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Confidence Intervals for the Power of Two-Sided Student's t-test

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Abstract. For the power of two-sided hypothesis testing about the mean of a normal population, we derive a $100(1-\alpha)\%$ confidence interval. Then by using a numerical method we will find a shortest confidence interval and consider some special cases.

1 Introduction

Suppose $X_1, ..., X_n$ is a random sample of size n from a normal population with mean μ and variance σ^2. The sample mean, $\bar{X}$, and the sample variance, S^2, are respectively defined as

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad and \quad S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

It is well-known that for testing $H_\circ : \mu = \mu_\circ$ against $H_1 : \mu \neq \mu_\circ$, with fixed μ_0 and significance level α, the relevant test rejects H_0

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whenever

$$|\bar{X}-\mu_\circ| > \frac{S}{\sqrt{n-1}} t_{1-\alpha/2,n-1}$$

where $t_{p,v}$ is the pth quantiles of the Student's t distribution with v degrees of freedom.

The power function of this test (Lehmann, 1991) is

$$\begin{aligned}\beta(\mu,\sigma) &= P(|\bar{X}-\mu_\circ| > \frac{S}{\sqrt{n-1}} t_{1-\alpha/2,n-1}) \\ &= 1 - G_{n-1,\sqrt{n}\frac{\mu-\mu_\circ}{\sigma}}(t_{1-\alpha/2,n-1}) \\ &\quad + G_{n-1,\sqrt{n}\frac{\mu-\mu_\circ}{\sigma}}(-t_{1-\alpha/2,n-1}). \qquad (1)\end{aligned}$$

where $G_{v,\delta}(\cdot)$ is the cumulative distribution function of the noncentral t distribution with v degrees of freedom and noncentrality parameter δ.

When σ is unknown, we cannot calculate the value of the power function $\beta(\mu,\sigma)$, but it can be estimated by using the invariant properties of Maximum Likelihood Estimators (MLE). Since the MLE of σ is S, so

$$\begin{aligned}\hat{\beta}(\mu,\sigma) &= \beta(\mu,S) \\ &= 1 - G_{n-1,\sqrt{n}\frac{\mu-\mu_\circ}{S}}(t_{1-\alpha/2,n-1}) \\ &\quad + G_{n-1,\sqrt{n}\frac{\mu-\mu_\circ}{S}}(-t_{1-\alpha/2,n-1}).\end{aligned}$$

Tarasińska (2005) proposed a minimum length method for determining confidence intervals for the power of one-sided t-test at fixed alternative means. He derived

$$P(1-G_{n-1,\frac{\Delta\sqrt{n}}{b}}(t_{1-\alpha,n-1}) < \beta(\Delta) < 1+G_{n-1,\frac{\Delta\sqrt{n}}{a}}(t_{1-\alpha,n-1})) = 1-\gamma,$$

where $\beta(\Delta) = \beta^*(\mu,\sigma) = 1 - G_{n-1,\frac{\Delta\sqrt{n}}{\sigma}}(t_{1-\alpha,n-1})$ is the power function of the one-sided Student's t-test, and $\Delta = |\mu-\mu_\circ|$.

He takes $a = \frac{\sqrt{n}S}{\sqrt{B}}$ and $b = \frac{\sqrt{n}S}{\sqrt{A}}$. So, for fixed values of n, $\frac{\mu-\mu_\circ}{S}$, α, and γ, values of A and B are found for minimizing

$$\left[\beta^*(\mu, \frac{\sqrt{n}S}{\sqrt{B}}) - \beta^*(\mu, \frac{\sqrt{n}S}{\sqrt{A}})\right],$$

under the condition

$$\int_A^B f(x)dx = 1-\gamma,$$

where $f(x)$ is the probability density function of chi-squared distribution with $(n-1)$ degrees of freedom. With this method the values of A and B are found numerically.

Now, in this paper we first drive a $100(1-\alpha)\%$ confidence interval for the power of two-sided hypothesis testing about the mean of a normal population. Then by the above minimizing method, the shortest confidence intervals are obtained for some special cases.

2 Confidence intervals for the power of the test

The main results of this section are presented by the following lemma and theorem.

Lemma 2.1. *Let us define*

$$h(\delta) = G_{v,\delta}(t) - G_{v,\delta}(-t),$$

in which $t > 0$, is constant. Then

i) For all $\delta > 0$, the function $h(\delta)$ is a decreasing function .

ii) For all $\delta < 0$, the function $h(\delta)$ is an increasing function.

iii) $h(\delta)$ is an even function, that is $h(-\delta) = h(\delta)$.

Proof.
i) Following Owen (1968), we have

$$G_{v,\delta}(t) = \frac{\sqrt{2\pi}}{\Gamma(v/2)2^{(v-2)/2}} \int_0^\infty u^{v-1}\Phi(\frac{tu}{\sqrt{v}} - \delta)\phi(u)du,$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cdf and pdf of the standard normal distribution, respectively. Let $C = \frac{\sqrt{2\pi}}{\Gamma(v/2)2^{(v-2)/2}}$, then

$$h(\delta) = C\int_0^\infty u^{v-1}\phi(u)\left[\Phi(\frac{tu}{\sqrt{v}} - \delta) - \Phi(\frac{-tu}{\sqrt{v}} - \delta)\right] du. \tag{2}$$

In (2), let $k(\delta) = \Phi(x-\delta) - \Phi(-x-\delta)$ and $x = \frac{tu}{\sqrt{v}}$. We have

$$k'(\delta) = \phi(x+\delta) - \phi(x-\delta) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(x^2+\delta^2)}(e^{-x\delta} - e^{x\delta}).$$

Since $\delta > 0$ and $x > 0$, so $k'(\delta) < 0$. Therefore $k(\delta)$ and also $h(\delta)$ are decreasing functions of δ.

ii) The proof of this part is similar to the proof of part (i).

iii) By direct use of (2), we get

$$\begin{aligned} h(-\delta) &= C\int_0^\infty u^{v-1}\phi(u)\left[\Phi(\frac{tu}{\sqrt{v}}+\delta)-\Phi(\frac{-tu}{\sqrt{v}}+\delta)\right]du \\ &= C\int_0^\infty u^{v-1}\phi(u)\left[\Phi(\frac{tu}{\sqrt{v}}+\delta)-\left(1-\Phi(\frac{tu}{\sqrt{v}}-\delta)\right)\right]du \\ &= C\int_0^\infty u^{v-1}\phi(u)\left[\Phi(\frac{tu}{\sqrt{v}}-\delta)-\Phi(\frac{-tu}{\sqrt{v}}-\delta)\right]du \\ &= h(\delta). \end{aligned}$$

Since the value of $\beta(\mu,\sigma)$ for $\mu > \mu_\circ$ is equal to the value of $\beta(\mu,\sigma)$ for $\mu < \mu_\circ$, we will find the confidence interval for $\beta(\mu,\sigma)$, when $\mu > \mu_\circ$.

Theorem 2.1. *Let (a,b) be any* $100(1-\gamma)\%$ *confidence interval for* σ*, then*

$$\begin{aligned} &P\Big\{1-G_{n-1,\sqrt{n}\frac{\mu-\mu_\circ}{b}}(t_{1-\alpha/2,n-1})+G_{n-1,\sqrt{n}\frac{\mu-\mu_\circ}{b}}(-t_{1-\alpha/2,n-1}) \\ &\qquad\qquad < \beta(\mu,\sigma) < \\ &\quad 1-G_{n-1,\sqrt{n}\frac{\mu-\mu_\circ}{a}}(t_{1-\alpha/2,n-1})+G_{n-1,\sqrt{n}\frac{\mu-\mu_\circ}{a}}(-t_{1-\alpha/2,n-1})\Big\} \\ &= 1-\gamma. \end{aligned} \tag{3}$$

Proof. Since $\mu > \mu_\circ$, by part (i) of Lemma 2.1, we have

$$\begin{aligned} 1-\gamma &= P(a<\sigma<b) \\ \Longleftrightarrow\quad 1-\gamma &= P(\frac{\mu-\mu_\circ}{b}\sqrt{n}<\frac{\mu-\mu_\circ}{\sigma}\sqrt{n}<\frac{\mu-\mu_\circ}{a}\sqrt{n}) \\ \Longleftrightarrow\quad 1-\gamma &= P(\beta(\mu,b)<\beta(\mu,\sigma)<\beta(\mu,a)). \end{aligned}$$

Therefore, by using (1), we obtain (3).

Using R software version 2.7.0., the minimum length method was carried out numerically, with steps equal to .001. Table 1 presents the estimates of the power of the test (the middle value in the cell), and the bounds for 95% confidence intervals of the power with $\alpha = 0.05$. The curve of the bounds and of the estimate as functions of $\frac{\mu-\mu_\circ}{S}$ for $n = 10$ are given in Fig 1.

We note again that all the entries in Table 1 are also good for the case $\mu < \mu_\circ$.

Table 1. Values of lower bound of CI, estimate and upper bound CI for the power of the test, $\alpha = \gamma = 0.05$

	n							
$\frac{\Delta}{S}$	4	5	6	7	10	12	13	15
	0.050	0.050	0.050	0.051	0.052	0.053	0.053	0.055
0.1	0.052	0.054	0.055	0.056	0.059	0.062	0.063	0.065
	0.055	0.057	0.059	0.061	0.066	0.070	0.071	0.075
	0.050	0.050	0.051	0.052	0.057	0.061	0.064	0.069
0.2	0.060	0.064	0.069	0.074	0.088	0.097	0.102	0.112
	0.069	0.077	0.085	0.093	0.116	0.131	0.138	0.152
	0.050	0.051	0.052	0.054	0.066	0.076	0.081	0.093
0.3	0.072	0.082	0.093	0.103	0.136	0.158	0.169	0.192
	0.093	0.112	0.130	0.149	0.201	0.235	0.251	0.283
	0.050	0.051	0.056	0.063	0.097	0.128	0.146	0.184
0.5	0.111	0.141	0.171	0.201	0.293	0.353	0.382	0.438
	0.169	0.223	0.275	0.326	0.463	0.542	0.578	0.644
	0.050	0.055	0.069	0.093	0.218	0.323	0.376	0.474
0.8	0.206	0.281	0.356	0.428	0.616	0.714	0.754	0.821
	0.344	0.469	0.580	0.675	0.873	0.944	0.966	0.989
	0.050	0.060	0.091	0.146	0.368	0.496	0.556	0.664
1.0	0.289	0.401	0.506	0.600	0.803	0.883	0.910	0.949
	0.482	0.646	0.774	0.872	0.995	1.000	1.000	1.000
	0.051	0.056	0.112	0.192	0.433	0.573	0.637	0.746
1.1	0.335	0.466	0.583	0.681	0.871	0.933	0.952	0.977
	0.552	0.727	0.855	0.944	1.000	1.000	1.000	1.000
	0.051	0.073	0.145	0.247	0.497	0.674	0.712	0.816
1.2	0.383	0.530	0.655	0.754	0.920	0.950	0.977	0.991
	0.619	0.798	0.924	0.991	1.000	1.000	1.000	1.000
	0.054	0.144	0.258	0.365	0.683	0.831	0.882	0.946
1.5	0.533	0.711	0.833	0.907	0.987	0.996	0.998	1.000
	0.792	0.969	0.999	1.000	1.000	1.000	1.000	1.000

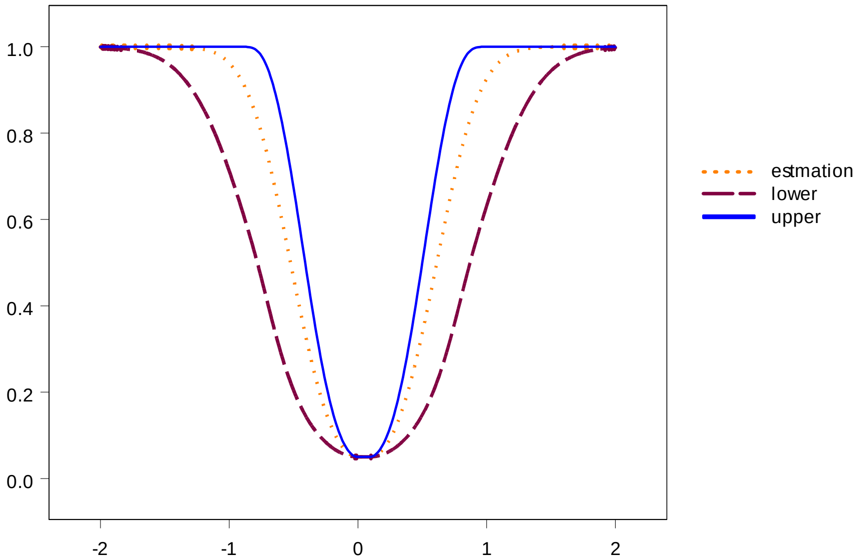

Fig. 1. CI curves (95%) for the test power and the estimate of the power (the middle one) as the functions of $\frac{\mu-\mu_\circ}{S}$, $n = 10$, $\alpha = 0.05$

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