Estimation of Subpopulation Parameters in One-stage Cluster Sampling Design

Mostafa Hossaini ¹, and Abdolhamid Rezaei Roknabady ¹

¹Department of Statistics, Ferdowsi University of Mashhad.

Received: 17/02/2020, Revision received: 02/09/2021, Published online: 21/10/2021

Abstract. Sometimes in order to estimate population parameters such as mean and total values, we extract a random sample by cluster sampling method, and after completing sampling, we are interested in using the same sample to estimate the desired parameters in a subset of the population, which is said subpopulation. In this paper, we try to estimate subpopulation parameters in different cases when one-stage cluster sampling design is used.

Keywords. Finite Population, Subpopulation, Cluster Sampling, Unbiased

MSC: 62D05, 62P12.

1 Introduction

The purpose of a sample survey design is to estimate the population parameters with the lowest cost and the highest accuracy. One of the most used and low cost methods in sampling schemes is cluster sampling, which does not require a population framework.
that it is generally costly. On the other hand, many sampling schemes can be cited where in addition to the inference about the population parameters, it is important to infer the parameters of a part of the population that share a particular feature (subpopulation). Since subpopulation estimators are based on the same sample, there is no additional cost to the researcher. In this paper, we attempt to discuss a subpopulation parameter estimation in a cluster sampling scheme.

To estimate the subpopulation parameters in a simple random sampling scheme, we recall some notion. Suppose that the members of a finite population are $U_1, U_2, \ldots, U_N$ and that $Y_i$ denotes the $Y$-value associated with $U_i$ for $i = 1, \ldots, N$. Hence $Y = \sum_{i=1}^{N} Y_i$ and $\bar{Y} = \frac{Y}{N}$ represent the total and mean values of the population, respectively. Now, suppose that $u_1, u_2, \ldots, u_n$ are sample members and the random variable $y_i$ is the value of the desired attribute $u_i$ for $i = 1, \ldots, n$. Consider the attribute $c$ that some members of the population possess, and call them a subpopulation. After simple random sampling for estimating population parameters, we are interested in estimating subpopulation parameters, using the same simple random sample. For this purpose, we assume that $N_c$ is the size of the subpopulation and that $Y_{ci}$ for $i = 1, \ldots, N_c$, is the value of the attribute of $i$th subpopulation member. Then, $Y_c = \sum_{i=1}^{N_c} Y_{ci}$ and $\bar{Y}_c = \frac{Y_c}{N_c}$ will represent the total and mean values of the subpopulation, respectively. For $j = 1, \ldots, n_c$, suppose that the random variable $y_{cj}$ represents the value of the attribute desired for $j$th sample member in the subpopulation $c$, where $n_c$ is the sample size in the subpopulation. Obviously $n_c$ is a random variable with hyper-geometric distribution and expectation $\frac{nN_c}{N}$.

To estimate $\bar{Y}_c$, Cochran (1977) introduced the following estimator:

$$\hat{y}_c = \frac{1}{n_c} \sum_{j=1}^{n_c} y_{cj},$$

and

$$V(\hat{y}_c) = S^2_c \left[ E\left( \frac{1}{n_c} \right) - \frac{1}{N_c} \right], \quad (1.1)$$

where $S^2_c$ is the subpopulation variance. Similarly $Y_c$ is estimated by

$$\bar{Y}_c = \frac{N_c}{N} \hat{y}_c, \quad (1.2)$$
but \( \hat{Y}_c \) is an estimator of \( Y_c \) if \( N_{c0} \) is known. If \( N_{c0} \) is unknown, by using \( \hat{N}_c = \frac{N n_c}{n} \) as an unbiased estimator for \( N_{c0} \), the following estimator can be introduced for estimating \( Y_c \):

\[
\hat{Y}_c = \frac{N n_c}{n} \bar{y}_c. \tag{1.3}
\]

We can prove that all estimators \( \bar{y}_c, \hat{Y}_c, \) and \( \hat{\hat{Y}}_c \) are unbiased for their corresponding parameters. Let us assume that

\[
Y^*_c = \begin{cases} 
Y_i, & u_i \in c, \\
0, & o.w.,
\end{cases}
\]

for \( i = 1, \ldots, N \), and that, in this case,

\[
S^2_c = \frac{1}{N - 1} \sum_{i=1}^{N} (Y^*_c - \bar{Y}_c)^2, \tag{1.4}
\]

where \( \bar{Y}_c \) is the average of \( Y^*_c \)’s. Then, it can be shown that

\[
V(\hat{Y}_c) = N^2 \frac{S^2_c}{n} \left( 1 - \frac{n}{N} \right). \tag{1.5}
\]

Also, unbiased estimators for \( V(\hat{Y}_c) \) and \( V(\hat{\hat{Y}}_c) \) are, respectively,

\[
v(\hat{Y}_c) = N^{2 \sigma^2_c} \left( \frac{1}{n_c} \right) \left( \frac{1}{N_{c0}} \right),
\]

and

\[
v(\hat{\hat{Y}}_c) = N^2 \frac{s^2_c}{n} \left( 1 - \frac{n}{N} \right),
\]

where \( s^2_c \) and \( s^*_c \) are the sample variances of \( y_{ci} \) and

\[
y^*_c = \begin{cases} 
y_i, & u_i \in c, \\
0, & o.w.
\end{cases},
\]

for \( i = 1, \ldots, n \).
Note 1. Note that $s^2_c$ and $s^*^2_c$ are unbiased estimators for $S^2_c$ and $S^*^2_c$, respectively.

We refer readers to Cochran (1977) and Thompson (2012) for more information on these estimators. Besides, Salehi and Chang (2005) provided another estimator for the parameter of the total subpopulation based on the inverse sampling method and Clark (2009) introduced a regression estimator for total value of subpopulation in two stage sampling. It should be noted that the method of two-stage sampling is different from that two-stage cluster sampling. Two-stage sampling is useful when the variable of interest $y$ is relatively expensive to measure but a correlated variable $x$ can be measured fairly easily and used to improve the precision of the estimation of parameters of interest. It may also be used to adjust for nonresponse, to sample rare populations, or to improve the sampling frame (see Lohr (2010)). But in two-stage cluster, we cannot census all of elements of clusters selected in the first stage, or it may be expensive to measure all of them. So we select a sample with two-stage cluster sampling.

2 Estimation of Subpopulation Parameters in One-stage Cluster Sampling Design

Consider a population with $N$ clusters that members of the $i$th cluster with the size of $M_i$ are denoted by $U_{i1}, \ldots, U_{iM_i}$ for $i = 1, \ldots, N$. Assume that $M = \sum_{i=1}^{N} M_i$ is the size of population and that $Y_{ij}$ is the value of the desired attribute for the unit $U_{ij}$.

The parameters $\tau = \sum_{i=1}^{N} \sum_{j=1}^{M_i} Y_{ij}$ and $\mu = \frac{\tau}{M}$ represent the total and the mean of the population, respectively. Accordingly, for $i = 1, \ldots, N$, the parameters $Y_i = \sum_{j=1}^{M_i} Y_{ij}$ and $\bar{Y}_i = \frac{Y_i}{M_i}$ represent the total value and the mean of the $i$-th cluster, respectively. Now for this population, suppose that $n$ clusters are selected under the simple random sampling design and that the clusters are censused. Assume that $m_i$ is the sample size of $i$-th cluster and that $y_{ij}$ indicates the sample values of the desired attribute in the $i$-th cluster of sample ($i = 1, \ldots, n$; $j = 1, \ldots, m_i$).

We know that the unbiased estimator of the population total reads as follows:

$$\hat{\tau} = \frac{N}{n} \sum_{i=1}^{n} y_i = N\bar{y}, \quad (2.1)$$
where \( y_i = \sum_j^{M_i} y_{ij} \), for \( i = 1, \ldots, n \), is the total value of the cluster \( i \) in the sample.

The variance of this estimator is

\[
V(\bar{\tau}) = N^2 S^2_b \left[ \frac{1}{n} - \frac{1}{N} \right],
\]

(2.2)

where considering \( \bar{Y} = \frac{1}{N} \sum_i Y_i \), we have

\[
S^2_b = \frac{1}{N-1} \sum_i (Y_i - \bar{Y})^2.
\]

(2.3)

If \( s^2_b = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2 \) is the sample variance \( y_1, \ldots, y_n \), then the unbiased estimator \( V(\bar{\tau}) \) is

\[
v(\bar{\tau}) = s^2_b \left[ \frac{1}{n} - \frac{1}{N} \right].
\]

(2.4)

However, the estimator \( \hat{\mu} = \frac{\bar{Y}}{\bar{M}} \) is an unbiased estimator for \( \mu \) with variance \( \frac{V(\bar{\tau})}{M^2} \).

We also know that \( \hat{\mu}_R = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} m_i} \) is a biased estimator of \( \mu \) and it is used when \( M \) is unknown. The approximate variance of this estimator is

\[
V(\hat{\mu}_R) \approx \frac{(N-n)}{nNM} \sum_{i=1}^{N} (Y_i - \bar{M}_i \mu)^2,
\]

where \( \bar{M} = \frac{1}{N} \sum_{i=1}^{N} M_i \) is the average of the cluster sizes. For more information on ratio estimators, see Rao (1988) and Royal (1988). Moreover, Salehi and Seber (2001) presented an unbiased estimator in adaptive cluster sampling.

Now suppose among the \( N \) clusters, \( N_c \) clusters have members of the subpopulation. Obviously \( N_c \) is a natural number and \( 1 \leq N_c \leq N \). For \( i = 1, \ldots, N_c \), let \( M_{c,i} \) be the size of members of the subpopulation in \( i \)-th cluster and let \( M_c = \sum_{i=1}^{N_c} M_{c,i} \) be the total number of subpopulation members. In concordance with the parameters of the population, let \( \tau_c = \sum_{i=1}^{N_c} \sum_{j=1}^{M_{c,i}} Y_{c,ij} \) and \( \mu_c = \frac{\tau_c}{M_c} \) represent the total and the mean of

\[\text{Estimation of Subpopulation Parameters} \]
the subpopulation, respectively, where $Y_{c,ij}$ is the attribute value for the $j$-th member of subpopulation in the $i$-th cluster.

Also, let $Y_{c,i} = \sum_{j=1}^{M_{c,i}} Y_{c,ij}$ and $\bar{Y}_{c,i} = \frac{Y_{c,i}}{M_{c,i}}$ indicate the total value and mean of the subpopulation in $i$-th cluster for ($i = 1, \ldots, N_c$), respectively, and $\bar{Y}_c = \frac{1}{N_c} \sum_{i=1}^{N_c} Y_{c,i}$ is the average of $Y_{c,i}$'s. Now, if we consider among $n$ clusters in the sample, $n_c$ clusters contain subpopulation's members, then $n_c$ is a random variable with hyper-geometric distribution and $E(n_c) = \frac{nN_c}{N}$. Similarly, suppose that $m_{c,i}$ is the sample size of subpopulation’s members in the cluster $i$ and that $y_{c,ij}$ represents the sample values of the attribute desired in the $i$-th cluster containing subpopulation members in the sample ($i = 1, \ldots, n_c; j = 1, \ldots, m_{c,i}$). The following theorem presented an unbiased estimator of $\tau_c$.

**Theorem 2.1.** a) By considering $y_{c,i} = \sum_{j=1}^{m_{c,i}} y_{c,ij}$ as the total value of the subpopulation in the $i$-th cluster of sample, the estimator

$$\hat{\tau}_c = \frac{N_c}{n_c} \sum_{i=1}^{n_c} y_{c,i} = N_c \bar{y}_c,$$

is an unbiased estimator for $\tau_c$.

b) The variance of this estimator is as follows:

$$V(\hat{\tau}_c) = N_c^2 S_{cb}^2 \left[ E\left(\frac{1}{n_c}\right) - \frac{1}{N_c}\right],$$

where

$$S_{cb}^2 = \frac{1}{N_c - 1} \sum_{i=1}^{N_c} (Y_{c,i} - \bar{Y}_c)^2.$$

c) The unbiased estimator for the variance in part b) is

$$v(\hat{\tau}_c) = S_{cb}^2 \left[ E\left(\frac{1}{n_c}\right) - \frac{1}{N_c}\right],$$

where

$$s_{cb}^2 = \frac{1}{n_c - 1} \sum_{i=1}^{n_c} (y_{c,i} - \bar{y}_c)^2.$$
Proof. a) We know that by given $n_c$, every possible combination of $n_c$ of the $N_c$ subpopulation units has equal probability of being included in the sample, so

$$E(\bar{\tau}_c) = E\left( \frac{N_c}{n_c} \sum_{j=1}^{n_c} Y_{c,j} \right)$$

$$= N_c E\left( \frac{1}{n_c} \sum_{j=1}^{n_c} y_{c,j|n_c} \right)$$

$$= N_c E(\bar{Y}_c)$$

$$= \tau_c.$$

b) Using the well-known decomposition of $Var(X) = E(Var(X|n_c)) + Var(E(X|n_c))$, we can write

$$V(\bar{\tau}_c) = N_c^2 \left[ E\left( Var\left( \frac{1}{n_c} \sum_{j=1}^{n_c} y_{c,j|n_c} \right) \right) + Var\left( \frac{1}{n_c} \sum_{j=1}^{n_c} y_{c,j|n_c} \right) \right]$$

$$= N_c^2 \left( S_{cb}^2 \left( \frac{1}{n_c(N_c - n_c)} \right) + 0 \right)$$

$$= N_c^2 S_{cb}^2 \left( \frac{1}{n_c} - \frac{1}{N_c} \right). \quad (2.5)$$

(c) Since in the cluster sampling scheme, the clusters are selected under simple random sampling, according to Note 1, the desired result is obtained. □

About the estimator $\hat{\tau}_c$, the following note is important.

Note 2. a) To obtain the variance of $\bar{\tau}_c$, we need to calculate $E\left( \frac{1}{n_c} \right)$, which is difficult in practice, but given the positive value for $n_c$, a good approximation of $E\left( \frac{1}{n_c} \right)$ is as follows (Stephen 1945):

$$E\left( \frac{1}{n_c} \right) \approx \frac{N}{nN_c} + \frac{N(N - N_c)}{n^2 N_c^2}. \quad (2.6)$$

b) If $N_c$ is unknown, then $\hat{\tau}_c$ is not an estimator for $\tau_c$.

The following theorem introduces an estimator for $\tau_c$ when $N_c$ is unknown.
Theorem 2.2. a) Estimator

\[ \tilde{\tau}_c = \frac{N}{n} \sum_{i=1}^{n_c} y_{c,i}, \]

is an unbiased estimator for \( \tau_c \).

b) The variance of this estimator is as follows:

\[ V(\tilde{\tau}_c) = N^2 S_{cb}^2 \left[ \frac{1}{n} - \frac{1}{N} \right]. \]

If

\[ Y_{ij} = \begin{cases} Y_{c,ij}, & \text{if } U_{ij} \in c, \\ 0, & \text{o.w.} \end{cases} \quad \text{for } (i = 1, \ldots, N ; j = 1, \ldots, m_i), \]

and \( Y_i = \sum_{j=1}^{M_i} Y_{ij} \), then

\[ s_{cb}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (Y_i - \bar{Y})^2, \]

where \( \bar{Y} \) is the average of \( Y_1, \ldots, Y_N \). Obviously, the value of \( Y_i \) is zero for clusters that are empty from subpopulation members.

c) An unbiased estimator for variance of part b) is,

\[ v(\tilde{\tau}_c) = s_{cb}^2 \left[ \frac{1}{n} - \frac{1}{N} \right], \]

where if the sample units are \( u_{ij} \) (\( i = 1, \ldots, n ; j = 1, \ldots, m_i \)),

\[ y_{ij} = \begin{cases} y_{c,ij}, & \text{if } u_{ij} \in c, \\ 0, & \text{o.w.} \end{cases} \]

and

\[ y_i = \sum_{j=1}^{m_i} y_{ij}, \]

then

\[ s_{cb}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2, \]

and \( \bar{y} \) is the average of \( y_1, \ldots, y_n \).
Proof. a) It can be easily shown that
\[ \tilde{\tau}_c = \frac{N}{n} \sum_{i=1}^{n} y_{c,i} = \frac{N}{n} \sum_{i=1}^{n} y^*_i, \]
so
\[ E(\tilde{\tau}_c) = N \mathbb{E}\left(\frac{1}{n} \sum_{i=1}^{n} y^*_i\right) = N \left(\frac{1}{N} \sum_{i=1}^{N} Y^*_i\right) = \sum_{i=1}^{N_c} Y_{c,i} = \tau_c. \]

b)
\[ V(\tilde{\tau}_c) = N^2 g\left(\frac{1}{n} \sum_{i=1}^{n} Y^*_i\right) = N^2 S^2_{c} \left[ \frac{1}{n} - \frac{1}{N} \right]. \]

c) Since the clusters are selected by simple random sampling and according to Note 1, this part is easily established. \(\square\)

Proposition 2.1. The estimator \(\hat{\mu}_c = \frac{\tilde{\tau}_c}{M_c}\) with variance \(V(\hat{\mu}_c)\) is an unbiased estimator for \(\mu_c\), and also \(\tilde{\mu}_c = \frac{\sum_{i=1}^{n_c} Y_{c,i}}{\sum_{i=1}^{m_{c,i}}}\) is a biased estimator of \(\mu_c\), and an approximation value for the variance of this estimator is
\[ V(\tilde{\mu}_c) \approx \frac{E(\frac{1}{n_c}) - \frac{1}{N_c}}{M^2_c (N - 1)} \sum_{i=1}^{N_c} (Y_{c,i} - M_{c,i} \mu_c)^2, \quad (2.7) \]
where \(M_c = \frac{1}{N_c} \sum_{i=1}^{N_c} M_{c,i}\).
By pasting $\overline{m}_c = \frac{1}{n_c} \sum_{i=1}^{n_c} m_{c,i}$ and $\overline{N}_c = \frac{n_c}{n} N$, respectively, as unbiased estimators of $M_c$ and $N_c$ in Equation (2.7), a good estimator of $V(\hat{\mu}_{cR})$ is as follows:

$$v(\hat{\mu}_{cR}) \approx \frac{1}{n_c^2 (n_c - 1)} \sum_{i=1}^{n_c} (y_{c,i} - m_{c,i}\hat{\mu}_{cR})^2.$$ 

**Examples 2.1.** In 2016, a one-stage cluster sampling with 60 clusters was selected to estimate the total amount of cereal (wheat, barley, and maize) produced in South Khorasan province comprising 183 rural areas (clusters). Each farm in the selected villages in the sample was examined as a small unit. The sample is the weight of cereal in tons and is given in Table 1.

Table 1: Quantities of cereal produced at tonnes in clusters of sample

<table>
<thead>
<tr>
<th>Cluster</th>
<th>Grain</th>
<th>Cluster</th>
<th>Grain</th>
<th>Cluster</th>
<th>Grain</th>
<th>Cluster</th>
<th>Grain</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>276</td>
<td>16</td>
<td>466</td>
<td>31</td>
<td>690</td>
<td>46</td>
<td>794</td>
</tr>
<tr>
<td>2</td>
<td>738</td>
<td>17</td>
<td>540</td>
<td>32</td>
<td>842</td>
<td>47</td>
<td>476</td>
</tr>
<tr>
<td>3</td>
<td>686</td>
<td>18</td>
<td>748</td>
<td>33</td>
<td>754</td>
<td>48</td>
<td>457</td>
</tr>
<tr>
<td>4</td>
<td>704</td>
<td>19</td>
<td>1049</td>
<td>34</td>
<td>654</td>
<td>49</td>
<td>685</td>
</tr>
<tr>
<td>5</td>
<td>529</td>
<td>20</td>
<td>730</td>
<td>35</td>
<td>794</td>
<td>50</td>
<td>563</td>
</tr>
<tr>
<td>6</td>
<td>919</td>
<td>21</td>
<td>714</td>
<td>36</td>
<td>408</td>
<td>51</td>
<td>448</td>
</tr>
<tr>
<td>7</td>
<td>911</td>
<td>22</td>
<td>625</td>
<td>37</td>
<td>685</td>
<td>52</td>
<td>555</td>
</tr>
<tr>
<td>8</td>
<td>527</td>
<td>23</td>
<td>637</td>
<td>38</td>
<td>693</td>
<td>53</td>
<td>706</td>
</tr>
<tr>
<td>9</td>
<td>460</td>
<td>24</td>
<td>316</td>
<td>39</td>
<td>304</td>
<td>54</td>
<td>469</td>
</tr>
<tr>
<td>10</td>
<td>259</td>
<td>25</td>
<td>564</td>
<td>40</td>
<td>421</td>
<td>55</td>
<td>787</td>
</tr>
<tr>
<td>11</td>
<td>1018</td>
<td>26</td>
<td>359</td>
<td>41</td>
<td>380</td>
<td>56</td>
<td>570</td>
</tr>
<tr>
<td>12</td>
<td>421</td>
<td>27</td>
<td>681</td>
<td>42</td>
<td>668</td>
<td>57</td>
<td>663</td>
</tr>
<tr>
<td>13</td>
<td>793</td>
<td>28</td>
<td>726</td>
<td>43</td>
<td>838</td>
<td>58</td>
<td>704</td>
</tr>
<tr>
<td>14</td>
<td>549</td>
<td>29</td>
<td>829</td>
<td>44</td>
<td>398</td>
<td>59</td>
<td>812</td>
</tr>
<tr>
<td>15</td>
<td>640</td>
<td>30</td>
<td>457</td>
<td>45</td>
<td>457</td>
<td>60</td>
<td>931</td>
</tr>
</tbody>
</table>

Based on this sample, the estimated total cereal production of the province is as follows:

$$\tilde{\tau} = \frac{183}{60} (276 + 738 + \cdots + 931) = 114245,$$
and the estimation of its variance is
\[v(\hat{\tau}) = N^2 \hat{s}^2 \left( \frac{1}{n} - \frac{1}{N} \right) = 12836085.\]

After estimating the total amount of cereal, we are interested in estimating the total amount of wheat and the total amount of barley produced in the province. Sample data for total wheat and barley at tonnes are given in Table 2.

<table>
<thead>
<tr>
<th>Cluster</th>
<th>Wheat</th>
<th>Barley</th>
<th>Cluster</th>
<th>Wheat</th>
<th>Barley</th>
<th>Cluster</th>
<th>Wheat</th>
<th>Barley</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>276</td>
<td>0</td>
<td>21</td>
<td>608</td>
<td>46</td>
<td>41</td>
<td>243</td>
<td>95</td>
</tr>
<tr>
<td>2</td>
<td>521</td>
<td>65</td>
<td>22</td>
<td>625</td>
<td>0</td>
<td>42</td>
<td>518</td>
<td>78</td>
</tr>
<tr>
<td>3</td>
<td>489</td>
<td>131</td>
<td>23</td>
<td>472</td>
<td>87</td>
<td>43</td>
<td>698</td>
<td>62</td>
</tr>
<tr>
<td>4</td>
<td>594</td>
<td>72</td>
<td>24</td>
<td>316</td>
<td>0</td>
<td>44</td>
<td>215</td>
<td>74</td>
</tr>
<tr>
<td>5</td>
<td>422</td>
<td>66</td>
<td>25</td>
<td>482</td>
<td>36</td>
<td>45</td>
<td>420</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>679</td>
<td>187</td>
<td>26</td>
<td>219</td>
<td>68</td>
<td>46</td>
<td>587</td>
<td>105</td>
</tr>
<tr>
<td>7</td>
<td>820</td>
<td>46</td>
<td>27</td>
<td>527</td>
<td>105</td>
<td>47</td>
<td>312</td>
<td>120</td>
</tr>
<tr>
<td>8</td>
<td>434</td>
<td>93</td>
<td>28</td>
<td>508</td>
<td>124</td>
<td>48</td>
<td>428</td>
<td>29</td>
</tr>
<tr>
<td>9</td>
<td>205</td>
<td>155</td>
<td>29</td>
<td>689</td>
<td>75</td>
<td>49</td>
<td>568</td>
<td>46</td>
</tr>
<tr>
<td>10</td>
<td>185</td>
<td>38</td>
<td>30</td>
<td>457</td>
<td>0</td>
<td>50</td>
<td>413</td>
<td>73</td>
</tr>
<tr>
<td>11</td>
<td>719</td>
<td>128</td>
<td>31</td>
<td>517</td>
<td>58</td>
<td>51</td>
<td>302</td>
<td>98</td>
</tr>
<tr>
<td>12</td>
<td>325</td>
<td>56</td>
<td>32</td>
<td>654</td>
<td>131</td>
<td>52</td>
<td>321</td>
<td>82</td>
</tr>
<tr>
<td>13</td>
<td>612</td>
<td>112</td>
<td>33</td>
<td>590</td>
<td>78</td>
<td>53</td>
<td>378</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>480</td>
<td>42</td>
<td>34</td>
<td>417</td>
<td>45</td>
<td>54</td>
<td>318</td>
<td>112</td>
</tr>
<tr>
<td>15</td>
<td>517</td>
<td>102</td>
<td>35</td>
<td>594</td>
<td>112</td>
<td>55</td>
<td>619</td>
<td>58</td>
</tr>
<tr>
<td>16</td>
<td>387</td>
<td>0</td>
<td>36</td>
<td>320</td>
<td>61</td>
<td>56</td>
<td>420</td>
<td>8</td>
</tr>
<tr>
<td>17</td>
<td>487</td>
<td>25</td>
<td>37</td>
<td>685</td>
<td>0</td>
<td>57</td>
<td>518</td>
<td>78</td>
</tr>
<tr>
<td>18</td>
<td>650</td>
<td>84</td>
<td>38</td>
<td>504</td>
<td>85</td>
<td>58</td>
<td>341</td>
<td>218</td>
</tr>
<tr>
<td>19</td>
<td>981</td>
<td>0</td>
<td>39</td>
<td>228</td>
<td>0</td>
<td>59</td>
<td>620</td>
<td>142</td>
</tr>
<tr>
<td>20</td>
<td>482</td>
<td>212</td>
<td>40</td>
<td>313</td>
<td>108</td>
<td>60</td>
<td>778</td>
<td>67</td>
</tr>
</tbody>
</table>

Based on the available information, we know that wheat was produced in all population clusters, so \(N_c = N = 183\) and to estimate the total amount of province’s
wheat production, we use $\widehat{\tau}_c$,

$$\widehat{\tau}_c = \frac{183}{60}(276 + 521 + \cdots + 778) = 90057.$$  

Hence $n_c$ is a constant number and the estimation of variance of this estimator is

$$v(\widehat{\tau}_c) = N_c^2 s_{cb}^2 \left( \frac{1}{n_c} - \frac{1}{N_c} \right) = 10482468.$$  

There is no information about the number of villages producing barley, so to estimate province’s barley production, $\tilde{\tau}_c$ is used, and we have

$$\tilde{\tau}_c = \frac{183}{60}(65 + 131 + \cdots + 67) = 13575.$$  

The estimation of the variance of this estimator is as follows:

$$v(\tilde{\tau}_c) = N_c^2 s_{cb}^2 \left( \frac{1}{n} - \frac{1}{N} \right) = 988189.$$  

One of the topics discussed in this section is the comparison of $MSE(\hat{\mu}_{cR})$ and $V(\hat{\mu}_c)$. Since $n_c$ is a random variable, this is not theoretically feasible. But they can be compared using a hypothetical example.

### 3 Simulated Study

In this section, we conduct a simulation study to assess the performance of the proposed estimators. Consider a population with 100 clusters. We randomly select 100 numbers from 500 to 2000 and consider them as the size of clusters in this population. Suppose that in this population there are 40 clusters containing members of the sub-population so that the number and size of members of the sub-population of these clusters are randomly selected from 1-100 and 1- $M_i$, respectively, where $M_i$ for $i = 1, 2, \ldots, 100$ is size of $i$-th cluster. We simulate the members of each cluster from Normal distribution with different means and variance equal to 1. Then we select samples with sizes ($n = 20, 40, 70$), repeated 1000 times, by simple random sampling method from the clusters of this population. Estimator $T = (\bar{\tau}_c, \bar{\tau}_c, \bar{\mu}_c, \bar{\mu}_{cR})'$ is obtained as follow:

$$T = \frac{1}{1000} \sum_{i=1}^{1000} T_i,$$
where $T_i$ is the value of the estimator in the $i$-th iteration. Details of the values of the estimators and their variances are given in Table 3 and Table 4. It should be noted that $\tau_c = 147354.6$, $M_c = 29726$ and $\mu_c = 4.957$.

Table 3: the values of $\tau_c$, $\tilde{\tau}_c$ and their variances

<table>
<thead>
<tr>
<th>n</th>
<th>$\tau_c$</th>
<th>$\tilde{\tau}_c$</th>
<th>$V(\tau_c)$</th>
<th>$V(\tilde{\tau}_c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>147237.9</td>
<td>146228.8</td>
<td>1760.04</td>
<td>2896.47</td>
</tr>
<tr>
<td>2</td>
<td>146653.5</td>
<td>146539.5</td>
<td>654.21</td>
<td>1184.59</td>
</tr>
<tr>
<td>3</td>
<td>147498.6</td>
<td>147659.6</td>
<td>187.88</td>
<td>326.48</td>
</tr>
</tbody>
</table>

Table 4: the values of $\mu_c$, $\hat{\mu}_c$ and their variances

<table>
<thead>
<tr>
<th>n</th>
<th>$\mu_c$</th>
<th>$\hat{\mu}_c$</th>
<th>$V(\mu_c)$</th>
<th>$V(\hat{\mu}_c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.953</td>
<td>5.000</td>
<td>1.991</td>
<td>1.212</td>
</tr>
<tr>
<td>2</td>
<td>4.933</td>
<td>4.964</td>
<td>0.740</td>
<td>0.444</td>
</tr>
<tr>
<td>3</td>
<td>4.961</td>
<td>4.961</td>
<td>0.212</td>
<td>0.124</td>
</tr>
</tbody>
</table>

According to Tables 3 and 4, it seems that estimator $\tilde{\tau}_c$ is more accurate than $\bar{\tau}_c$, and estimator $\hat{\mu}_c$ performs better than $\bar{\mu}_c$.

4 Conclusions

We introduced some estimators for the mean and total value subpopulation parameters under one-stage cluster sampling where the subpopulation size is known or unknown. The introduced estimators do not have a double cost because these estimators are based on a general sample taken from the population. In an example we also estimated the total value of wheat and barley in South Khorasan, in 2016, using a real data related to the amount of grain in this province. In addition, based on a simulation study, we observed the precision of the estimators and saw that $\tilde{\tau}_c$ and $\hat{\mu}_c$ are better estimators for $\tau_c$ and $\mu_c$, respectively.
References


