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The Optimal Design of the VSI T^2 Control Chart

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Abstract. Recent studies have shown that the variable sampling interval (VSI) scheme helps practitioners detect process shifts more quickly than the classical scheme (FRS). In this paper, the economically and statistically optimal design of the VSI T^2 control chart for monitoring the process mean vector is investigated. The cost model proposed by Lorenzen and Vance (1986) is minimized through a genetic algorithm (GA) approach. Then the effects of the costs and operating parameters on the optimal design (OD) of the chart parameters and resulting operating loss through a fractional factorial design is systematically studied and finally, based on the ANOVA results, a Meta model to facilitate implementation in industry is proposed to determine the OD of the VSI T^2 control chart parameters from the process and cost parameters.

Key words and phrases: Economic statistical design, Hotelling's T^2 control chart, Markov chain and genetic algorithm, variable sampling interval scheme.

1 Introduction

A common multivariate control chart is the Hotelling s T^2 control chart (Hotelling 1947). The traditional sampling strategy in the Hotelling s T^2 control chart is the fixed ratio sampling (FRS) scheme in which samples of fixed size are obtained at constant intervals to monitor a process. A major deficiency of the FRS T^2 control scheme is that its efficiency to detect small and moderate shifts or drifts in the process mean is poor. Consequently several modifications have been suggested in the quality control literature to improve the performance of the FRS policy.

One procedure to improve the statistical performance of the FRS control schemes is a Variable Sampling Interval (VSI) scheme that varies the sampling interval between successive samples as a function of prior sample results. In this procedure, the area between the control limits and the origin has been divided into two zones by a warning line for the use of two different sampling intervals $(h_1 > h_2)$. If the current sample value falls in a particular zone, then the next sample is to be drawn from the process after according to corresponding sampling interval. The use of the VSI control schemes requires the user to select five design parameters: the long and short sampling intervals h_1 and h_2, the fixed sample size n, the warning limit w and the control limit k. Traditionally, the design of VSI schemes involves the selection of convenient sample size and the control limit is then determined upon a maximum probability of a Type I error (false alarm) and/or a Type II error (failure to sound an alarm). The parameters w, h_1 and h_2 are determined such that the statistical performance or the speed with which process mean shifts are detected is minimized. Faraz et al (2010) provides a literature review on the statistical design of the VSI schemes.

The economic statistical design (ESD) of control charts is of great importance. Based on the ESD procedure, the chart is designed in such a way that the overall costs associated with maintaining current control of a process is minimized while keeping good statistical properties. This procedure is first developed by Saniga (1989) and was so well received that Montgomery (1996) strongly endorsed it for practice; It is called the optimal design (OD) in the literature. A more detailed literature review and discussion on the economic design of control charts can be found in Montgomery (1980) containing fifty one references on the topic. A study evaluating the optimal design of the VSI T^2 control chart has not been found in the literature; this is the contribution of this paper. This paper is organized as follows: In section 2 the VSI T^2 control scheme and Markov chain approach are briefly reviewed. In section 3, the cost model proposed by Lorenzen and Vance (1986) based on the Markov chain approach is modified as the objective function. Section 4 is devoted to GA procedure for solving the cost model and the solution procedure is illustrated in section 5. The meta model to construct the VSI T^2 scheme is proposed in section 6 to determine the optimal values of the control chart parameters directly from the process and cost parameters which also shall facilitate implementation in industry. Besides, the proposed model can be acting as a guide line for practitioners to specify the important process and cost parameters and finally, concluding remarks make up the last section.

2 VSI T^2 control scheme and Markov chain approach

In order to control a process with p correlated characteristics using the T^2 scheme, it is first assumed that the joint probability distribution of the quality characteristics is a p-variate normal distribution with in-control mean vector $\mu_0' = (\mu_{01}, \cdots, \mu_{0p})$ and variance-covariance matrix $\sum$. Then the subgroups (each of size n) statistics $T_i^2 = n(\bar{x}_i - \mu_0)' \sum^{-1}(\bar{x}_i - \mu_0)$ are plotted in sequential order to form the T^2 control chart. The chart signals as soon as $T_i^2 \geq k$. In statistical design methodology, if the process parameters (μ_0 and $\sum$) are known, k is given by the upper a percentage point of chi-square variable with p degrees of freedom. However μ_0 and $\sum$ are generally unknown and have to be estimated through m initial samples when the process is in control. In this case, the parameter k is obtained upon the $1 - a$ percentage point F distribution with p and v degrees of freedom as follows:

$$k = c(m, n, p) F_a(p, v) \tag{1}$$

where

$$c(m, n, p) = \begin{cases} \frac{p(m+1)(n-1)}{m(n-1)-p+1} & n > 1 \\ \frac{p(m+1)(m-1)}{m(m-p)} & n = 1 \end{cases} \tag{2}$$

$$v = \begin{cases} m(n-1) - p + 1 & n > 1 \\ (m - p) & n = 1 \end{cases} \tag{3}$$

In this paper, it is assumed that the process starts in a state of statistical control with mean vector μ_0 and covariance matrix $\sum$ and then an assignable cause occurs resulting in a shift in the process mean (μ_1). The magnitude of the shift is measured by $d = (\mu_1 - \mu_0)' \sum^{-1}(\mu_1 - \mu_0)$. Further it is assumed that the time before the assignable cause occurs has an exponential distribution with parameter λ. Thus, the mean time that the process remains in state of statistical control is λ^{-1} (Faraz and Parsian, 2006).

Now, upon the VSI scheme, at each sampling stage, one of the following transient states is met according to the status of the process (in or out of control) and the size of the sample (small or large).
State 1: $0 \le T^2 < w$ and the process is in control;
State 2: $w \le T^2 < k$ and the process is in control;
State 3: $T^2 \le k$ and the process is in control (false alarm);
State 4: $0 \le T^2 < w$ and the process is out of control;
State 5: $w \le T^2 < k$ and the process is out of control;

The control chart produces a signal when $T^2 \ge k$. If the current state is 3, the signal is a false alarm; the absorbing state (state 6) is reached when the true alarm occurs. The transition probability matrix is given by

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{11} \times \frac{q_2}{q_1} & p_{12} \times \frac{q_2}{q_1} & p_{13} \times \frac{q_2}{q_1} & p_{14} \times \frac{1-q_2}{1-q_1} & p_{15} \times \frac{1-q_2}{1-q_1} & p_{16} \times \frac{1-q_2}{1-q_1} \\ p_{11} \times \frac{q_2}{q_1} & p_{12} \times \frac{q_2}{q_1} & p_{13} \times \frac{q_2}{q_1} & p_{14} \times \frac{1-q_2}{1-q_1} & p_{15} \times \frac{1-q_2}{1-q_1} & p_{16} \times \frac{1-q_2}{1-q_1} \\ 0 & 0 & 0 & \frac{p_{14}}{1-q_1} & \frac{p_{15}}{1-q_1} & \frac{p_{16}}{1-q_1} \\ 0 & 0 & 0 & \frac{p_{14}}{1-q_1} & \frac{p_{15}}{1-q_1} & \frac{p_{16}}{1-q_1} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where p_{ij} denotes the transition probability that i is the prior state and j is the current state. In what follows, $F(x, p, v, \eta)$ will denote the cumulative probability distribution function of a non-central F distribution with p and v degrees of freedom and non-centrality parameter $\eta = nd^2$, where, $q_i = \exp(-\lambda h_i); 1, 2$ and $p'_{1j}s$ are

$$p_{11} = F(\frac{w}{c(m,n,p)}, p, v, \eta = 0) \times q_1$$
$$p_{12} = F(\frac{k}{c(m,n,p)}, p, v, \eta = 0) \times q_1 - p_{11}$$
$$p_{13} = q_1 - p_{12} - p_{11}$$

$$p_{14} = F(\frac{w}{c(m,n,p)}, p, v, \eta = nd^2) \times (1 - q_1)$$
$$p_{15} = F(\frac{k}{c(m,n,p)}, p, v, \eta = nd^2) \times (1 - q_1) - p_{14}$$
$$p_{16} = q_1 - p_{15} - p_{14}$$

The speed with which a control chart detects process mean shifts measures its statistical efficiency and is calculated as follows:

$$AATS = ATC - \frac{1}{\lambda} \tag{4}$$

where the AATS and ATC are the adjusted average time to signal and the average time from the start of the production until the first signal after the process shift, respectively. Figure 1 illustrates the ATC and AATS measures. According to the elementary Markov chain properties, the average time of the cycle (ATC) or the average time from the start of the production until the first signal after the process shift is calculated as follows:

$$ATC = \mathbf{b}'(\mathbf{I} - \mathbf{Q})^{-\mathbf{1}}\mathbf{h} \tag{5}$$

where $\mathbf{b}$ is a vector of initial probabilities, $\mathbf{I}$ is the identity matrix of order 5, $\mathbf{Q}$ is the 5×5 matrix obtained from $\mathbf{P}$ on deleting the elements corresponding to the absorbing state and $\mathbf{h}'=(h_1, h_2, h_2, h_1, h_2)$ is the vector of sampling time intervals. In this paper the vector $\mathbf{b}'$ is set to $(0, 1, 0, 0, 0)$, for providing an extra protection and preventing problems that are encountered during start-up.

3 The cost model

Faraz et al (2009) modified the Lorenzen and Vance (1986) economic model based on some common assumptions and used the Markov chain approach. In this paper, the same approach is applied to study the OD of the VSI T^2 control chart. Figure 1 illustrates a quality cycle observed by Duncan, which is divided into four time intervals of in-control period, out-of-control period, time to take a sample and interpret the results and time to find and repair an assignable cause. The average time of a quality cycle is calculated as follows:

$$\begin{aligned} E(T) &= \frac{1}{\lambda} + (1-\gamma_1)T_0ANF + AATS + nE + T_1 + T_2 \\ &= ATC + (1-\gamma_1)T_0ANF + nE + T_1 + T_2 \end{aligned} \tag{6}$$

where $\gamma_1 = 1$ if the process is not shut down during false alarms and 0 otherwise, T_0 stands for the expected time spent searching for a false alarm, E stands for the expected time to plot and chart the sample which triggers an out-of-control signal. The expected time to find the assignable cause and repair the process are given as T_1 and T_2 respectively. ANF is the expected number of false alarms in each quality cycle and is calculated as follows:

$$ANF = \mathbf{b}'(\mathbf{I} - \mathbf{Q})^{-\mathbf{1}}(0, 0, 1, 0, 0)' \tag{7}$$

The costs of a quality cycle is categorized into four main components: the cost of producing nonconformities while the process is in control (C_0), the cost of producing nonconformities while the process is out of control (C_1), the cost of evaluating alarms - both false alarms (a_3') and repairing the process (a_3), and the cost of sampling (a_1 and a_2 as the fixed and variable cost components of sampling and testing, respectively). Then the expected cost per quality cycle, $E(C)$, is defined as:

$$E(C) = \frac{C_0}{\lambda} + C_1[AATS + nE + \gamma_1 T_1 + \gamma_2 T_2] + a_3' ANF + a_3 + (a_1 + a_2 n) ANS \tag{8}$$

where γ_2 is an indicator function for if production continues during the repair of the process, the stand for the expected number of inspected samples taken from the start of the process until the chart signals and is calculated as follows:

$$ANS = \mathbf{b}'(\mathbf{I} - \mathbf{Q})^{-\mathbf{1}}(1, 1, 1, 1, 1)' \tag{9}$$

It is noted that when the process goes out of control, the sampling procedure stops even if the process continues. Now, based on the renewal reward process assumption (see Ross, 1995), the expected cost per hour is just defined as follows:

$$E(A) = \frac{E(C)}{E(T)} \tag{10}$$

4 The optimization problem and genetic algorithm approach

In the ED of control charts, it is assumed that the nine process parameters $(p, \lambda, d, T_0, T_1, T_2, \gamma_1, \gamma_2, E)$ and the six cost parameters $(C_0, C_1, a_1, a_2, a_3, a_3')$ are previously estimated. Then, the procedure continues to find the five chart parameters (k, w, n, h_1, h_2) which minimize (10). Among these five chart parameters, the sample size n is always a discrete variable and the other four variables are continuous where $0 \leq w < k$. To keep the chart practical, the minimum and maximum value of sampling intervals are considered as the possible minimum time between successive samples and maximum hours available in a work shift, respectively. $i \cdot e \cdot 0 \cdot 1 \leq h_2 \leq h_1 \leq 8$. The sampling intervals less than $0 \cdot 1$ hour may be problematic in the field. Therefore, the general optimization problem is defined as follows:

$$\begin{aligned} &\min E(A) \\ &s.t: \\ &\alpha \leq \alpha_0 \\ &k > 0 \\ &0 \leq w < k, \\ &0 \cdot 1 \leq h_2 \leq h_1 \leq 8 \\ &n \in Z^+ \end{aligned} \tag{11}$$

For offering the best protection against false alarms, the Type I error constraint $\alpha \leq \alpha_0$ is added to form the optimal design. The optimization problem (11) has both continuous and discrete decision variables and a discontinuous and non-convex solution space. In this paper, the problem is solved via genetic algorithm (GA) approach which is the most widely used tool in this area; for example see Faraz et al (2009). Using GA requires one to determine the values of the most significant GA parameters, i.e. the crossover rate (r_C), number of elites (N_{elit}), the initial population size (N_{pop}) and the mutation rate (r_M). Faraz et al (2009) found that the low crossover rate and large mutation fraction values result in great explorations and refrain from trapping many local minimums. Hence, they proposed the optimal values of $N_{pop} = 100, N_{elit} = 5, r_C = 0.05$ and $r_M = 0.9$. These values are used here to study the OD of the VSI T^2 control scheme. The procedure is as follows:

Step 1: generate a population of size N_{pop} chromosomes to form initial generation. Each chromosome is an arbitrary solution to optimization problem (12) and usually is represented by a numerical string.

Step 2: find the expected cost per hour corresponding to each chromosome

Step 3: scale chromosomes based on their expected cost per hour to obtain fitness values and assign each chromosome the selection prob-

ability corresponding to its fitness value. A lower expected cost per hour causes a higher fitness value and consequently the corresponding chromosome will have a higher chance for survive to next generation.
Step 4: select N_{elit} chromosomes with the best fitness values in the current generation to survive to the next generation.
Step 5: select (randomly but biased by the fitness values) two chromosomes from the mating pool of N_{pop} chromosomes. An individual can be selected more than once as a parent, in which case it contributes its genes to more than one child.
Step 6: recombine these two chromosomes (parents) using the crossover and mutation operators to produce two new chromosomes (children). Repeat steps 5 and 6 until $N_{pop} - N_{elit}$ children are born to form the new generation.
Step 7: repeat the steps from 2 to 6 until the termination conditions are met, i.e. when the number of generations is large enough or no more optimization in $E(A)$ value is observed.

This procedure is illustrated through an industrial application in the following section.

5 An illustrative example

In this section the proposed approach to the OD of the VSI T^2 control chart is illustrated through the industrial example taken from Faraz et al (2009) which considers the GM Company casting operation. The model estimated parameters is given in Table 1. The optimization problem (12) with Type I error constraint $\alpha \leq 0.005$ s considered and the optimal designs are given in Table 2 for different values of mean shift $d = 0.5(0.25)1.5(0.5)3$ with a cost comparison to the corresponding optimal FRS scheme. The results indicate that the VSI scheme is consistently cheaper than the FRS scheme while possessing a good statistical performance (AATS and α). The parameter h_2 is always set to a minimum value 0.1 and hence the practitioner should take samples when the chart measure falls in the corresponding zone. If the mean shift d increases, the values of sample size n and large sampling interval h_1 decrease with an increase in the values of parameter w. i.e., the sampling rate decreases as the value of parameter d increases. It is intuitive that less effort is needed for detecting larger amount of shifts in the process mean. In the example, consider the case where the objective is to provide a good protection over the shift $d = 1$. The optimal design of the VSI T2 control chart is set to $k = 13.09, w = 2.93, n = 9, h1 = 1.57$ (nearly 95 minutes) and $h_2 = 0.1$ (6 minutes) having a low Type I error $\alpha = 0.002$ and a good power $AATS = 1.21$ (73 minutes). Note that the optimally designed VSI scheme has a smaller Type I error rate than the statistical designs ($\alpha_0 = 0.005$). When $T^2 < 2.93$ the next sample of size 9 is taken after 95 minutes. Otherwise the next sample is taken after 6 minutes. This design imposes \$264.68 per hour to the company which results in 7% savings per hour when it is compared to the optimally designed FRS scheme. Considering the process works 20 days a month, establishing the OD of the VSI T2 chart results in more than $\$110,000$ annual savings with respect to the OD of the FRS scheme.

In the next section, a Meta model for designing the OD of the VSI T^2 control chart will be derived which can facilitate the application of the chart in industry. Besides, it may be helpful to thoroughly understand the effects of the cost and process parameters' changes on the optimal design of the VSI T^2 scheme.

6 A Sensitivity analysis for optimally designed the VSI T^2 scheme

In the OD of the VSI T2 control charts, it is assumed that economic information is readily available. However, in practice it is usually difficult to estimate all fifteen process and cost parameters and also the process of estimating needed parameters is often costly. Keats et al. (1997) mentioned that difficulties in estimating the economic model parameters are a substantial barrier for practitioners in ED of the control charts implementation, but performing sensitivity analysis can alleviate this problem. In this way, practitioners can spend most of their efforts estimating the critical parameters. For example, if it can be shown that the cost of repairing a process plays a small role in determining the optimal design, then fewer resources can be used to estimate repair cost.

Therefore, the fractional factorial design, resolution V, is used here to fully examine the effects of all fifteen parameters on the OD of the VSI T^2 control charts. Using a resolution V design ensures that no main effects and no two-factor interactions are aliased with each other, but the two factor interactions are confounded with higher level interactions. Hence, it is assumed that all three-way and higher interactions are zero. See Montgomery (2001) for a detailed discussion of factorial designs, fractional factorial designs, and design resolution. Table 3 provides the high and low level settings for the fifteen factors considered. High and low values for each cost and process parameters were determined based on previous studies investigated.

A computer program called Design-Expert is used to perform the analysis. By comparing the sum of squares among the fifteen factors for each response, the significant factors can be determined. Tables 4 — 7 show the ANOVA tables as well as the regression models to estimate the control chart parameters. The significant factors are marked in bold face. However, the insignificant factors (not counting those required to support hierarchy) are removed to improve the model accuracy as there are many insignificant model terms. Besides, the nonlinear effects are captured in the regression models by moving up and down the ladder of power transformations and appropriate transformation for each variable is selected using Box-Cox plots to analytically calculate the best power law transformation (See Montgomery, 2001 for details).

6.1 ANOVA for Control Limit k

The ANOVA Table 4 indicates that the three process parameters (p, λ, d) and the three cost parameters (C_1, a_1, a_2) have the largest impact on the optimal value for the upper control limit k. The most significant term is p, the number of variables. It is intuitive that as the number of variables increases, the control limit k increases. The impact of the variable cost of sampling a_2 is the second most significant term. A smaller variable sampling cost makes it economical to increase the sample size. A larger sample size in turn makes it easier to distinguish between in control and out of control states, and this decreases the upper control limit. Finally, as λ increases, the process remains less under control and therefore k decreases to quickly detect out-of-control states .

The statistical measures "Adj R-Squared" indicates that the defined regression equation in Table 4 is significant for predicting the $k^{0.4}$. The "Pred R-Squared" value of 0.89 is in a reasonable agreement with the model "Adj R-Squared" value of 0.89. Finally, the "Adeq Precision" measures the signal to noise ratio and a ratio greater than 4 is always desirable. The ratio of 48.27 indicates an adequate signal and therefore a reasonable and accurate prediction can be made by just considering main effects and two-way interactions

6.2 ANOVA for warning line w

The final ANOVA table for the significant model terms (and those required to support hierarchy) is given in Table 5. The most significant term for determining w is p, the number of variables. It is intuitive that as the number of variables increases, the warning limit w increases. The larger mean shift d, the easier it is to discover and hence fewer samples are needed and the warning limit w decreases. The variable cost of sampling and the cost of producing nonconformities when the process is out of control form the largest interaction. In fact when the value of C_1 is high then the warning limit w is decreased to increase the sampling frequency to detect the out of control state as soon as possible regarding the matter of variable cost of sampling.

The regression model in Table 5 indicates that the parameter $(w+0.17)^{0.37}$ can be estimated accurately with the "Pred R-Squared" value of 0.64 which is in a reasonable agreement with the model "Adj R-Squared" value of 0.67. Also, the "Adeq Precision" value of 22.60 indicates an adequate signal to noise ratio.

6.3 ANOVA for sample size n

The final ANOVA result for the significant model terms (and those required to support hierarchy) is given in Table 6. The process parameters (p, λ, d, E) and the cost parameters (C_0, C_1, a_1, a_2) have the significant impact on the optimal value for the sample size n with d, E, λ and a_2 having the greatest impact. The presented regression model with high values of "Pred R-Squared", "Adj R-Squared" and the "Adeq Precision" can accurately predict the power transformation $n^{-0.01}$. The positive sign of the coefficient d indicates that a smaller mean shift d requires one to use a larger sample size n which is consistent with the principle of statistical hypothesis testing. E has intuitive appeal for affecting n since it is the proportionality constant between the sample size and the time associated with plotting each point on the control chart. The parameter a_2 also has intuitive appeal since it is the variable cost associated with sampling. The ANOVA table also indicates that the largest interaction effect is between a_2 and C_1.

6.4 ANOVA for large sampling interval h_1

From the ANOVA Tables 7, the main effects a2 and C_1 have the greatest impact in predicting the sampling interval h_1. It seems intuitive that when the variable cost of sampling increases the sampling intervals increase to decrease sampling frequencies. Also, a high value of cost of producing defective products when the process is out of control causes a reduction in sampling intervals to detect out-of-control states as quickly as possible. The largest interaction effect is between a2 and a1. The value of parameter h_2 is always set to the minimum possible value, 0.1 in this paper.

The regression models presented in Tables 7 with high values of "Pred R-Squared", "Adj R-Squared" and the "Adeq Precision" can be used to significantly predict the power transformations $h_1^{-0.17}$.

7 Concluding remarks

In the present paper, the optimal design of the T^2 control chart with VSI scheme is developed based on the cost model proposed by Lorenzen and Vance (1986) and the expected total cost per hour is minimized using GA. An illustrative example is provided and a sensitivity analysis is then carried out to study the effect of model parameters on the solution of the optimal design. The ANOVA results indicate that the model parameters $\lambda, d, T_1, C_1, E, a_1$ and a_3 play a significant role in designing the chart parameters. In addition, the variable cost of sampling plays an important role in determining all the control char parameters. This paper also provides regression equations in Table 4-7 which can be considered as the basis of an efficient and effective Meta model for the OD of the VSI T^2 control chart from combinations of model parameters. The high values of "Pred R-Squared" and "Adj R-Squared" measures indicated that the regression equations provide a good approximation and also provide a much richer interpretation by considering nonlinear transformations. The provided regression equations - easy computational methods - make it easier to determine the optimal design of the VSI T2 chart and facilitate implementation in industry. This approach provides practitioners with a solution they can understand, and hence will be more willing to adopt.

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Figure and Tables

Figure 1. A quality Cycle

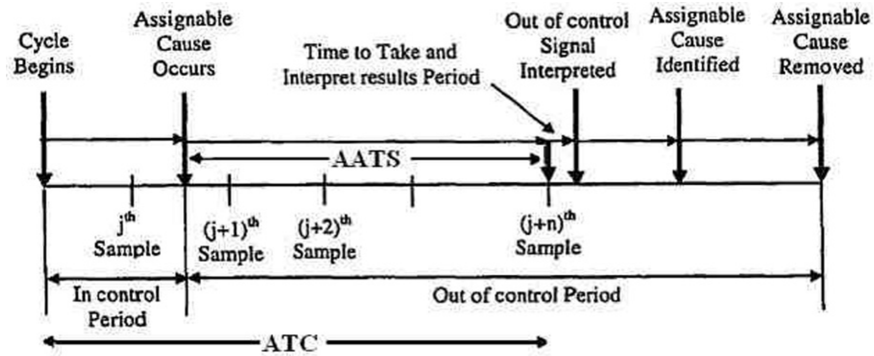

Table 1. Data adapted from General Motors

$p = 2$	$m = 25$	$\lambda = 0.05$	$\gamma_1 = 1$	$\gamma_2 = 0$
$T_0 = T_1 = 0.0833$	$E = 0.0833$	$T_2 = 0.75$	$C_0 = 114.24$	$C_1 = 949.2$
$a_1 = 5$	$a_2 = 4.22$	$a_3 = 977.4$	$a'_3 = 977.4$	$d = 1$

Table 2. The optimal parameters of ESD of the FRS and VSI schemes for different values of d

d	VSI scheme								Frs scheme	%
	k	w	n	h_1	h_2	a	AATS	$E(A)$	$E(A)$	
0.50	10.86	2.00	22	2.40	0.10	0.005	2.49	368.94*	408.64	11%
0.75	10.86	2.64	13	1.89	0.10	0.005	1.54	297.31*	325.41	9%
1.00	13.09	2.93	9	1.57	0.10	0.002	1.21	264.68*	283.67	7%
1.25	11.50	3.28	6	1.37	0.10	0.5	0.97	245.00*	259.57	6%
1.5	12.59	3.83	5	1.26	0.10	0.004	0.83	232.94*	244.13	5%
2.00	13.95	3.84	3	1.10	0.10	0.003	0.72	218.09*	225.96	4%
2.50	16.32	5.87	3	1.10	0.10	0.001	0.62	210.45*	215.78	3%
3.0	17.95	5.26	2	1.10	0.10	0.001	0.64	205.24*	209.14	2%

Table 3. High and low levels for the model parameters

Factor	γ_1	γ_2	T_0	T_1	T_2	C_0	C_1	a_1
Low	0	0	0.1	0.1	1	50	250	0.5
High	1	1	5	5	15	200	1000	5

Factor	a_2	a_3	a'_3	E	λ	p	d
Low	0.1	25	50	0.1	0.01	2	0.5
High	10	1000	1000	1	0.05	10	2

Table 4. ANOVA and Regression model for control limit k

Response: $k^{0.4} - 2.37$

Source		Sum of Squares	DF	Mean Square	F Value	$Prob > F$
Model		87.83	12.00	7.32	175.11	< 0.0001
Coefficients	Parameters					
8.07E-02	γ_2	0.01	1.00	0.01	0.32	0.5742
-4.24E-02	a_1	0.41	1.00	0.41	9.78	0.0020
-6.88E-02	a_2	10.00	1.00	10.00	239.35	< 0.0001
-3.46E+00	λ	2.64	1.00	2.64	63.04	< 0.0001
-8.80E-03	d	1.23	1.00	1.23	29.31	< 0.0001
2.92E-03	T_2	0.12	1.00	0.12	2.75	0.0984
2.86E-01	C_1	2.96	1.00	2.96	70.70	< 0.0001
1.28E-01	p	67.61	1.00	67.61	1617.39	< 0.0001
-1.19E-02	$\gamma_2 \times T_2$	0.44	1.00	0.44	10.60	0.0013
4.87E-03	$a_1 \times a_2$	0.75	1.00	0.75	18.06	< 0.0001
-3.20E-01	$a_2 \times \lambda$	0.26	1.00	0.26	6.15	0.0138
2.00E-02	$a_2 \times d$	1.41	1.00	1.41	33.81	< 0.0001
Residual		10.16	243.00	0.04		
Total		97.99	255.00	7.32	175.11	< 0.0001

Model Adequacy Measures

R-Squared	0.90	Pred R-Squared	0.89
Adj R-Squareed	0.89	Adeq Precision	48.27

Table 5. ANOVA and Regression model for control limit w

Response: $(w+0.17)^{0.37}-1.23$		Sum of Squares	DF	Mean Square	F Value	$Prob > F$
Model		97.77	26.00	3.76	31.92	< 0.0001
Coefficients	Parameters					
-2.32 E-01	γ_1	0.46	1.00	0.46	3.91	0.0493
4.87E-01	γ_2	2.34	1.00	2.34	19.89	< 0.0001
1.53E-02	T_1	0.55	1.00	0.55	4.65	0.0321
-9.01E-02	a_2	7.38	1.00	7.38	62.67	< 0.0001
3.38E-04	C_0	1.06	1.00	1.06	9.03	0.0321
-2.16E-01	E	2.42	1.00	2.42	20.54	< 0.0001
-7.01E+00	λ	4.08	1.00	4.08	34.61	< 0.0001
3.65E-01	d	12.81	1.00	12.81	108.70	< 0.0001
3.97E-03	T_2	0.35	1.00	0.35	2.98	0.085
-5.14E-04	C_1	6.67	1.00	6.67	56.65	< 0.0001
1.08E-01	P	38.60	1.00	38.60	327.67	< 0.0001
-1.46E-04	a_3'	0.05	1.00	0.05	0.39	0.5315
2.35E-04	$\gamma_1 \times C_1$	0.50	1.00	0.50	4.23	0.0407
-1.26E-03	$\gamma_2 \times C_0$	0.57	1.00	0.57	4.83	0.0290
-4.72E+00	$\gamma_2 \times \lambda$	0.57	1.00	0.57	4.83	0.0289
-1.85E-01	$\gamma_2 \times d$	1.23	1.00	1.23	10.48	0.0014
-1.85E-02	$\gamma_2 \times T_2$	1.08	1.00	1.08	9.13	0.0028
-2.73E-04	$T_1 \times C_0$	0.65	1.00	0.65	5.48	0.0201
2.61E-02	$a_2 \times d$	2.40	1.00	2.40	20.35	< 0.0001
8.55E-05	$a_2 \times C_1$	6.44	1.00	6.44	54.67	< 0.0001
-5.04E-03	$a_2 \times p$	2.55	1.00	2.55	21.61	< 0.0001
1.97E-06	$C_0 \times C_1$	0.78	1.00	0.78	6.65	0.0105
-1.84E-04	$C_0 \times p$	0.78	1.00	0.78	6.60	0.010
5.82E-03	$\lambda \times a_3'$	0.78	1.00	0.78	6.64	0.0106
-1.70E-04	$d \times C_1$	0.58	1.00	0.58	4.96	0.0268
6.03E-05	$C_1 \times p$	2.09	1.00	2.09	17.67	< 0.0001
Residual		26.98	229.00	0.12		
Total		124.75	255.00			

Model Adequacy Measures			
R-Squared	0.78	Pred R-Squared	0.73
Adj R-Squared	0.76	Adeq Precision	24.37

Table 6. ANOVA and Regression model for sample size n

Response: $n^{-0.01} - 0.96$						
Source		Sum of Squares	DF	Mean Square	F Value	$Prob > F$
Model		2.27E-02	26.00	8.74E-04	11.57	< 0.0001
Coefficients	Parameters					
1.92E-04	T_1	2.68E-05	1.00	2.68E-05	3.42	0.0656
-3.97E-04	a_1	6.48E-05	1.00	6.48E-05	8.27	0.0044
9.74E-04	a_2	1.45E-03	1.00	1.45E-03	185.56	< 0.00001
-7.47E-07	a_3	1.67E-06	1.00	1.67E-06	0.21	0.6450
2.30E-05	C_0	8.868E-05	1.00	8.86E-05	11.31	0.0009
1.18E-02	E	2.42E-03	1.00	2.42E-03	309.32	< 0.0001
1.68E-01	λ	1.64E-03	1.00	1.64E-03	208.62	< 0.0001
1.49E-02	d	1.34E-02	1.00	1.34E-02	1713.62	< 0.0001
-9.76E-07	C_1	2.71E-04	1.00	2.71E-04	34.60	< 0.0001
-6.50E-04	p	5.72E-04	1.00	5.72E-04	72.96	< 0.0001
3.58E-07	$T_1 \times a_3$	4.68E-05	1.00	4.68E-05	5.97	0.0153
-8.10E-03	$T_1 \times \lambda$	4.03E-05	1.00	4.03E-05	5.14	0.0243
3.43E-05	$a_2 \times a_1$	3.74E-05	1.00	3.74E-05	4.77	0.0299
1.56E-06	$a_2 \times C_0$	8.56E-05	1.00	8.56E-05	10.92	0.0011
-3.76E-04	$a_2 \times E$	1.79E-04	1.00	1.79E-04	22.90	< 0.0001
5.76E-03	$a_2 \times \lambda$	8.33E-05	1.00	8.33E-05	10.62	0.0013
-2.83E-04	$a_2 \times d$	2.83E-04	1.00	2.83E-04	36.06	< 0.0001
-8.01E-07	$a_2 \times C_1$	5.66E-04	1.00	5.66E-04	72.20	< 0.0001
1.78E-05	$a_2 \times p$	3.16E-05	1.00	3.16E-05	4.04	< 0.0457
-9.74E-06	$C_0 \times d$	7.68E-05	1.00	7.68E-05	9.80	0.0020
-1.74E-08	$C_0 \times C_1$	6.15E-05	1.00	6.15E-05	7.85	0.0055
5.21E-02	$E \times \lambda$	5.62E-05	1.00	5.62E-05	7.17	0.0079
-3.71E-03	$E \times d$	4.02E-04	1.00	4.02E-04	51.29	< 0.0001
-9.28E-02	$\lambda \times d$	4.96E-04	1.00	4.96E-04	63.22	< 0.0001
6.22E-03	$\lambda \times p$	6.34E-05	1.00	6.34E-05	8.09	0.0048
3.56E-06	$d \times C_1$	2.57E-04	1.00	2.57E-04	32.82	< 0.0001
Residual		1.79E-03	229.00	7.84E-06		
Total		2.45E-02	255.00			
Model Adequacy Measures						
R-Squared		0.93		Pred R-Squared	0.91	
Adj R-Squared		0.92		Adeq Precision	43.09	

Table 7. ANOVA and Regression model for short sampling interval h_1

Response: $h_1^{-0.17} - 0.93$						
Source		Sum of Squares	DF	Mean Square	F Value	$Prob > F$
Model		8.53E+00	36.00	2.37E-01	145.51	< 0.0001
Coefficients	Parameters					
1.13E-02	γ_1	1.95E-02	1.00	1.95E-02	11.99	0.0006
1.19E-02	γ_2	7.47E-02	1.00	7.47E-02	45.90	< 0.0001
8.30E-03	T_1	7.50E-04	1.00	7.50E-04	0.46	0.4979
-1.47E-02	a_1	3.21E-01	1.00	3.21E-01	196.90	< 0.0001
-2.88E-02	a_2	3.70E+00	1.00	3.70E+00	2273.34	< 0.0001
-7.06E-04	C_0	1.47E-01	1.00	1.47E-01	90.18	< 0.0001
-4.90E-02	E	5.90E-03	1.00	5.90E-03	3.63	0.0582
2.61E+00	λ	4.09E-01	1.00	4.09E-04	251.41	< 0.0001
5.59E-02	d	1.13E+00	1.00	1.13E+00	693.56	< 0.0001
7.69E-04	T_2	2.63E-04	1.00	2.63E-04	0.16	0.6884
1.44E-04	C_1	1.69E+00	1.00	1.69E+00	1040.06	< 0.0001
-2.69E-03	p	1.14E-01	1.00	1.14E-01	70.20	< 0.0001
-6.51E-03	$\gamma_1 \times T_1$	1.63E-02	1.00	1.63E-02	10.00	0.0018
2.16E-03	$\gamma_1 \times a_2$	7.32E-03	1.00	7.32E-03	4.50	0.0351
-7.69E-01	$\gamma_1 \times \lambda$	1.51E-02	1.00	1.51E-02	9.30	0.0026
2.68E-03	$\gamma_2 \times a_2$	1.13E-02	1.00	1.13E-02	6.94	0.0090
-1.21E+00	$\gamma_2 \times \lambda$	3.76E-02	1.00	3.67E-02	23.08	< 0.0001
-2.90E-03	$\gamma_2 \times T_2$	2.65E-02	1.00	2.65E-02	16.25	< 0.0001
-1.91E-01	$T_1 \times \lambda$	2.25E-02	1.00	2.25E-02	13.84	0.0003
2.75E-03	$a_1 \times a_2$	2.40E-01	1.00	2.40E-01	147.18	< 0.0001
-7.84E-03	$a_1 \times d$	4.48E-02	1.00	4.48E-02	27.52	< 0.0001
-8.12E-06	$a_1 \times C_1$	1.20E-02	1.00	1.20E-02	7.38	0.0071
2.66E-05	$a_2 \times C_0$	2.49E-02	1.00	2.49E-02	15.30	0.0001
2.62E-03	$a_2 \times E$	8.73E-03	1.00	8.73E-03	5.36	0.0215
-7.38E-02	$a_2 \times \lambda$	1.37E-02	1.00	1.37E-02	8.39	0.0042
-1.28E-05	$a_2 \times C_1$	1.45E-01	1.00	1.45E-01	89.06	< 0.0001
-1.14E-04	$C_0 \times d$	1.05E-02	1.00	1.05E-02	6.48	0.0116
6.33E-07	$C_0 \times C_1$	8.11E-02	1.00	8.11E-02	49.82	< 0.0001
3.14E-03	$E \times T_2$	2.50E-02	1.00	2.50E-02	15.38	0.0001
5.49E-01	$\lambda \times d$	1.74E-02	1.00	1.74E-02	10.67	0.0013
1.52E-03	$\lambda \times C_1$	3.34E-02	1.00	3.34E-02	20.49	< 0.0001
-6.65E-02	$\lambda \times p$	7.24E-03	1.00	7.24E-02	4.45	0.0361
6.40E-05	$d \times C_1$	8.29E-02	1.00	8.29E-02	50.93	< 0.0001
2.00E-03	$d \times p$	9.21E-03	1.00	9.21E-03	5.65	0.0183
-1.90E-06	$T_2 \times C_1$	6.38E-03	1.00	6.38E-03	3.92	0.0491
-4.96E-06	$C_1 \times p$	1.41E-02	1.00	1.41E-02	8.69	0.0036
Residual		3.57E-01	219.00	1.21E-02		
Total		8.89E+00	255.00			

Model Adequacy Measures			
R-Squared	0.96	Pred R-Squared	0.95
Adj R-Squared	0.95	Adeq Precision	56.13

Table 8. ANOVA and Regression model for long sampling interval h

Response: $\ell n(h_2) + 1.60$						
Source		Sum of Squares	DF	Mean Square	F Value	$Prob > F$
Model		722.41	36.00	20.07	98.13	< 0.0001
Coefficients	Parameters					
-2.91E-01	γ_1	7.81	1.00	7.81	38.17	< 0.0001
-1.61E-01	γ_2	31.15	1.00	31.15	152.31	< 0.0001
3.90E-02	T_0	0.09	1.00	0.09	0.46	0.4978
-7.53E-02	T_1	2.59	1.00	2.59	12.65	0.0005
2.54E-01	a_1	27.90	1.00	27.90	136.43	< 0.0001
3.76E-01	a_2	317.96	1.00	317.96	1554.93	< 0.0001
1.27E-04	a_3	0.99	1.00	0.99	4.82	0.0292
9.00E-03	C_0	14.22	1.00	14.22	69.52	< 0.0001
-4.73E-01	E	2.29	1.00	2.29	11.19	0.0010
-1.29E+01	λ	0.12	1.00	0.12	0.57	0.4503
-1.89E-01	d	33.76	1.00	33.76	165.12	< 0.0001
-1.78E-03	T_2	3.68	1.00	3.68	18.01	< 0.0001
-1.96E-03	C_1	175.41	1.00	175.41	857.82	< 0.0001
3.61E-02	p	2.05	1.00	2.05	10.02	0.0018
1.37E-01	$\gamma_1 \times T_1$	7.22	1.00	7.22	35.31	< 0.0001
-2.53E-02	$\gamma_1 \times a_2$	1.00	1.00	1.00	4.89	0.0280
7.10E+00	$\gamma_1 \times \lambda$	1.29	1.00	1.29	6.31	0.0127
1.64E-01	$\gamma_1 \times d$	0.97	1.00	0.97	4.73	0.0308
-6.27E-02	$\gamma_2 \times a_1$	1.27	1.00	1.27	6.23	0.0133
8.45E+00	$\gamma_2 \times \lambda$	1.83	1.00	1.83	8.93	0.0031
2.53E-01	$\gamma_2 \times d$	2.30	1.00	2.30	11.23	0.0009
5.78E-02	$\gamma_2 \times T_2$	10.47	1.00	10.47	51.22	< 0.0001
-3.90E-03	$T_0 \times T_2$	1.14	1.00	1.14	5.59	0.0189
1.59E+00	$T_1 \times \lambda$	1.56	1.00	1.56	7.63	0.0062
-2.49E-02	$a_1 \times a_2$	19.74	1.00	19.74	96.51	< 0.0001
4.01E-02	$a_1 \times d$	1.17	1.00	1.17	5.73	0.0175
-1.82E-04	$a_2 \times C_0$	1.17	1.00	1.17	5.73	0.0176
5.21E-02	$a_2 \times E$	3.44	1.00	3.44	16.84	< 0.0001
-9.23E-02	$a_2 \times d$	30.06	1.00	30.06	16.84	< 0.0001
-1.75E-03	$a_2 \times T_2$	0.94	1.00	0.94	4.59	0.0333
5.46E-05	$a_2 \times C_1$	2.63	1.00	2.63	12.87	0.0004
3.36E-03	$a_2 \times p$	1.13	1.00	1.13	5.53	0.0195
-1.10E-04	$C_0 \times T_2$	0.85	1.00	0.85	4.17	0.0424
-6.49E-06	$C_0 \times C_1$	8.52	1.00	8.52	41.69	< 0.0001
-2.46E-02	$d \times p$	1.39	1.00	1.39	6.80	0.0097
3.60E-05	$T_2 \times C_1$	2.29	1.00	2.29	11.20	0.0010
Residual		44.78	219.00	2.20		
Total		767.19	255.00			

Model Adequacy Measures			
R-Squared	0.94	Pred R-Squared	0.92
Adj R-Squared	0.93	Adeq Precision	44.66