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Ageing Orders of Series-Parallel and Parallel-Series Systems with Independent Subsystems Consisting of Dependent Components

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Abstract. In this paper, we consider series-parallel and parallel-series systems with independent subsystems consisting of dependent homogeneous components whose joint lifetimes are modeled by an Archimedean copula. Then, by considering two such systems with different numbers of components within each subsystem, we establish hazard rate and reversed hazard rate orderings between the two system lifetimes, and also discuss how these systems age relative to each other in terms of hazard rate and reversed hazard rate functions.

Keywords. Relative Ageing Orders, Hazard Rate Order, Reversed Hazard Rate Order, Series-Parallel Systems, Parallel-Series Systems, Archimedean Copulas.

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1 Introduction

The study of system reliability and comparison of system performance have traditionally focused on parallel and series systems due to their practical use and with components in the system functioning independently due to the simplicity it presents. However, many larger systems would involve both series and parallel configurations in the overall design of the system to provide a certain level of redundancy and also to achieve reliability to a required level.

In the present work, we focus on series-parallel and parallel-series systems with many subsystems. To be specific, a series-parallel system would contain k (say) subsystems connected in series with each subsystem consisting of n (say) homogeneous components in parallel. Similarly, a parallel-series system would contain k subsystems connected in parallel with each subsystem consisting of n homogeneous components in series. We then consider a general scenario in which the n components within each subsystem are dependent with the joint distribution of their lifetimes being modeled by the flexible family of Archimedean copulas. In this setting, we consider two different systems with similar structures (i.e., either both are series-parallel or parallel-series systems) with possibly different numbers of dependent components within each subsystem, and then discuss hazard rate and reversed hazard rate orderings between the two systems and also as to how they age relative to each other in terms of hazard rate and reversed hazard rate functions.

In order to provide a motivation for the problem considered here, we first point out that series-parallel and parallel-series structures are often used to construct larger systems. For example, four-engine-flying systems with two engines operating on either side in parallel, and k segmented pump irrigation systems in which the pumps in each of k locations may be in series but with the irrigation in the k locations being in parallel, are two good examples of such larger systems. Another classic example is in network circuits which are often formed as series-parallel or parallel-series systems so that the series circuit configurations can manage voltage drops to add to equal voltage, all components to share the same equal current and the resistance to add to equal total resistance, while the parallel circuit configurations can facilitate all components to share the same equal voltage, the branch currents to add to equal total current and resistance to diminish to equal total resistance.

Next, with regard to the components within each subsystem, it is quite reasonable to assume then to be homogeneous (like engines, pumps, circuits, etc.), but it may not be reasonable to assume independence between their lifetimes. After all, the components are functioning simultaneously within each subsystem and the functioning of one is likely to have an impact on the functioning of others; moreover, as is often the case, these components may all come from the same producer sharing the same production environment, thus possibly inducing dependence between them. So, to model the dependence between components within each subsystem, though these are many ways to model dependence [see, for example, Kotz et al. (2000)], one convenient and popular way is through the use of copulas. In the present work, we therefore use the flexible family of Archimedean copulas to model the joint distribution of lifetimes of components within each subsystem. Interested readers may refer to Nelsen (2006) for a detailed account on various copulas, their properties and applications.

In a pioneering work, El-Neweihi et al. (1986) used the concepts of majorization and Schur-convex functions to address the problem of optimal allocation of components to parallel-series and series-parallel systems, for maximizing the reliability of the whole system. They described the optimal allocation depending only on the ordering of component reliabilities in parallel-series systems, and a partial ordering among allocations that could lead to the optimal allocation in series-parallel systems. Several authors have subsequently discussed this issue for parallel-series and series-parallel systems, including Coit and Smith (1996), Ramirez-Marquez et al. (2004), Sarhan et al. (2004), Billionnet (2008), Levitin and Amari (2009), Sun et al. (2017), Ling et al. (2018) and Fang et al. (2020).

The rest of this paper proceeds as follows. In Section 2, we present basic definitions of some reliability notions and copulas that will be used in subsequent sections. In Section 3, we consider series-parallel systems with different numbers of components within each subsystem and establish hazard and reversed hazard rate orders between the lifetimes of two systems. We also examine how these two systems age relative to each other in terms of hazard rate and reversed hazard rate functions. Next, in Section 4, we consider parallel-series systems and develop analogous results. We present several examples to show that the conditions considered on the Archimedean generator function ϕ are quite general and are satisfied by many forms of Archimedean copulas. Finally, we present some concluding remarks in Section 5.

2 Preliminaries

We briefly introduce some well-known concepts about stochastic orders, majorization and related orders, and copulas in this section. We assume all random variables under consideration are continuous and nonnegative, and use “increasing” to mean "nondecreasing" and similarly "decreasing" to mean "nonincreasing". All involved expectations are assumed to exist wherever they appear. Also, for convenience, we use $a \overset{sgn}{=} b$ to denote that both sides of an equality have the same sign.

2.1 Stochastic Orders

Let X and Y be two random variables with density functions f_X and f_Y, distribution functions F_X and F_Y, survival functions $\bar{F}_X = 1 - F_X$ and $\bar{F}_Y = 1 - F_Y$, hazard rate functions $h_X = f_X/\bar{F}_X$ and $h_Y = f_Y/\bar{F}_Y$, and reversed hazard rate functions $\tilde{h}_X = f_X/F_X$ and $\tilde{h}_Y = f_Y/F_Y$, respectively.

Definition 2.1. Then, X is said to be larger than Y in the

(i) usual stochastic order (denoted by $X \geq_{\rm st} Y$) if $\bar{F}_X(t) \geq \bar{F}_Y(t)$, for all $t \in \mathbb{R}$, or equivalently, $\mathbb{E}[\phi(X)] \geq \mathbb{E}[\phi(Y)]$ for all increasing functions $\phi : \mathbb{R} \to \mathbb{R}$;

(ii) hazard rate order (denoted by $X \geq_{\rm hr} Y$) if and only if $\bar{F}_X(t)/\bar{F}_Y(t)$ is increasing in $t \in \mathbb{R}$, or equivalently, $h_Y(t) \geq h_X(t)$ for all $t \in \mathbb{R}$;

(iii) reversed hazard rate order (denoted by $X \geq_{\rm rh} Y$) if and only if $F_X(t)/F_Y(t)$ is increasing in $t \in \mathbb{R}$, or equivalently, $\tilde{h}_X(t) \geq \tilde{h}_Y(t)$ for all $t \in \mathbb{R}$.

The following implication is well-known between the above orders:

$$X \leq_{\rm hr[rh]} Y \Longrightarrow X \leq_{\rm st} Y.$$

One may refer to Müller and Stoyan (2002) and Shaked and Shanthikumar (2007) for extensive discussions on various stochastic orderings, and their inter-relationships and properties.

2.2 Ageing Concepts

The notion of ageing, describing the variation of the performance of an unit over time, plays an important role in survival and reliability analyses. Many different measures and measure-based stochastic orders have been discussed in the literature to describe ageing characteristics of life distributions. The following definition introduces the relative ageing by increasing hazard and reversed hazard ratios.

Definition 2.2. The random variable X is said to be ageing faster than Y in

(i) hazard rate (denoted by $X \geq_{c} Y$) if $h_X(t)/h_Y(t)$ is decreasing in $t \in \mathbb{R}$ (Kalashnikov and Rachev, 1986);

(ii) reversed hazard rate (denoted by $X \geq_{b} Y$) if $\tilde{h}_X(t)/\tilde{h}_Y(t)$ is increasing in $t \in \mathbb{R}$ (Rezaei et al., 2015).

2.3 Dependence Through Archimedean Copulas

Stochastic comparisons of univariate random variables have been discussed rather extensively in the literature; see, Müller and Stoyan (2002) and Shaked and Shanthikumar (2007) for pertinent details. Most of the univariate stochastic orders are based on comparisons of marginal distributions of the underlying variables, without taking dependence between variables into account. Here, we discuss relative ageing orders of series-parallel and parallel-series systems with independent subsystems consisting of components which are dependent with Archimedean copulas representing the joint distribution of components within each subsystem.

Archimedean copulas have been used widely due to their mathematical tractability as well as their ability to capture a wide range of dependence. To be specific, for a decreasing and continuous function $\phi : [0, \infty) \longrightarrow [0, 1]$ with $\phi(0) = 1$ and $\phi(+\infty) = 0$ and $\psi = \phi^{-1}$ being the pseudo-inverse,

$$C_\phi(u_1, \cdots, u_n) = \phi(\psi(u_1) + \cdots + \psi(u_n)) \qquad for\ all\ u_i \in [0, 1],\ \ i = 1, \cdots, n, \tag{2.1}$$

is said to be an Archimedean copula with generator ϕ if $(-1)^k \phi^{[k]}(x) \geq 0$ for $k = 0, \cdots, n-2$ and $(-1)^{n-2}\phi^{[n-2]}(x)$ is decreasing and convex; here, $\phi^{[k]}(x)$ denotes the k-th derivative of the function $\phi(x)$ with respect to x. The Archimedean copula family is a rich family of dependence models that includes many well-known copulas such as independence (product) copula, Clayton copula, Frank copula, Gumbel-Hougaard copula, and Ali-Mikhail-Haq (AMH) copula.

3 Results for Series-Parallel Systems

We focus in this section on a series-parallel system consisting of k subsystems, each with n homogeneous components functioning in a dependent manner. We then use Archimedean copulas to model the dependence between these components. One question that arises naturally is that between two series-parallel systems, one with n components and another with m components in each of the k subsystems, which one is more reliable in terms of ageing orders. We address questions of this nature here and establish several results concerning ageing orders.

The survival function, density function, hazard rate function and reversed hazard rate function of a series-parallel system, with k independent subsystems and n homogeneous dependent components within each subsystem, are given by

$$\begin{aligned} \bar{F}_{S_{n,k}}(x) &= \Big\{1-\phi\left(n\psi\left[F(x)\right]\right)\Big\}^{k}, \qquad x \geq 0, && (3.1)\\ f_{S_{n,k}}(x) &= nkf(x)\,\psi'\left[F(x)\right]\phi'\left(n\psi\left[F(x)\right]\right)\Big\{1-\phi\left(n\psi\left[F(x)\right]\right)\Big\}^{k-1}, \qquad x \geq 0, && (3.2)\\ h_{S_{n,k}}(x) &= \frac{nkf(x)\,\psi'\left[F(x)\right]\phi'\left(n\psi\left[F(x)\right]\right)}{1-\phi\left(n\psi\left[F(x)\right]\right)}, \qquad x \geq 0, && (3.3)\\ \tilde{h}_{S_{n,k}}(x) &= \frac{nkf(x)\,\psi'\left[F(x)\right]\phi'\left(n\psi\left[F(x)\right]\right)\Big\{1-\phi\left(n\psi\left[F(x)\right]\right)\Big\}^{k-1}}{1-\Big\{1-\phi\left(n\psi\left[F(x)\right]\right)\Big\}^{k}}, \qquad x \geq 0, && (3.4) \end{aligned}$$

respectively. In the above expressions, $S_{n,k}$ denotes the lifetime of a series-parallel system with k independent subsystems, each consisting of n dependent components whose joint distribution is given by the Archimedaen copula in (2.1).

Theorem 3.1. *If* $t\frac{d}{dt}\ln\Big[1-\phi(t)\Big] = t\ln'\Big[1-\phi(t)\Big]$ *is decreasing, then for* $m \geq n$*, we have* $S_{m,k} \geq_{hr} S_{n,k}$.

Proof. Consider the hazard rate function of $S_{n,k}$ given by [see (3.3)]

$$h_{S_{n,k}}(x) = \frac{nkf(x)\,\psi'\left[F(x)\right]\phi'\left(n\psi\left[F(x)\right]\right)}{1-\phi\left(n\psi\left[F(x)\right]\right)}, \qquad x \geq 0.$$

Then, for establishing the desired result, we need to show that $h_{S_{n,k}}(x) - h_{S_{m,k}}(x) \geq 0$, for any $x \geq 0$. We have, for $x \geq 0$,

$$\begin{aligned} h_{S_{n,k}}(x) - h_{S_{m,k}}(x) &= \frac{kf(x)\psi'\left(F(x)\right)}{\psi\left(F(x)\right)}\left[\frac{n\psi\left(F(x)\right)\phi'\left(n\psi\left(F(x)\right)\right)}{1-\phi\left(n\psi\left(F(x)\right)\right)} - \frac{m\psi\left(F(x)\right)\phi'\left(m\psi\left(F(x)\right)\right)}{1-\phi\left(m\psi\left(F(x)\right)\right)}\right]\\ &\overset{sgn}{=} t\ln'\Big[1-\phi(t)\Big]\Big|_{t=n\psi(F(x))} - t\ln'\Big[1-\phi(t)\Big]\Big|_{t=m\psi(F(x))}. \qquad (3.5) \end{aligned}$$

Upon using the decreasing property of $t\ln'\Big[1-\phi(t)\Big]$, for $m \geq n$, we readily observe from (3.5) that $h_{S_{n,k}}(x) \geq h_{S_{m,k}}(x)$, for $x \geq 0$. Hence, the theorem. □

Remark 1. Theorem 3.1 shows that for some types of Archimedean copulas, series-parallel systems with less redundancy is more reliable in the sense of hazard rate order meaning that a series-parallel system with subsystems being parallel with more (dependent) components will possess higher hazard than a corresponding system with less number of components in the subsystems.

Examples 3.1. It needs to be mentioned that the condition "$t\ln'\left[1-\phi(t)\right]$ is decreasing in Theorem 3.1 is quite general and holds for many Archimedean copulas, as seen in the following cases:

(1) If $\phi_1(t) = e^{-t^\theta}$, for $\theta \in [0, \infty)$, we have

$$t\ln'\left[1-\phi_1(t)\right] = -\frac{t\phi_1'(t)}{1-\phi_1(t)} = \frac{\theta t^\theta e^{-t^\theta}}{1-e^{-t^\theta}},$$

to be decreasing in $t \geq 0$;

(2) If $\phi_2(t) = 1-\left(1-e^{-t}\right)^\theta$ for $\theta \in [0, 1)$, we have

$$t\ln'\left[1-\phi_2(t)\right] = -\frac{t\phi_2'(t)}{1-\phi_2(t)} = \frac{\theta t e^{-t}}{1-e^{-t}},$$

to be decreasing in $t \geq 0$;

(3) If $\phi_3(t) = \frac{1}{\sqrt{t}+1}$, we have

$$t\ln'\left[1-\phi_3(t)\right] = -\frac{t\phi_3'(t)}{1-\phi_3(t)} = \frac{1}{4\left(\sqrt{t}+1\right)},$$

to be decreasing in $t \geq 0$;

(4) If $\phi_4(t) = \frac{1}{ln(t+e)}$, we have

$$t\ln'\left[1-\phi_4(t)\right] = -\frac{t\phi_4'(t)}{1-\phi_4(t)} = -\frac{t}{(t+e)ln(t+e)},$$

to be decreasing in $t \geq 0$.

Theorem 3.2. *If $t\frac{d}{dt}\ln\left[1-\left(1-\phi(t)\right)^k\right] = t\ln'\left[1-\left(1-\phi(t)\right)^k\right]$ is decreasing, then for $m \geq n$, we have $S_{n,k} \leq_{rh} S_{m,k}$.*

Proof. Consider the reversed hazard rate function of $S_{n,k}$ given by [see (3.4)]

$$\tilde{h}_{S_{n,k}}(x) = \frac{nkf(x)\,\psi'[F(x)]\,\phi'(n\psi[F(x)])\left[1-\phi(n\psi[F(x)])\right]^{k-1}}{1-\left[1-\phi(n\psi[F(x)])\right]^{k}}, \qquad x \geq 0.$$

Then, for establishing the desired result, we need to show that $\tilde{h}_{S_{n,k}}(x) - \tilde{h}_{S_{m,k}}(x) \leq 0$, for any $x \geq 0$. We have, for $x \geq 0$,

$$\begin{aligned}
I(x) &= \tilde{h}_{S_{n,k}}(x) - \tilde{h}_{S_{m,k}}(x) \\
&= \frac{kf(x)\psi'(F(x))}{\psi(F(x))} \\
&\times \left\{\frac{n\psi(F(x))\,\phi'(n\psi(F(x)))\left[1-\phi(n\psi(F(x)))\right]^{k-1}}{1-\left[1-\phi(n\psi(F(x)))\right]^{k}}\right. \\
&- \left.\frac{m\psi(F(x))\,\phi'(m\psi(F(x)))\left[1-\phi(m\psi(F(x)))\right]^{k-1}}{1-\left[1-\phi(m\psi(F(x)))\right]}\right\} \\
&\overset{sgn}{=} t\ln'\left[1-\left(1-\phi(t)\right)^{k}\right]\Big|_{t=m\psi(F(x))} - t\ln'\left[1-\left(1-\phi(t)\right)^{k}\right]\Big|_{t=n\psi(F(x))}. \qquad (3.6)
\end{aligned}$$

Upon using the decreasing property of $t\ln'\left[1-\left(1-\phi(t)\right)^{k}\right]$, for $m \geq n$, we readily observe from (3.6) that $I(x) \leq 0$, for $x \geq 0$. Hence, the theorem. □

Remark 2. Theorem 3.2 shows that for some types of Archimedean copulas, series-parallel systems with less redundancy is more reliable in the sense of reversed hazard rate order meaning that a series-parallel system with subsystems being parallel with more (dependent) components will possess higher reversed hazard rate than a corresponding system with less number of components in the subsystems.

Examples 3.2. It needs to be mentioned that the condition"$t\ln'\left[1-\left(1-\phi(t)\right)^{k}\right]$ is decreasing" in Theorem 3.2 is general and holds for a number of Archimedean copulas, as seen in the following cases:

(1) If $\phi_1(t) = e^{-t}$, for $k = 3$, we have

$$t\ln'\left[1-\left(1-\phi_1(t)\right)^{3}\right] = \frac{3t\phi_1'(t)\left(1-\phi_1(t)\right)^{2}}{1-\left(1-\phi_1(t)\right)^{3}} = -\frac{3t\left(1-e^{-t}\right)^{2}}{3-3e^{-t}+e^{-2t}},$$

to be decreasing in $t \geq 0$;

(2) If $\phi_2(t) = \frac{1}{\sqrt{t}+1}$, for $k = 2$, we have

$$t \ln' \left[1 - \left(1 - \phi_2(t)\right)^2\right] = \frac{2t\phi_2'(t)\left(1 - \phi_2(t)\right)}{1 - \left(1 - \phi_2(t)\right)^2} = -\frac{t}{\left(\sqrt{t}+1\right)\left(2\sqrt{t}+1\right)},$$

to be decreasing in $t \geq 0$.

Theorem 3.3. *If* $t\frac{d}{dt}\ln\left[-\frac{\phi'(t)}{1-\phi(t)}\right] = t\ln'\left[-\frac{\phi'(t)}{1-\phi(t)}\right]$ *is decreasing in* $t \geq 0$*, then for* $m \geq n$*, we have* $S_{n,k} \geq_c S_{m,k}$.

Proof. By using arguments similar to those used in Theorem 3.1 of Ding and Zhang (2018), we can show that, if $t\ln'\left[-\frac{\phi'(t)}{1-\phi(t)}\right]$ is decreasing in $t \geq 0$, then $\frac{h_{S_{m,k}}(t)}{h_{S_{n,k}}(t)}$ is increasing in $t \geq 0$, as required. □

Remark 3. Theorem 3.3 shows that for some forms of Archimedean copulas, a series-parallel system with more redundancy ages faster in hazard rate meaning that a series-parallel system with subsystems being parallel with more (dependent) components will age faster in terms of hazard rate than a corresponding system with less number of components in the subsystems.

Theorem 3.4. *If* $t\frac{d}{dt}\ln\left[-\frac{\phi'(t)\left(1-\phi(t)\right)^{k-1}}{1-\left(1-\phi(t)\right)^k}\right] = t\ln'\left[-\frac{\phi'(t)\left(1-\phi(t)\right)^{k-1}}{1-\left(1-\phi(t)\right)^k}\right]$ *is decreasing (increasing), then for* $m \geq n$*, we have* $P_{n,k} \geq_c (\leq_c) P_{m,k}$.

Proof. The reversed hazard rate functions of $S_{n,k}$ and $S_{m,k}$ given by [see (3.4)]

$$\tilde{h}_{S_{m,k}}(x) = \frac{mkf(x)\,\psi'\left[F(x)\right]\phi'\left(m\psi\left[F(x)\right]\right)\left\{1 - \phi\left(m\psi\left[F(x)\right]\right)\right\}^{k-1}}{1 - \left\{1 - \phi\left(m\psi\left[F(x)\right]\right)\right\}^k}, \qquad x \geq 0,$$

and

$$\tilde{h}_{S_{n,k}}(x) = \frac{nkf(x)\,\psi'\left[F(x)\right]\phi'\left(n\psi\left[F(x)\right]\right)\left\{1 - \phi\left(n\psi\left[F(x)\right]\right)\right\}^{k-1}}{1 - \left\{1 - \phi\left(n\psi\left[F(x)\right]\right)\right\}^k}, \qquad x \geq 0,$$

respectively. Let us set $u = F(x)$, and consider the function

$$\begin{aligned} I(u) &= \frac{\tilde{h}_{S_{m,k}}(x)}{\tilde{h}_{S_{n,k}}(x)} \\ &= \frac{m}{n} \times \frac{\phi'(m\psi(u))\left\{1-\phi(m\psi(u))\right\}^{k-1}}{1-\left\{1-\phi(m\psi(u))\right\}^{k}} \times \left[\frac{\phi'(n\psi(u))\left\{1-\phi(n\psi(u))\right\}^{k-1}}{1-\left\{1-\phi(n\psi(u))\right\}^{k}}\right]^{-1} \quad (3.7) \end{aligned}$$

for $u \in [0,1]$. Upon differentiating (3.7) with respect to u, we find

$$\begin{aligned} I'(u) &\overset{sgn}{=} \left[\frac{\phi'(m\psi(u))\left\{1-\phi(m\psi(u))\right\}^{k-1}}{1-\left\{1-\phi(m\psi(u))\right\}^{k}}\right]' \times \frac{\phi'(n\psi(u))\left\{1-\phi(n\psi(u))\right\}^{k-1}}{1-\left\{1-\phi(n\psi(u))\right\}^{k}} \\ &\quad - \frac{\phi'(m\psi(u))\left\{1-\phi(m\psi(u))\right\}^{k-1}}{1-\left\{1-\phi(m\psi(u))\right\}^{k}} \times \left[\frac{\phi'(n\psi(u))\left\{1-\phi(n\psi(u))\right\}^{k-1}}{1-\left\{1-\phi(n\psi(u))\right\}^{k}}\right]' \\ &\overset{sgn}{=} \frac{\psi'(u)\left\{1-\left(1-\phi(m\psi(u))\right\}^{k}\right)}{\psi(u)\left(\phi'(m\psi(u))\left\{1-\phi(m\psi(u))\right\}^{k-1}\right)} \\ &\quad \times \left[\frac{m\psi(u)\phi''(m\psi(u))\left\{1-\phi(m\psi(u))\right\}^{k-1} - (k-1)m\psi(u)\phi'^2(m\psi(u))\left\{1-\phi(m\psi(u))\right\}^{k-2}}{\left[1-\left\{1-\phi(m\psi(u))\right\}^{k}\right]} \right. \\ &\quad \left. - \frac{km\psi(u)\phi'^2(m\psi(u))\left\{1-\phi(m\psi(u))\right\}^{2(k-1)}}{\left[1-\left\{1-\phi(m\psi(u))\right\}^{k}\right]^2}\right] \\ &\quad - \frac{\psi'(u)\left(1-\left\{1-\phi(n\psi(u))\right\}^{k}\right)}{\psi(u)\left(\phi'(n\psi(u))\left\{1-\phi(n\psi(u))\right\}^{k-1}\right)} \end{aligned}$$

$$\times\left[\frac{n\psi(u)\phi''(n\psi(u))\left\{1-\phi(n\psi(u))\right\}^{k-1}-(k-1)n\psi(u)\phi'^2(n\psi(u))\left\{1-\phi(n\psi(u))\right\}^{k-2}}{\left[1-\left\{1-\phi(n\psi(u))\right\}^{k}\right]}\right.$$
$$\left.-\frac{kn\psi(u)\phi'^2(n\psi(u))\left\{1-\phi(n\psi(u))\right\}^{2(k-1)}}{\left[1-\left\{1-\phi(n\psi(u))\right\}^{k}\right]^2}\right]$$
$$\begin{aligned}
&\overset{sgn}{=} && n\psi(u)\left[\frac{\phi''(n\psi(u))}{\phi'(n\psi(u))}-\frac{(k-1)\phi'(n\psi(u))}{1-\phi(n\psi(u))}-\frac{k\phi'(n\psi(u))\left\{1-\phi(n\psi(u))\right\}^{k-1}}{1-\left\{1-\phi(n\psi(u))\right\}^{k}}\right]\\
& && -m\psi(u)\left[\frac{\phi''(m\psi(u))}{\phi'(m\psi(u))}-\frac{(k-1)\phi'(m\psi(u))}{1-\phi(m\psi(u))}-\frac{k\phi'(m\psi(u))\left\{1-\phi(m\psi(u))\right\}^{k-1}}{1-\left\{1-\phi(m\psi(u))\right\}^{k}}\right]\\
&= && t\ln'\left[-\frac{\phi'(t)\left\{1-\phi(t)\right\}^{k-1}}{1-\left\{1-\phi(t)\right\}^{k}}\right]\Bigg|_{t=n\psi(u)}-t\ln'\left[-\frac{\phi'(t)\left\{1-\phi(t)\right\}^{k-1}}{1-\left\{1-\phi(t)\right\}^{k}}\right]\Bigg|_{t=m\psi(u)}\\
&\geq && (\leq)\ 0,
\end{aligned}$$

according as whether $t\ln'\left[-\frac{\phi'(t)(1-\phi(t))^{k-1}}{1-(1-\phi(t))^{k}}\right]$ is decreasing (increasing) in $t \geq 0$. Hence, the theorem. □

Remark 4. Theorem 3.4 shows that for some types of Archimedean copulas, under the decreasing (increasing) property of the function $t\ln'\left[-\frac{\phi'(t)(1-\phi(t))^{k-1}}{1-(1-\phi(t))^{k}}\right]$, a series-parallel system with more redundancy ages faster (ages slower) in terms of reversed hazard rate than a corresponding system with less redundancy.

Examples 3.3. The condition "$t\ln'\left[-\frac{\phi'(t)(1-\phi(t))^{k-1}}{1-(1-\phi(t))^{k}}\right]$ is decreasing" in Theorem 3.4 holds for a number of Archimedean copulas, as seen in the following cases:

(1) If $\phi_1(t) = \frac{1}{\sqrt{t}+1}$, for $k = 2$, we have

$$t \ln\left[-\frac{\phi'(t)\left(1-\phi(t)\right)^{k-1}}{1-\left(1-\phi(t)\right)^{k}}\right] = \frac{\left(-\frac{\left(\sqrt{t}+1\right)^{-3}}{2}-\frac{\left(\sqrt{t}+1\right)^{-2}}{4\sqrt{t}}\right)}{-\frac{\left(\sqrt{t}+1\right)^{-2}}{2\sqrt{t}}} + \frac{\frac{\sqrt{t}\left(\sqrt{t}+1\right)^{-2}}{2}}{1-\left(\sqrt{t}+1\right)^{-1}} + \frac{t\left(\frac{\left(\sqrt{t}+1\right)^{-2}}{2\sqrt{t}}\right)\left(1-\left(\sqrt{t}+1\right)^{-1}\right)}{1-\left(1-\left(\sqrt{t}+1\right)^{-1}\right)^{2}},$$

to be decreasing in $t \geq 0$;

(2) If $\phi_2(t) = e^{\frac{1-e^t}{\theta}}$, $\theta \in (0, 0.5(3-\sqrt{5})]$, when $m = n = 3$ and $k = 1$, we have

$$t \ln'\left[-\frac{\phi_2'(t)}{\phi_2(t)}\right] = \frac{t\phi_2''(t)}{\phi_2(t)} - \frac{t\phi_2'(t)}{\phi_2(t)} = \left(\frac{1}{\theta}-1\right)te^t + t,$$

to be increasing in $t \geq 0$. .

4 Results for Parallel-Series Systems

In this section, we focus on a parallel-series system consisting of k subsystems, each with n homogeneous components functioning in a dependent manner with Archimedean copulas for the joint distribution of lifetimes of these components. It is then of interest to examine between two parallel-series systems, one with n components and another with m components in each of the k subsystems, which one is more reliable in terms of ageing orders.

The distribution function, density function, hazard rate function and reversed hazard rate function of a parallel-series system, with k independent subsystems and n

homogeneous dependent components in any subsystem, are given by

$$F_{P_{n,k}}(x) = \left[1-\phi\left(n\psi\left[\bar{F}(x)\right]\right)\right]^k, \qquad x \geq 0, \tag{4.1}$$

$$f_{P_{n,k}}(x) = nkf(x)\,\psi'\left[\bar{F}(x)\right]\phi'\left(n\psi\left[\bar{F}(x)\right]\right)\left[1-\phi\left(n\psi\left[\bar{F}(x)\right]\right)\right]^{k-1}, \qquad x \geq 0, \tag{4.2}$$

$$h_{P_{n,k}}(x) = \frac{nkf(x)\psi'\left[\bar{F}(x)\right]\phi'\left(n\psi\left[\bar{F}(x)\right]\right)\left[1-\phi\left(n\psi\left[\bar{F}(x)\right]\right)\right]^{k-1}}{1-\left[1-\phi\left(n\psi\left[\bar{F}(x)\right]\right)\right]^k}, \qquad x \geq 0, \tag{4.3}$$

$$\tilde{h}_{P_{n,k}}(x) = \frac{nkf(x)\,\psi'\left[\bar{F}(x)\right]\phi'\left(n\psi\left[\bar{F}(x)\right]\right)}{1-\phi\left(n\psi\left[\bar{F}(x)\right]\right)}, \qquad x \geq 0, \tag{4.4}$$

respectively. In the above expressions, $P_{n,k}$ denotes the lifetime of a parallel-series system with k independent subsystems, each consisting of n dependent components whose joint distribution is given by the Archimedean copula in (2.1).

Theorem 4.1. *If* $t\frac{d}{dt}\ln\left[1-\left\{1-\phi(t)\right\}^k\right] = t\ln'\left[1-\left\{1-\phi(t)\right\}^k\right]$ *is decreasing, then for* $m \geq n$, *we have* $P_{n,k} \leq_{hr} P_{m,k}$.

Proof. Using the hazard rate functions of $P_{n,k}$ and $P_{m,k}$ in (4.3), let us consider, for $x \geq 0$,

$$\begin{aligned} I(x) &= h_{P_{n,k}}(x) - h_{P_{m,k}}(x) \\ &= \frac{kf(x)\psi'\left(\bar{F}(x)\right)}{\psi\left(\bar{F}(x)\right)} \\ &\times \left\{\frac{n\psi\left(\bar{F}(x)\right)\phi'\left(n\psi\left(\bar{F}(x)\right)\right)\left\{1-\phi\left(n\psi\left(\bar{F}(x)\right)\right)\right\}^{k-1}}{1-\left\{1-\phi\left(n\psi\left(\bar{F}(x)\right)\right)\right\}^k}\right. \\ &- \left.\frac{m\psi\left(\bar{F}(x)\right)\phi'\left(m\psi\left(\bar{F}(x)\right)\right)\left\{1-\phi\left(m\psi\left(\bar{F}(x)\right)\right)\right\}^{k-1}}{1-\left\{1-\phi\left(m\psi\left(\bar{F}(x)\right)\right)\right\}^k}\right\} \\ &\overset{sgn}{=} t\ln'\left[1-\left\{1-\phi(t)\right\}^k\right]\Big|_{t=m\psi(\bar{F}(x))} - t\ln'\left[1-\left\{1-\phi(t)\right\}^k\right]\Big|_{t=n\psi(\bar{F}(x))}. \end{aligned} \tag{4.5}$$

Upon using the decreasing property of $t\ln'\left[1-\left\{1-\phi(t)\right\}^k\right]$, for $m \geq n$, we readily observe from (4.5) that $I(x) \leq 0$. Hence, the theorem. □

Remark 5. Theorem 4.1 shows that for some Archimedean copulas, a parallel-series system with subsystems being series with less (dependent) components is more reliable in the sense of hazard rate order, meaning that it will possess a lower hazard function, than a corresponding system with more number of components in the subsystems.

Theorem 4.2. *If* $t\frac{d}{dt}\ln'\left[1-\phi(t)\right] = t\ln'\left[1-\phi(t)\right]$ *is decreasing, then for* $m \geq n$, *we have* $P_{n,k} \geq_{rh} P_{m,k}$.

Proof. Using the reversed hazard rate functions of $P_{n,k}$ and $P_{m,k}$ in (4.4), let us consider the function, for $x \geq 0$,

$$\begin{aligned} I(x) &= \tilde{h}_{P_{n,k}}(x) - \tilde{h}_{P_{m,k}}(x) \\ &= \frac{kf(x)\psi'\left(\bar{F}(x)\right)}{\psi\left(\bar{F}(x)\right)}\left[\frac{n\psi\left(\bar{F}(x)\right)\phi'\left(n\psi\left(\bar{F}(x)\right)\right)}{1-\phi\left(n\psi\left(\bar{F}(x)\right)\right)} - \frac{m\psi\left(\bar{F}(x)\right)\phi'\left(m\psi\left(\bar{F}(x)\right)\right)}{1-\phi\left(m\psi\left(\bar{F}(x)\right)\right)}\right] \\ &\overset{sgn}{=} t\ln'\left[1-\phi(t)\right]|_{t=n\psi(\bar{F}(x))} - t\ln'\left[1-\phi(t)\right]|_{t=m\psi(\bar{F}(x))}\,. \end{aligned} \tag{4.6}$$

Upon using the decreasing property of $t\ln'\left[1-\phi(t)\right]$, for $m \geq n$, we readily observe from (4.6) that $I(x) \geq 0$. Hence, the theorem. □

Remark 6. Theorem 4.2 shows that for some Archimedean copulas, a parallel-series system with subsystems being series with less (dependent) components will possess lower reversed hazard rate than a corresponding system with more number of components in the subsystems.

Theorem 4.3. *If* $t\ln\left[-\frac{\phi'(t)\left(1-\phi(t)\right)^{k-1}}{1-\left(1-\phi(t)\right)^{k}}\right]$ *is decreasing (increasing), then for* $m \geq n$, *we have* $P_{n,k} \geq_c (\leq_c) P_{m,k}$.

Proof. Using the hazard rate functions of $P_{n,k}$ and $P_{m,k}$ in (4.3), let us consider the function, for $x > 0$,

$$\begin{aligned} I(x) &= \frac{h_{P_{n,k}}(x)}{h_{P_{m,k}}(x)} \\ &= \frac{n}{m} \times \frac{\phi'\left(n\psi(\bar{F}(x))\right)\left\{1-\phi\left(n\psi(\bar{F}(x))\right)\right\}^{k-1}}{1-\left\{1-\phi\left(n\psi(\bar{F}(x))\right)\right\}^{k}} \\ &\times \left[\frac{\phi'\left(m\psi(\bar{F}(x))\right)\left\{1-\phi\left(m\psi(\bar{F}(x))\right)\right\}^{k-1}}{1-\left\{1-\phi\left(m\psi(\bar{F}(x))\right)\right\}^{k}}\right]^{-1}. \end{aligned} \tag{4.7}$$

Upon setting $v = \bar{F}(x)$ in (4.7) and then differentiating it with respect to v, we find

$$\begin{aligned}
I'(v) &\overset{sgn}{=} \left[\frac{\phi'(n\psi(v))\left\{1-\phi(n\psi(v))\right\}^{k-1}}{1-\left\{1-\phi(n\psi(v))\right\}^{k}}\right]' \times \frac{\phi'(m\psi(v))\left\{1-\phi(m\psi(v))\right\}^{k-1}}{1-\left\{1-\phi(m\psi(v))\right\}^{k}} \\
&\quad - \frac{\phi'(n\psi(v))\left\{1-\phi(n\psi(v))\right\}^{k-1}}{1-\left\{1-\phi(n\psi(v))\right\}^{k}} \times \left[\frac{\phi'(m\psi(v))\left\{1-\phi(m\psi(v))\right\}^{k-1}}{1-\left\{1-\phi(m\psi(v))\right\}^{k}}\right]' \\
&\overset{sgn}{=} \frac{\psi'(v)\left\{1-\left(1-\phi(m\psi(v))\right\}^{k}\right)}{\psi(v)\left(\phi'(m\psi(v))\left\{1-\phi(m\psi(v))\right\}^{k-1}\right)} \\
&\quad \times \Bigg[\frac{\left(m\psi(v)\phi''(m\psi(v))\left\{1-\phi(m\psi(v))\right\}^{k-1} - (k-1)m\psi(v)\phi'^{2}(m\psi(v))\left\{1-\phi(m\psi(v))\right\}^{k-2}\right)}{\left[1-\left\{1-\phi(m\psi(v))\right\}^{k}\right]} \\
&\quad - \frac{km\psi(v)\phi'^{2}(m\psi(v))\left\{1-\phi(m\psi(v))\right\}^{2(k-1)}}{\left[1-\left\{1-\phi(m\psi(v))\right\}^{k}\right]^{2}}\Bigg] \\
&\quad - \frac{\psi'(v)\left(1-\left\{1-\phi(n\psi(v))\right\}^{k}\right)}{\psi(v)\left(\phi'(n\psi(v))\left\{1-\phi(n\psi(v))\right\}^{k-1}\right)} \\
&\quad \times \Bigg[\frac{\left(n\psi(v)\phi''(n\psi(v))\left\{1-\phi(n\psi(v))\right\}^{k-1} - (k-1)n\psi(v)\phi'^{2}(n\psi(v))\left\{1-\phi(n\psi(v))\right\}^{k-2}\right)}{\left[1-\left\{1-\phi(n\psi(v))\right\}^{k}\right]} \\
&\quad - \frac{kn\psi(v)\phi'^{2}(n\psi(v))\left\{1-\phi(n\psi(v))\right\}^{2(k-1)}}{\left[1-\left\{1-\phi(n\psi(v))\right\}^{k}\right]^{2}}\Bigg] \\
&\overset{sgn}{=} n\psi(v)\left[\frac{\phi''(n\psi(v))}{\phi'(n\psi(v))} - \frac{(k-1)\phi'(n\psi(v))}{1-\phi(n\psi(v))} - \frac{k\phi'(n\psi(v))\left\{1-\phi(n\psi(v))\right\}^{k-1}}{1-\left\{1-\phi(n\psi(v))\right\}^{k}}\right] \\
&\quad - m\psi(\bar{F}(x))\left[\frac{\phi''(m\psi(v))}{\phi'(m\psi(v))} - \frac{(k-1)\phi'(m\psi(v))}{1-\phi(m\psi(v))} - \frac{k\phi'(m\psi(v))\left\{1-\phi(m\psi(v))\right\}^{k-1}}{1-\left(1-\phi(m\psi(v))\right)^{k}}\right] \\
&= t\ln'\left[-\frac{\phi'(t)\left\{1-\phi(t)\right\}^{k-1}}{1-\left\{1-\phi(t)\right\}^{k}}\right]\Bigg|_{t=n\psi(v)} - t\ln'\left[-\frac{\phi'(t)\left\{1-\phi(t)\right\}^{k-1}}{1-\left\{1-\phi(t)\right\}^{k}}\right]\Bigg|_{t=m\psi(v)} \\
&\geq (\leq)\ 0,
\end{aligned}$$

according as whether $t\ln\left[-\frac{\phi'(t)\left(1-\phi(t)\right)^{k-1}}{1-\left(1-\phi(t)\right)^{k}}\right]$ is decreasing (increasing) in $t \geq 0$, for $m \geq n$. Hence, the theorem. □

Remark 7. Theorem 4.3 shows that for some Archimedean copulas, under the decreasing (increasing) property of the function $t\ln\left[-\frac{\phi'(t)\left(1-\phi(t)\right)^{k-1}}{1-\left(1-\phi(t)\right)^{k}}\right]$, a parallel-series system with subsystems being series with less (dependent) components ages faster (ages slower) in terms of hazard rate than a corresponding system with more number of components in the subsystems.

Theorem 4.4. *If* $t\frac{d}{dt}\ln\left[-\frac{\phi'(t)}{1-\phi(t)}\right] = t\ln'\left[-\frac{\phi'(t)}{1-\phi(t)}\right]$ *is decreasing, then for* $m \geq n$, *we have* $P_{n,k} \geq_b P_{m,k}$.

Proof. By using arguments similar to those used for proving Theorem 3.1 of Ding and Zhang (2018), we can show that if $t\ln'\left[-\frac{\phi'(t)}{1-\phi(t)}\right]$ is decreasing in $t \geq 0$, then $\frac{\tilde{h}_{P_{n,k}}(t)}{\tilde{h}_{P_{m,k}}(t)}$ is increasing in $t \geq 0$. Hence, the theorem. □

Remark 8. Theorem 4.4 shows that for some Archimedean copulas, a parallel-series system with subsystems being series with less (dependent) components ages faster (ages slower) in terms of reversed hazard rate than a corresponding system with more number of components in the subsystems.

5 Conclusion

In this paper, we have considered series-parallel and parallel-series systems with k independent subsystems, each consisting of n homogeneous dependent components having a joint distribution as an Archimedean copula. Then, by considering two similar systems with different numbers of components within each subsystem, we have established hazard rate and reversed hazard rate orderings between the two systems, and also have discussed how these systems age relative to each other in terms of hazard rate and reverse hazard rate functions.

One assumption that we have made in developing these results is that the subsystems within each system are independent of each other, even through the components within each subsystem are dependent. It will naturally be of interest to consider a more general situation in which the subsystems also function dependently, and then consider developing analogous comparison results between two such systems. The model involved will naturally be quite complex in this case, and we are currently looking into this problem, and hope to report the findings in a future paper. Another extension of the present work that will be of interest is to consider series-l-out-of-n and parallel-l-out-of-n systems and develop analogous ordering and ageing results.

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