad T_2 = \frac{k(k+2)}{8u_a^{(k+4)/2}}\left(\frac{S'_{t_a}}{\sqrt{a}}\right)^k\left(\frac{t_a - a}{\sqrt{a}}\right)^2.$$

We first give the following result about the uniform integrabilities of T_1 and T_2.

Lemma 3.1. *Under (C1)-(C4), T_1 and T_2 are uniformly integrable for $k = 1$ and 2, if $p \geq 4$ and $E(Y_1^4) < \infty$. Furthermore, T_1 and T_2 are uniformly integrable for $k = 3$ and 4, if $p \geq 6$ and $E(Y_1^8) < \infty$.*

Proof. Note that $\left(S'_{t_a}/\sqrt{a}\right)^{2k}$ is uniformly integrable by Lemma 3.2 of Chow and Yu, 1981 and $(R_{t_a} - \zeta_{t_a})^2$ is uniformly integrable by Theorem 2.1. So the uniform integrability of T_1 in both cases follows simply from Holder's inequality. For T_2, we divide it into two parts

$$\begin{aligned} T'_2 &= \frac{k(k+2)}{8u_a^{(k+4)/2}} \left(\frac{S'_{t_a}}{\sqrt{a}}\right)^k \left(\frac{t_a - a}{\sqrt{a}}\right)^2 I_{\{t_a < \delta a\}}, \\ T''_2 &= \frac{k(k+2)}{8u_a^{(k+4)/2}} \left(\frac{S'_{t_a}}{\sqrt{a}}\right)^k \left(\frac{t_a - a}{\sqrt{a}}\right)^2 I_{\{t_a \geq \delta a\}}. \end{aligned}$$

On $\{t_a \geq \delta a\}$, u_a is bounded away for zero, so the uniform integrability of T''_2 in both cases follows directly from Holder's inequality as above for T_1. On $\{t_a < \delta a\}$, consider $k = 2$ first. By checking the Taylor expansion of $(t_a/a) - 1$, one will see that actually $1/u_a^3 = a/t$. So

$$\begin{aligned} &\int_\Omega \frac{1}{u_a^3} \left(\frac{S'_{t_a}}{\sqrt{a}}\right)^2 \left(\frac{t_a - a}{\sqrt{a}}\right)^2 I_{\{t_a < \delta a\}} dP \\ \leq\ &\int_\Omega a^2 \left(\frac{S'_{t_a}}{t_a}\right)^2 I_{\{t_a < \delta a\}} dP \\ \leq\ &\sqrt{a^4 P\{t_a < \delta a\} E \left(\frac{S'_{t_a}}{t_a}\right)^4} \to 0 \end{aligned}$$

by (C2) and the property

$$E\left\{\sup_{n \geq m} \left|\frac{Y_1 + \cdots + Y_n}{n}\right|^q\right\} \leq \left(\frac{q}{q-1}\right)^q E\left|\bar{Y}_m\right|^q$$

if $E\left|Y_1\right|^q$ exists (This is the Doob's maximal inequality applying to the reverse martingale $\{\bar{Y}_n, n \geq 1\}$). The uniform integrability of T'_2 for $k = 2$ then follows. For $k = 1, 3, 4$, checking the Taylor expansion we have in general $C(a/t_a)^k > u_a^{-(k+2)/2}$ where C is constant depending only on k and δ. So

$$\begin{aligned} \int_\Omega \frac{k(k+2)}{8u_a^{(k+4)/2}} \left(\frac{S'_{t_a}}{\sqrt{a}}\right)^k \left(\frac{t_a - a}{\sqrt{a}}\right)^2 I_{\{t_a < \delta a\}} dP &\leq \\ \frac{Ck(k+2)}{8} \int_\Omega a^{k/2+1} \left(\frac{S'_{t_a}}{t}\right)^k I_{\{t_a < \delta a\}} dP &\leq \end{aligned}$$

$$\sqrt{a^{k+2}P\{t_a < \delta a\}E\left(\frac{S'_{t_a}}{t_a}\right)^{2k}} \to 0$$

by (C2), Doob's maximal inequality and the moment assumptions of the lemma. The proof is thus complete. □

In addition to (C1)-(C4) (or (C4')), the following additional condition (C5) is required.

(C5) $\zeta_n, n \geq 1$ are slowly changing and $(\mathbf{S}_n/\sqrt{n}, \zeta_n)$ has weak limit $(\mathbf{V}, \zeta)$. Here $\mathbf{S}_n = (S_n, S'_n)$.

It is obvious that $\mathbf{V} = (V_1, V_2)$ has a bivariate normal distribution with mean $(0, 0)$ and covariance matrix

$$(v_{ij})_{2\times 2} = \begin{pmatrix} E(X_1^2)\ , & E(X_1Y_1) \\ E(X_1Y_1)\ , & E(Y_1^2) \end{pmatrix}.$$

Suppose the distribution of X_1 is nonarithmetic — we keep this assumption throughout the rest of this paper. Then a consequence of (C4) and (C5) is that $(\mathbf{S}_{t_a}/\sqrt{t_a}, \zeta_{t_a}, R_{t_a})$ converges weakly to $(\mathbf{V}, \zeta, R)$, where R is the limit distribution of overshoot and independent of $(\mathbf{V}, \zeta)$. See Proposition 3 of Aras and Woodroofe (1993).

Lemma 3.2. *Let $\lambda = E(R)$ and $\nu = E(\zeta)$. Then, under (C1)-(C5),*

$$\lim_{a\to\infty} E(t_a - a) = \lambda - \nu.$$

Proof. From the equality $t_a - a = R_{t_a} - \zeta_{t_a} - S_{t_a}$, the Wald's lemma gives

$$E(t_a - a) = E(R_{t_a} - \zeta_{t_a}),$$

which converges to $\lambda - \nu$ since $R_{t_a} - \zeta_{t_a}$ is uniformly integrable and (R_{t_a}, ζ_{t_a}) has weak limit (R, ζ). □

We now consider the first two moments.

Theorem 3.1. *Suppose $p \geq 4$ and $E(Y_1^4) < \infty$. Then, under (C1)-(C5),*

$$\lim_{a\to\infty} \frac{1}{a} E(t_a S'_{t_a}) = -E(X_1Y_1), \qquad \lim_{a\to\infty} \frac{1}{a} E\left(t_a S_{t_a}\right) = -E(X_1^2),$$

$$E(\sqrt{t_a}\bar{Y}_{t_a}) = \frac{E(X_1Y_1)}{2\sqrt{a}} + \frac{E(\zeta V_2)}{2a} + o\left(\frac{1}{a}\right),$$

and

$$E[t_a(\bar{Y_{t_a}})^2] = E(Y_1^2) + \frac{1}{a}[E(\zeta V_2^2) + E(Y_1^2X_1) - \nu E(Y_1^2)] + o\left(\frac{1}{a}\right).$$

Proof. By Wald's lemma, $E(t_a S'_{t_a}) = E[(t_a - a)S'_{t_a}]$. Note that

$$\frac{(t_a - a)S'_{t_a}}{a} = \frac{[(R_{t_a} - \zeta_{t_a}) - S_{t_a}]S'_{t_a}}{a}$$

which converges weakly to $-V_1V_2$ as $a \to \infty$. Note also

$$\frac{(t_a - a)S'_{t_a}}{a} = \left(\frac{t_a - a}{\sqrt{a}}\right) \frac{S'_{t_a}}{\sqrt{a}}$$

which is uniformly integrable by Corollary 2.1, Lemma 3.2 of Chow and Yu (1981) and Holder's inequality. So

$$\frac{1}{a}E(t_a S'_{t_a}) \to -E(V_1V_2) = -v_{12} = -E(X_1Y_1).$$

The second limit follows a similar argument as

$$\frac{1}{a}E\left(t_a S_{t_a}\right) = E\left[\frac{(t_a - a)S_{t_a}}{a}\right] \to -E(V_1^2) = -v_{11} = -E(X_1^2).$$

As for $E(\sqrt{t_a}\bar{Y}_{t_a})$, from (2) and Wald's lemma, we have

$$\begin{aligned} E(\sqrt{t_a}\bar{Y}_{t_a}) &= \frac{E(X_1Y_1)E(t_a)}{2\sqrt{a^3}} + \frac{1}{2a}E\left(\frac{S'_{t_a}}{\sqrt{a}}(\zeta_{t_a} - R_{t_a})\right) \\ &+ \frac{3}{8a}E\left[\frac{S'_{t_a}}{u_a^{2.5}\sqrt{a}}\left(\frac{t_a - a}{\sqrt{a}}\right)^2\right]. \end{aligned}$$

Observe that $a^{-1/2}S'_{t_a}(\zeta_{t_a} - R_{t_a})$ and $u_a^{-2.5}a^{-1/2}S'_{t_a}[a^{-1/2}(t_a - a)]^2$ converge weakly to $V_2(\zeta - R)$ and $V_2V_1^2$, and both are uniformly integrable from Lemma 3.1. Hence

$$E\left(\frac{S'_{t_a}}{\sqrt{a}}(\zeta_{t_a} - R_{t_a})\right) \to E[V_2(\zeta - R)],$$

$$E\left[\frac{S'_{t_a}}{u_a^{2.5}\sqrt{a}}\left(\frac{t_a - a}{\sqrt{a}}\right)^2\right] \to E(V_2V_1^2).$$

A simple calculation by noting the result of Lemma 3.2 and the independence of R and $\mathbf{V}$ then gives the desired result.

Finally consider $E[t_a(\bar{Y_{t_a}})^2]$. By (2),

$$t_a\bar{Y_{t_a}}^2 = \frac{1}{a}\left(S'_{t_a}\right)^2 + \frac{1}{a^2}S_{t_a}\left(S'_{t_a}\right)^2 + \frac{1}{a}(T_1 + T_2)$$

where

$$T_1 = \frac{1}{a}(\zeta_{t_a} - R_{t_a})\left(S'_{t_a}\right)^2, \quad T_2 = \frac{1}{au_a^3}\left(\frac{t_a - a}{\sqrt{a}}\right)^2\left(S'_{t_a}\right)^2.$$

From Wald's lemmas,

$$E(S'_{t_a})^2 = E(Y_1^2)E(t_a)$$

and

$$E[S_{t_a}(S'_{t_a})^2] = E(Y_1^2)E(t_aS_{t_a}) + 2E(X_1Y_1)E(t_aS'_{t_a}) + E(X_1Y_1^2)E(t_a).$$

Therefore, by Lemma 3.2 and the first part of this theorem, we have

$$E(S'_{t_a})^2 = E(Y_1^2)(a + \lambda - \nu) + o(1)$$

and

$$\frac{1}{a}E\left[S_{t_a}\left(S'_{t_a}\right)^2\right] \to E(X_1Y_1^2) - E(X_1^2)E(Y_1^2) - 2[E(X_1Y_1)]^2.$$

It is clear that

$$T_1 \to V_2^2(\zeta - R), \quad T_2 \to V_1^2V_2^2.$$

So, by Lemma 3.1,

$$ET_1 \to EV_2^2(\zeta - R), \quad ET_2 \to EV_1^2V_2^2.$$

Combining all these results gives the expansion of $E[t_a(\bar{Y_{t_a}})^2]$. □

The expansions for high order moments can be obtained in the same way as above by using Lemma 3.1, Lemma 3.2 and Wald's lemmas, except that the moment requirements are higher in order to ensure the validity of Lemma 3.1 and the validity of high order Wald's lemmas. We give only the assumptions and expansions for the third and forth moment but not the proofs as they are similar and involve much more lengthy and tedious manipulations.

Theorem 3.2. *Suppose $p \geq 6$, $E(X_1^8) < \infty$ and $E(Y_1^8) < \infty$. Then, under (C1)-(C5),*

$$\begin{aligned} E\left[\sqrt{t_a}\bar{Y}_{t_a}\right]^3 &= \frac{1}{\sqrt{a}}\left(E(Y_1^3) + \frac{3}{2}v_{12}v_{22}\right) \\ &\quad + \frac{1}{a}\left(\frac{3}{2}E(\zeta V_2^3) - 3v_{22}E(\zeta V_2)\right) + o\left(\frac{1}{a}\right), \end{aligned}$$

$$\begin{aligned} E\left[\sqrt{t_a}\bar{Y}_{t_a}\right]^4 &= 3 + \frac{1}{a}[E(Y_1^4) + 4v_{12}E(Y_1^3) + 6E(X_1Y_1^2) - 3 \\ &\quad + 2E(\zeta V_2^4) - 6E(\zeta V_2^2)] + o\left(\frac{1}{a}\right). \end{aligned}$$

4 Appendix

In this appendix, we present some Wald's lemmas which may not be readily available in the literature but are required in Section 3.

Let $S_n = \sum_{i=1}^n X_i$ and $S_n' = \sum_{i=1}^n Y_i$ where $(X_n, Y_n), n \geq 1$ are i.i.d. random vectors with zero mean. Let η be a stopping time with respect to process $\{(X_nY_n), \mathcal{F}_n, n \geq 1\}$ where $\{\mathcal{F}_n, n \geq 1\}$ is an increasing σ-class such that $(X_nY_n) \in \mathcal{F}_n$ and $(X_{n+1}Y_{n+1})$ is independent of $\mathcal{F}_n$ for all n. We consider the Wald's lemmas for $E[{S_\eta}^k S'_\eta], k = 1, 2, 3, 4$.

Denote $m(s_1, s_2) = E\exp(is_1X_1 + is_2Y_1)$, the characteristic function of (X_1, Y_1). One can easily verify that $\{m(s_1, s_2)^{-n}\exp(is_1S_n + is_2S'_n), \mathcal{F}_n, n \geq 1\}$ is a martingale. Suppose η satisfy the conditions in Doob's optional sampling theorem. Then

$$\begin{aligned} E[m(s_1, s_2)^{-\eta}\exp(is_1S_\eta + is_2S'_\eta)] &= \\ E[m(s_1, s_2)^{-1}\exp(is_1X_1 + is_2Y_1)] &= 1. \end{aligned} \tag{3}$$

Differentiating both sides of (3) with respect to s_1 k (times) and s_2 (once) and then setting $s_1 = s_2 = 0$ will yield the following required Wald's lemma heuristically:

$$\begin{aligned} E(S_\eta S'_\eta) &= E(X_1Y_1)E(\eta), \\ E(S_\eta^2 S'_\eta) &= E(X_1^2)E(\eta S'_\eta) + 2E(X_1Y_1)E(\eta S_\eta) + E(X_1^2Y_1)E(\eta), \\ E(S_\eta^3 S'_\eta) &= E(X_1^3Y_1)E(\eta) - 3E(X_1^2)E(X_1Y_1)E[\eta(\eta+1)] \\ &\quad +3E(X_1^2Y_1)E(\eta S_\eta) + 3E(X_1Y_1)E(\eta S_\eta^2) \\ &\quad +E(X_1^3)E(\eta S'_\eta) + 3E(X_1^2)E(\eta S_\eta S'_\eta), \\ E(S_\eta^4 S'_\eta) &= E(X^4Y)E(\eta) - [4E(X^3)E(XY) \\ &\quad +6E(X^2)E(X^2Y)]E[\eta(\eta+1)] \\ &\quad -12E(XY)E(X^2)E[\eta(\eta+1)S_\eta] \\ &\quad -3[E(X^2)]^2E[\eta(\eta+1)S'_\eta] \\ &\quad +4E(X^3Y)E(\eta S_\eta) + 6E(X^2Y)E(\eta S_\eta^2) \\ &\quad +4E(XY)E(\eta S_\eta^3) + 4E(X^3)E(\eta S_\eta S'_\eta) \\ &\quad +6E(X^2)E(\eta S_\eta^2 S'_\eta). \end{aligned}$$

We give the conditions and rigorous proofs for the first two equations. For the last two equations we provide only the conditions since the proofs are similar.

Second order Wald's lemma. If $E\left|X_1^2\right|$, $E\left|Y_1^2\right|$ and $E(\eta)$ are all finite, then

$$E(S_\eta S'_\eta) = E(X_1Y_1)E(\eta). \tag{4}$$

Proof. Let $Z_n = S_nS'_n - nE(X_1Y_1)$. Then

$$\begin{aligned} E(Z_n \quad &| \quad \mathcal{F}_{n-1}) \\ &= E[(S_{n-1} + X_n)(S'_{n-1} + Y_n) \mid \mathcal{F}_{n-1}] - nE(X_1Y_1) \\ &= S_{n-1}S'_{n-1} + E(X_nY_n) - nE(X_1Y_1) \\ &= S_{n-1}S'_{n-1} - (n-1)E(X_1Y_1) = Z_{n-1}. \end{aligned}$$

So $\{Z_n, \mathcal{F}_n, n \geq 1\}$ is a martingale with $E(Z_n) = 0$. Set $\eta_n = \eta \wedge n$. Then by the optional sampling theorem, (4) is true for η_n. But under the conditions $EX_1^2 < \infty$, $EY_1^2 < \infty$ and $E\,|\eta| < \infty$, both $\{S_{\eta_n}, n \geq 1\}$ and $\{S'_{\eta_n}, n \geq 1\}$ are Cauchy series in $L^2\{\Omega, \mathcal{F}, P\}$—this can be seen by noting that (taking S_{η_n} for example)

$$S_{\eta_{n+1}} - S_{\eta_m} = \sum_{k=m}^{n} X_{k+1}I_{\{\eta \geq k+1\}}$$

and hence (observe that $I_{\{\eta \geq k+1\}} \in \mathcal{F}_k$ is independent of X_{k+1})

$$\begin{aligned} E(S_{\eta_{n+1}} - S_{\eta_m})^2 &= \sum_{k=m}^{n} E(X_{k+1}^2)P\{\eta \geq k+1\} \\ &= E(X_1^2)[E(\eta_{n+1}) - E(\eta_m)] \to 0 \quad \text{as } n, m \to \infty. \end{aligned}$$

So $E(S_{\eta_n}S'_{\eta_n}) \to E(S_\eta S'_\eta)$ as $n \to \infty$. Thus (4) is true for η. □

Third order Wald's lemma If $E\,|X_1|^4$, $E\,|Y_1|^4 < \infty$ and $\mathcal{F}_n$-stopping time η satisfies $E(\eta^2) < \infty$, then

$$E(S_\eta^2 S'_\eta) = E(X_1^2)E(\eta S'_\eta) + 2E(X_1Y_1)E(\eta S_\eta) + E(X_1^2Y_1)E(\eta). \tag{5}$$

Proof. Let $Z_n = S_n^2S'_n - E(X_1^2)nS'_n - 2E(X_1Y_1)nS_n - nE(Y_1^2X_1)$. As in the last theorem we can prove $\{Z_n, \mathcal{F}_n, n \geq 1\}$ is a martingale and $E(Z_n) = 0$. So (5) is true for $\eta_n = \eta \wedge n$ by the optional sampling theorem. Observe that

$$\lim_{n\to\infty} E(X_1^2)E(\eta_n S'_{\eta_n}) + 2E(X_1Y_1)E(\eta_n S_{\eta_n}) + E(X_1^2Y_1)E(\eta_n)$$
$$= E(X_1^2)E(\eta S'_\eta) + 2E(X_1Y_1)E(\eta S_\eta) + E(X_1^2Y_1)E(\eta),$$

since both $\{S_{\eta_n}, n \geq 1\}$ and $\{S'_{\eta_n}, n \geq 1\}$ are Cauchy series in $L^2\{\Omega, \mathcal{F}, P\}$ and $E(\eta^2) < \infty$. Therefore $\lim_{n\to\infty} E(S^2_{\eta_n} S'_{\eta_n})$ exists and

$$\lim_{n\to\infty} E(S^2_{\eta_n} S'_{\eta_n}) = E(X_1^2)E(\eta S'_\eta) + 2E(X_1Y_1)E(\eta S_\eta) + E(X_1^2Y_1)E(\eta).$$

Next we prove $E(S^2_{\eta_n} S'_{\eta_n} - S^2_\eta S'_\eta) \to 0$ as $n \to \infty$. Note that

$$\left|E(S^2_{\eta_n} S'_{\eta_n} - S^2_\eta S'_\eta)\right| = \left|ES^2_{\eta_n}(S'_{\eta_n} - S'_\eta) + E(S_{\eta_n} S'_\eta + S_\eta S'_\eta)(S_{\eta_n} - S_\eta)\right|$$

$$\leq \left(ES^4_{\eta_n} E(S'_{\eta_n} - S'_\eta)^2\right)^{1/2} + \left(E(S_{\eta_n} S'_\eta + S_\eta S'_\eta)^2 E(S_{\eta_n} - S_\eta)^2\right)^{1/2}.$$

So it is enough to show that $ES^4_{\eta_n}, ES^4_\eta$ and ES'^4_η are uniformly bounded since $E(S'_{\eta_n} - S'_\eta)^2$ and $E(S_{\eta_n} - S_\eta)^2$ converge to zero. The boundedness of these three moments is an immediate consequence of the assumptions $E\left|X_1\right|^4$, $E\left|Y_1\right|^4, E(\eta^2) < \infty$ and Gut, (1988)'s inequality (see remark in Section 1). $\square$

Forth and fifth order Wald's lemma The equations for $E(S^3_\eta S'_\eta)$ and $E(S^4_\eta S'_\eta)$ are valid under the conditions $E\left|X_1\right|^8$, $E\left|Y_1\right|^8$ and $E\left|\eta\right|^4 < \infty$.

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