

## Optimal Allocation of Policy Layers for Exponential Risks

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**Abstract.** In this paper, we study the problem of optimal allocation of insurance layers for a portfolio of i.i.d exponential risks. Using the first stochastic dominance criterion, we obtain an optimal allocation for the total retain risks faced by a policyholder. This result partially generalizes the known result in the literature for deductible as well as policy limit coverages.

**Keywords.** Deductible policy, First stochastic dominance, Majorization, Policy limit, Schur-convex functions, Utility functions.

**MSC:** 60E15; 62P05.

### 1 Introduction

Let  $X$  be a loss faced by a policyholder which is a non-negative random variable, hereafter called a risk. Consider an insurance agreement under which not the whole risk is insured, but only a part of it. An example of such a coverage is an insurance

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layer  $X(d, d + l]$  which is defined by a pay-off function

$$X(d, d + l] = \begin{cases} 0, & \text{if } 0 < X \leq d, \\ X - d, & \text{if } d < X \leq d + l, \\ l, & \text{if } d + l < X, \end{cases}$$

where  $d$  and  $l$  are pre-specified values called deductible (or retention) and the policy limit, respectively (Wang, 1996, 2000). It follows from the layer contract that the risk  $X(d, d + l] = ((X - d)_+ \wedge l)$  is covered by the insurer and the remaining risk,  $X - X(d, d + l] = (X \wedge d) + (X - (l + d))_+$  is self-insured by the policyholder, where  $x_+ = \max\{0, x\}$  and  $x \wedge y = \min\{x, y\}$ . Insurance coverages are known as the deductible and the policy limit are particular cases of layer coverage. That is, when  $l = \infty$ , then it is equivalent to the deductible coverage and if  $d = 0$ , it is equivalent to the policy limit coverage (cf. Klugman et al., 2004). The stochastic properties and applications of the layer policy in different actuarial aspects have been studied in the literature. Wang (1996) investigated the problem of determining the premium principle for insurance layers. Goovaerts and Dhaene (1998) characterized Wang's class of premium principle. Sung et al. (2011) studied the optimal insurance policy in which the insurer's decision-making behavior is modeled by Kahneman and Tversky's Cumulative Prospect Theory with convex probability distortions. They showed that, under a fixed premium rate, an insurance layer can be an optimal insurance policy. Cheung et al. (2012) studied the optimal reinsurance decision problem in which the Average Value-at-Risk of the retained risk is minimized under Wang's premium principle. They showed that an insurance layer is an optimal reinsurance design under a budget constraint on reinsurance premium. Some other work related to the insurance layer can be found in Cui et al. (2013), Cheung et al. (2014), Zheng and Cui (2014), Assa (2015), Zhang and Liang (2016) and references therein.

Assume a policyholder is facing with non-negative random risks  $X_1, X_2, \dots, X_n$  which are insured under an insurance layer coverage. Suppose the amounts  $d$  and  $l$  are respectively the total deductible and the total policy limit amounts corresponding to all risks. The policyholder more often has the right to divide  $d$  and  $l$  into  $n$  non-negative values  $d_1, d_2, \dots, d_n$  and  $l_1, l_2, \dots, l_n$ , respectively, such that  $\sum_{i=1}^n d_i = d$  and  $\sum_{i=1}^n l_i = l$ , for which  $d_i$  and  $l_i$  are respectively the deductible and the policy limit of  $X_i$ ,  $i = 1, 2, \dots, n$ . In view of these considerations, the covered amount by the insurer is given by  $\sum_{i=1}^n [(X_i - d_i)_+ \wedge l_i]$  and the retained risk which is not covered by the insurance layer coverage is given by  $\sum_{i=1}^n [X_i - (X_i - d_i)_+ \wedge l_i]$ . From the viewpoint of the policyholder, an allocation is optimal which maximizes his/her wealth in some senses. Let  $w$  denote the initial wealth of the policyholder after paying the required

premium which is assumed to be independent of the choice of  $\mathbf{d} = (d_1, d_2, \dots, d_n)$  and  $\mathbf{l} = (l_1, l_2, \dots, l_n)$ . It is of great importance for the policyholder to determine the optimal vectors  $\mathbf{d}'$  and  $\mathbf{l}'$  in the set

$$s_n(d, l) = \{(d_1, d_2, \dots, d_n), (l_1, l_2, \dots, l_n) \mid \sum_{i=1}^n d_i = d, \sum_{i=1}^n l_i = l\}.$$

such that the amount  $w - \sum_{i=1}^n [X_i - (X_i - d_i)_+ \wedge l_i]$  is maximized in some senses. Several optimization criteria such as maximizing the expected utility, minimizing the variance, minimizing the probability of ruin can be considered to deal with this problem. For more details on these optimization criteria, the readers are referred to Van Heerwaarden *et al.* (1989), Denuit and Vermandele (1998). Using the notion of majorization and various types of stochastic orderings, the above problem for the particular cases of the policy limit ( $d_i = 0, i = 1, 2, \dots, n$ ) and the deductible policy ( $l_i = \infty, i = 1, 2, \dots, n$ ) have been studied by many researchers. Cheung (2007) investigated the optimal allocation of the deductibles and the policy limits when the risks are independent and ordered in the sense of the hazard rate order. Lu and Meng (2011) consider the same problems for the case when the risks are ordered according to the likelihood ratio order and each risk has log-concave density. Hu and Wang (2014) further investigated the optimal allocation of the deductibles and the policy limits and generalized several results of Cheung (2007) and Lu and Meng (2011). Fathi Manesh and Khaledi (2015) and Fathi Manesh *et al.* (2016) recently studied these optimization problems with the assumption that the risks are exchangeable with decreasing joint density function. For more results on the optimal allocation of the deductibles and the policy limits, we refer the reader to Hua and Cheung (2008a,b), Zhuang *et al.* (2009), Xu and Hu (2012) and references therein.

In this paper, we use to the maximization of the expected utility of wealth criterion to find the optimal allocation of the deductibles and the policy limits in layer policies. Let  $u(x)$  be the increasing utility function of the policyholder. Then, from the viewpoint of the policyholder, the optimization allocation problem of layers is formalized as

$$\begin{cases} \max_{(\mathbf{d}, \mathbf{l}) \in s_n(d, l)} E[u(w - \sum_{i=1}^n [X_i - (X_i - d_i)_+ \wedge l_i])], \\ \text{where } u \text{ is an increasing utility function.} \end{cases} \quad (1.1)$$

or, equivalently,

$$\begin{cases} \min_{(\mathbf{d}, \mathbf{l}) \in s_n(d, l)} E[\tilde{u}(\sum_{i=1}^n [X_i - (X_i - d_i)_+ \wedge l_i])], \\ \text{where } \tilde{u}(x) = u(w - x) \text{ is a decreasing function.} \end{cases} \quad (1.2)$$

In this paper, we consider the problem given in (1.1) for a particular case when  $X_1, X_2, \dots, X_n$  are identical and independent exponential risks. We use the notion of majorization which is one of the basic tools in probability and statistics to establish various inequalities. For any real vector  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , denote  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$  the increasing arrangement of  $x_1, x_2, \dots, x_n$ . A vector  $\mathbf{x} \in \mathbb{R}^n$  is said to be majorized by another vector  $\mathbf{y} \in \mathbb{R}^n$  (denoted by  $\mathbf{x} \leq_m \mathbf{y}$ ) if  $\sum_{i=1}^j x_{(i)} \geq \sum_{i=1}^j y_{(i)}$  for  $j = 1, 2, \dots, n-1$  and  $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$ . A real valued function  $\phi$  defined on a set  $\mathbb{A} \subset \mathbb{R}^n$  is said to be Schur-convex (Schur-concave) on  $\mathbb{A}$ , if  $\phi(\mathbf{x}) \leq (\geq) \phi(\mathbf{y})$  for any  $\mathbf{x}, \mathbf{y} \in \mathbb{A}$  such that  $\mathbf{x} \leq_m \mathbf{y}$ . For more details, the reader is referred to Marshall *et al.* (2011).

Now, we recall the definition of the first stochastic dominance ordering used later in this paper. Let  $X$  and  $Y$  be two random variables with distribution functions  $F$  and  $G$ , survival functions  $\bar{F}$  and  $\bar{G}$ , respectively. The random variable  $X$  is said to be smaller than the random variable  $Y$  in the first stochastic dominance order (denoted as  $X \leq_{st} Y$ ), if  $E[\phi(X)] \leq E[\phi(Y)]$  for all increasing functions  $\phi$  for which the expectations exist. It is well known that  $X \leq_{st} Y$  if and only if  $\bar{F}(t) \leq \bar{G}(t)$  for all  $t$ . For more details on the first stochastic dominance order, see, e.g. Müller and Stoyan (2002), Denuit *et al.* (2005) and Shaked and Shanthikumar (2007). The optimal allocation in (1.2) minimizes the retained risks, according to the first stochastic dominance order.

Let  $X_1, X_2, \dots, X_n$  be a set of  $n$  risks faced by a policyholder and  $\mathbf{d} = (d_1, d_2, \dots, d_n)$  and  $\mathbf{l} = (l_1, l_2, \dots, l_n)$  be the deductibles and limits vectors, respectively. In the following, we review some results related to the optimization problem for deductible ( $l_i = \infty, i = 1, \dots, n$ ) as well policy limit ( $d_i = 0, i = 1, 2, \dots, n$ ) coverages which are particular cases of layer coverage. For the case when  $X_i$ 's are independent and identically distributed, Xu and Hu (2012) proved that

$$\mathbf{l} \geq_m \mathbf{l}^* \implies \sum_{i=1}^n \phi(X_i - l_i) \geq_{st} \sum_{i=1}^n \phi(X_i - l_i^*), \quad (1.3)$$

where  $\phi$  is an arbitrary convex function. Fathi Manesh and Khaledi (2015) proved that

$$\mathbf{l} \geq_m \mathbf{l}^* \implies \sum_{i=1}^n (X_i - l_i)_+ \geq_{st} \sum_{i=1}^n (X_i - l_i^*)_+, \quad (1.4)$$

when  $X_i$ 's are exchangeable with decreasing joint density function. Fathi Manesh *et al.* (2016) showed that

$$\mathbf{d} \geq_m \mathbf{d}^* \implies \sum_{i=1}^n (X_i \wedge d_i) \leq_{st} \sum_{i=1}^n (X_i \wedge d_i^*), \quad (1.5)$$

when  $X_i$ 's are exchangeable with log-concave joint density function.

The exponential distribution is one of the simplest models for insurance risks which has been extensively considered in the actuarial context (see for example Dufresne and Gerber (1988), Panjer and Willmot (1992), Cheng *et al.* (2002), Borovkov and Dickson (2008) and Jin *et al.* (2016) ). In this paper, we partially generalize the above results for the case when the  $n$  risks  $X_1, X_2, \dots, X_n$  are independent and identical exponential risks and prove that

$$\mathbf{d} \geq_m \mathbf{d}^* \implies \sum_{i=1}^n [X_i - (X_i - d_i)_+ \wedge l_i] \leq_{st} \sum_{i=1}^n [X_i - (X_i - d_i^*)_+ \wedge l_i], \quad (1.6)$$

when  $l_1 = \dots = l_n = l'$  and

$$\mathbf{l} \geq_m \mathbf{l}^* \implies \sum_{i=1}^n [X_i - (X_i - d_i)_+ \wedge l_i] \geq_{st} \sum_{i=1}^n [X_i - (X_i - d_i)_+ \wedge l_i^*], \quad (1.7)$$

when  $d_1 = d_2 = \dots = d_n = d'$ .

## 2 Main Results

The following lemma, whose proof is given in Appendix, is about the optimal allocation problem given in (1.1), for the case when  $n = 2$ .

**Lemma 2.1.** *Let  $X_1$  and  $X_2$  be independent and identical exponential risks with common rate  $\lambda$ . Then,*

(a) *for  $(d_1, d_2), (d_1^*, d_2^*) \in \mathbb{R}^{+2}$  and  $l' \in \mathbb{R}^+$ ,*

$$(d_1^*, d_2^*) \leq_m (d_1, d_2) \implies X_{d_1, d_2}^{l', l'} \leq_{st} X_{d_1^*, d_2^*}^{l', l'}$$

(b) *for  $(l_1, l_2), (l_1^*, l_2^*) \in \mathbb{R}^{+2}$  and  $d' \in \mathbb{R}^+$ ,*

$$(l_1^*, l_2^*) \leq_m (l_1, l_2) \implies X_{d', d'}^{l_1^*, l_2^*} \leq_{st} X_{d', d'}^{l_1, l_2}$$

where  $X_{d_1, d_2}^{l_1, l_2} = \sum_{i=1}^2 [X_i - (X_i - d_i)_+ \wedge l_i]$ .

Next, we generalize the result of Lemma 2.1 (a) from  $n = 2$  to  $n > 2$ .

**Theorem 2.1.** Let  $X_1, X_2, \dots, X_n$  be independent and identical exponential risks with common rate  $\lambda$ . Then for  $\mathbf{d}, \mathbf{d}^* \in \mathbb{R}^{+n}$  and  $l' \in \mathbb{R}^+$ ,

$$\mathbf{d} \geq_m \mathbf{d}^* \implies \sum_{i=1}^n [X_i - (X_i - d_i)_+ \wedge l'] \leq_{st} \sum_{i=1}^n [X_i - (X_i - d_i^*)_+ \wedge l'].$$

*Proof.* Combining Lemma 2.1 and Theorem 1.A.3 of Shaked and Shanthikumar (2007), we obtain

$$\begin{aligned} \sum_{i=1}^2 [X_i - (X_i - d_i)_+ \wedge l'] + \sum_{i=3}^n [X_i - (X_i - d_i)_+ \wedge l'] &\leq_{st} \sum_{i=1}^2 [X_i - (X_i - d_i^*)_+ \wedge l'] \\ &\quad + \sum_{i=3}^n [X_i - (X_i - d_i)_+ \wedge l'], \end{aligned}$$

where  $\mathbf{d} = (d_1, d_2, d_3, \dots, d_n)$ ,  $\mathbf{d}^* = (d_1^*, d_2^*, d_3, \dots, d_n)$  and  $(d_1, d_2) \geq_m (d_1^*, d_2^*)$ . Now, the required result follows from Lemma 3.A.2.b of Marshall *et al.* (2011).  $\square$

Since,  $(\bar{d}, \dots, \bar{d}) \leq_m (d_1, d_2, \dots, d_n) \leq_m (\sum_{i=1}^n d_i, 0, \dots, 0)$ , it follows from Theorem 2.1 that, when the policy limits corresponding to the risks are equal, the vector  $(\sum_{i=1}^n d_i, 0, \dots, 0)$  maximizes the expected utility of the policyholder's wealth given in (1.1) which is the best allocation of deductibles. On the other hand  $(\bar{d}, \dots, \bar{d})$  minimizes the expected utility which is the worse allocation of the deductibles.

Suppose that  $X_1, X_2$  and  $X_3$  are independent exponential random variables with rate 0.1. In Figure 1, we graph the survival function of the random variable  $\sum_{i=1}^3 [X_i - (X_i - d_i)_+ \wedge l']$  for the selected deductibles allocations  $(8, 8, 8) \leq_m (4, 8, 12) \leq_m (4, 6, 14) \leq_m (0, 8, 16) \leq_m (0, 0, 24)$  and  $l' = 20$ . The figure demonstrates the concept of Theorem 2.1.

The following theorem generalizes the result of Lemma 2.1 (b) from  $n = 2$  to  $n > 2$ . Its proof is similar to that of Theorem 2.1 and hence is omitted.

**Theorem 2.2.** Let  $X_1, X_2, \dots, X_n$  be independent and identical exponential risks with common rate  $\lambda$ . Then for  $\mathbf{l}, \mathbf{l}^* \in \mathbb{R}^n$  and  $d' \in \mathbb{R}^+$ ,

$$\mathbf{l} \geq_m \mathbf{l}^* \implies \sum_{i=1}^n [X_i - (X_i - d')_+ \wedge l_i] \geq_{st} \sum_{i=1}^n [X_i - (X_i - d')_+ \wedge l_i^*]. \quad (2.1)$$

Since,  $(\bar{l}, \dots, \bar{l}) \leq_m (l_1, l_2, \dots, l_n) \leq_m (\sum_{i=1}^n l_i, 0, \dots, 0)$ , where  $\bar{l} = \frac{1}{n} \sum_{i=1}^n l_i$ , it follows from the above theorem that, when the deductibles are the same, the vector  $(\bar{l}, \dots, \bar{l})$  at

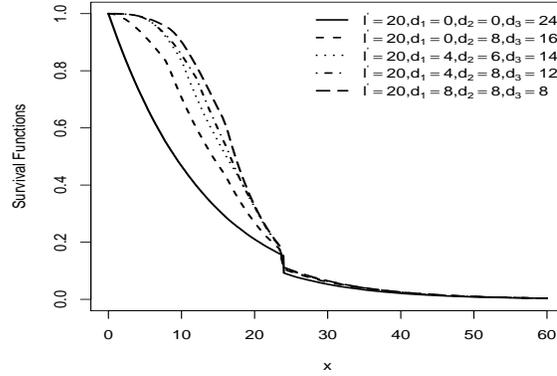


Figure 1: Survival functions of risk  $\sum_{i=1}^3 [X_i - (X_i - d_i)_+ \wedge l_i]$ .

which the expected utility of policyholder's wealth is maximized is the best allocation of the limits. On the other hand,  $(\sum_{i=1}^n l_i, 0, \dots, 0)$  is the worse allocation.

In Figure 2, we graph the survival function of  $\sum_{i=1}^3 [X_i - (X_i - d')_+ \wedge l_i]$  for the limits allocations  $(8, 8, 8) \leq_m (4, 8, 12) \leq_m (4, 6, 14) \leq_m (0, 8, 16) \leq_m (0, 0, 24)$  and  $d' = 8$ , where  $X_i$ 's are independent and identical exponential risks with rate  $\lambda = 0.1$ . The figure demonstrates the concept of Theorem 2.2.

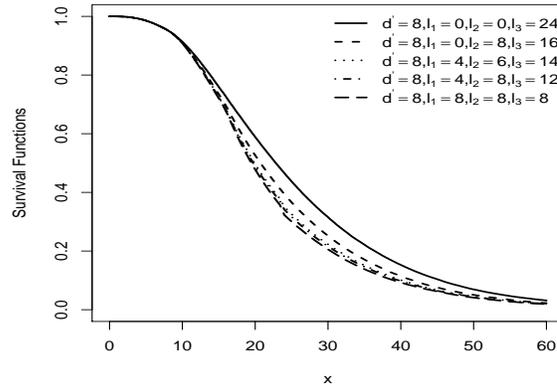


Figure 2: Survival functions of risk  $\sum_{i=1}^3 [X_i - (X_i - d')_+ \wedge l_i]$ .

*Remark 1.* The random variable  $X$  is said to be smaller than the random variable  $Y$  in the second stochastic dominance order (known as the increasing concave order and denoted by  $X \leq_{icv} Y$ ), if  $E[\phi(X)] \leq E[\phi(Y)]$  for all increasing concave functions  $\phi$  for which the expectations exist. It is clear that the first stochastic dominance order implies the second stochastic dominance order. Thus, the first stochastic dominance can be replaced by the second stochastic dominance in Theorems 2.1 and 2.2.

Now, a possible question is that whether there are a jointly optimal deductible and limit allocations for the problem given in (1.1). Combining the results of Theorems 2.1 and 2.2, the jointly optimal, consistent candidate of deductible and limit allocations is  $\mathbf{d}^* = (d, 0, \dots, 0)$  and  $\mathbf{l}^* = (l/n, \dots, l/n)$ . Next, a counterexample is given to illustrate that  $\mathbf{d}^*$  and  $\mathbf{l}^*$  might not be the jointly optimal allocation.

**Examples 2.1.** Let  $X_1$  and  $X_2$  be independent exponential risks with common rate  $\lambda = 0.1$ . Let  $d = d_1 + d_2 = 3$ ,  $l = l_1 + l_2 = 12$  and  $\bar{F}_{d_1, d_2}^{l_1, l_2}$  denote the survival function of  $\sum_{i=1}^2 [X_i - (X_i - d_i)_+ \wedge l_i]$ . We prepared a graph of  $\bar{F}_{0,3}^{6,6}(x)$  and  $\bar{F}_{0,3}^{12,0}(x)$  in Figure 3. We also evaluated  $\bar{F}_{0,3}^{6,6}(x) - \bar{F}_{0,3}^{12,0}(x)$  at different values of  $x$  to show that  $\mathbf{d}^* = (0, 3)$  and  $\mathbf{l}^* = (6, 6)$  is not the jointly optimal allocations. For example,  $\bar{F}_{0,3}^{6,6}(2.5) - \bar{F}_{0,3}^{12,0}(2.5) = -0.04726$  and  $\bar{F}_{0,3}^{6,6}(3) - \bar{F}_{0,3}^{12,0}(3) = 0.103079$  which shows that  $\bar{F}_{0,3}^{6,6}(x)$  is not less than  $\bar{F}_{0,3}^{12,0}(x)$  for all  $x \geq 0$ .

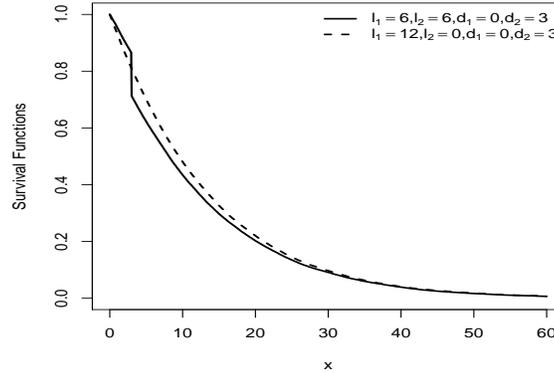


Figure 3: Survival functions of  $\sum_{i=1}^2 [X_i - (X_i - d_i)_+ \wedge l_i]$ .

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## Appendix

We need the following lemma to prove Lemma 2.1.

**Lemma 2.2.** *Let  $X_1$  and  $X_2$  be independent and identical exponential risks with common rate  $\lambda$ . Then for  $0 \leq d_1 \leq d_2$  and  $l_1, l_2 \geq 0$ ,*

$$F_{X_{d_1, d_2}^{l_1, l_2}}(t) = \begin{cases} 1 - (t\lambda + 1)e^{-\lambda t}, & \text{if } 0 \leq t < d_1, \\ 1 - (d_1\lambda + 1)e^{-\lambda t} - \lambda(t - d_1)e^{-\lambda(l_1+t)}, & \text{if } d_1 \leq t < d_2, \\ 1 - (\lambda(d_1 + d_2 - t) + 1)e^{-\lambda t} - \lambda(t - d_2)e^{-\lambda(t+l_2)} \\ - \lambda(t - d_1)e^{-\lambda(t+l_1)}, & \text{if } d_2 \leq t < d_1 + d_2, \\ 1 - (\lambda d_1 + 1)e^{-\lambda(t+l_2)} - (\lambda d_2 + 1)e^{-\lambda(t+l_1)} \\ - (\lambda(t - d_1 - d_2) - 1)e^{-\lambda(l_1+l_2+t)}, & \text{if } d_1 + d_2 \leq t. \end{cases}$$

*Proof.* It is easy to see that for  $x_1, x_2 \in \mathbb{R}^+$ ,

$$x_{d_1, d_2}^{l_1, l_2} = \begin{cases} x_1 + x_2, & x_1 < d_1, x_2 < d_2, \\ x_1 + d_2, & x_1 < d_1, d_2 \leq x_2 < d_2 + l_2, \\ x_1 + x_2 - l_2, & x_1 < d_1, d_2 + l_2 \leq x_2, \\ d_1 + x_2, & d_1 \leq x_1 < d_1 + l_1, x_2 < d_2, \\ d_1 + d_2, & d_1 \leq x_1 < d_1 + l_1, d_2 \leq x_2 < d_2 + l_2, \\ d_1 + x_2 - l_2, & d_1 \leq x_1 < d_1 + l_1, d_2 + l_2 \leq x_2, \\ x_1 - l_1 + x_2, & d_1 + l_1 \leq x_1, x_2 < d_2, \\ x_1 - l_1 + d_2, & d_1 + l_1 \leq x_1, d_2 \leq x_2 < d_2 + l_2, \\ x_1 - l_1 + x_2 - l_2, & d_1 + l_1 \leq x_1, d_2 + l_2 \leq x_2. \end{cases}$$

Now, the distribution function of  $X_{d_1, d_2}^{l_1, l_2}$  for  $t \in \mathbb{R}$  can be expressed as

$$F_{X_{d_1, d_2}^{l_1, l_2}}(t) = \int_0^\infty \int_0^\infty I(x_{d_1, d_2}^{l_1, l_2} \leq t) f(x_1, x_2) dx_2 dx_1,$$

where  $I(A) = 1$  if  $A$  occurs and  $I(A) = 0$  otherwise. If  $0 \leq t < d_1$ , then

$$\begin{aligned}
F_{X_{d_1, d_2}^{l_1, l_2}}(t) &= \int_0^{d_1} \int_0^{d_2} I(x_1 + x_2 \leq t) \lambda^2 e^{-\lambda(x_1 + x_2)} dx_2 dx_1 \\
&= \int_0^t \int_0^{t-x_1} \lambda^2 e^{-\lambda(x_1 + x_2)} dx_2 dx_1 \\
&= 1 - (t\lambda + 1)e^{-\lambda t}.
\end{aligned}$$

For  $d_1 \leq t < d_2$ ,

$$\begin{aligned}
F_{X_{d_1, d_2}^{l_1, l_2}}(t) &= \int_0^{d_1} \int_0^{d_2} I(x_1 + x_2 \leq t) \lambda^2 e^{-\lambda(x_1 + x_2)} dx_2 dx_1 \\
&\quad + \int_{d_1}^{d_1+l_1} \int_0^{d_2} I(d_1 + x_2 \leq t) \lambda^2 e^{-\lambda(x_1 + x_2)} dx_2 dx_1 \\
&\quad + \int_{d_1+l_1}^{\infty} \int_0^{d_2} I(x_1 - l_1 + x_2 \leq t) \lambda^2 e^{-\lambda(x_1 + x_2)} dx_2 dx_1 \\
&= \int_0^{d_1} \int_0^{t-x_1} \lambda^2 e^{-\lambda(x_1 + x_2)} dx_2 dx_1 + \int_{d_1}^{d_1+l_1} \int_0^{t-d_1} \lambda^2 e^{-\lambda(x_1 + x_2)} dx_2 dx_1 \\
&\quad + \int_{d_1+l_1}^{t+l_1} \int_0^{t+l_1-x_1} \lambda^2 e^{-\lambda(x_1 + x_2)} dx_2 dx_1 \\
&= 1 - (d_1\lambda + 1)e^{-\lambda t} - \lambda(t - d_1)e^{-\lambda(l_1+t)}.
\end{aligned}$$

Next, if  $d_2 \leq t < d_1 + d_2$ , then

$$\begin{aligned}
F_{X_{d_1, d_2}^{l_1, l_2}}(t) &= \int_0^{d_1} \int_0^{d_2} I(x_1 + x_2 \leq t) \lambda^2 e^{-\lambda(x_1 + x_2)} dx_2 dx_1 \\
&\quad + \int_0^{d_1} \int_{d_2}^{d_2+l_2} I(x_1 + d_2 \leq t) \lambda^2 e^{-\lambda(x_1 + x_2)} dx_2 dx_1 \\
&\quad + \int_0^{d_1} \int_{d_2+l_2}^{\infty} I(x_1 + x_2 - l_2 \leq t) \lambda^2 e^{-\lambda(x_1 + x_2)} dx_2 dx_1 \\
&\quad + \int_{d_1}^{d_1+l_1} \int_0^{d_2} I(d_1 + x_2 \leq t) \lambda^2 e^{-\lambda(x_1 + x_2)} dx_2 dx_1
\end{aligned}$$

$$\begin{aligned}
& + \int_{d_1+l_1}^{\infty} \int_0^{d_2} I(x_1 - l_1 + x_2 \leq t) \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
= & \int_0^{t-d_2} \int_0^{d_2} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 + \int_{t-d_2}^{d_1} \int_0^{t-x_1} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
& + \int_0^{t-d_2} \int_{d_2}^{d_2+l_2} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 + \int_0^{t-d_2} \int_{d_2+l_2}^{l_2+t-x_1} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
& + \int_{d_1}^{d_1+l_1} \int_0^{t-d_1} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 + \int_{d_1+l_1}^{t+l_1} \int_0^{l_1+t-x_1} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
= & 1 - (\lambda(d_1 + d_2 - t) + 1)e^{-\lambda t} - \lambda(t - d_2)e^{-\lambda(t+l_2)} - \lambda(t - d_1)e^{-\lambda(t+l_1)}.
\end{aligned}$$

Finally, if  $d_1 + d_2 \leq t < \infty$ , then

$$\begin{aligned}
F_{X_{d_1, d_2}^{l_1, l_2}}(t) = & \int_0^{d_1} \int_0^{d_2} I(x_1 + x_2 \leq t) \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
& + \int_0^{d_1} \int_{d_2}^{d_2+l_2} I(x_1 + d_2 \leq t) \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
& + \int_0^{d_1} \int_{d_2+l_2}^{\infty} I(x_1 + x_2 - l_2 \leq t) \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
& + \int_{d_1}^{d_1+l_1} \int_0^{d_2} I(d_1 + x_2 \leq t) \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
& + \int_{d_1}^{d_1+l_1} \int_{d_2}^{d_2+l_2} I(d_1 + d_2 \leq t) \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
& + \int_{d_1}^{d_1+l_1} \int_{d_2+l_2}^{\infty} I(d_1 + x_2 - l_2 \leq t) \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
& + \int_{d_1+l_1}^{\infty} \int_0^{d_2} I(x_1 - l_1 + x_2 \leq t) \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
& + \int_{d_1+l_1}^{\infty} \int_{d_2}^{d_2+l_2} I(x_1 - l_1 + d_2 \leq t) \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
& + \int_{d_1+l_1}^{\infty} \int_{d_2+l_2}^{\infty} I(x_1 - l_1 + x_2 - l_2 \leq t) \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
F_{X_{d_1, d_2}^{l_1, l_2}}(t) &= \int_0^{d_1+l_1} \int_0^{d_2+l_2} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 + \int_0^{d_1} \int_{d_2+l_2}^{l_2+t-x_1} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
&+ \int_{d_1}^{d_1+l_1} \int_{d_2+l_2}^{l_2+t-d_1} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 + \int_{d_1+l_1}^{l_1+t-d_2} \int_0^{d_2} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
&+ \int_{l_1+t-d_2}^{l_1+t} \int_0^{l_1+t-x_1} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 + \int_{d_1+l_1}^{l_1+t-d_2} \int_{d_2}^{d_2+l_2} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
&+ \int_{d_1+l_1}^{l_1+t-d_2} \int_{d_2+l_2}^{l_1+l_2+t-x_1} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\
&= 1 - (\lambda d_1 + 1)e^{-\lambda(t+l_2)} - (\lambda d_2 + 1)e^{-\lambda(t+l_1)} - (\lambda(t-d_1-d_2) - 1)e^{-\lambda(l_1+l_2+t)}.
\end{aligned}$$

Hence, the proof is completed.  $\square$

**Proof of Lemma 2.1.** We first prove part (a) of the lemma. We need to show that  $F_{X_{d_1, d_2}^{l', l'}}(t) \geq F_{X_{d_1^*, d_2^*}^{l', l'}}(t)$  for all  $t$ . Without loss of generality, assume that  $d_1 \leq d_2$  and  $d_1^* \leq d_2^*$ . From Lemma 2.2,

$$F_{X_{d_1, d_2}^{l', l'}}(t) = \begin{cases} 1 - (t\lambda + 1)e^{-\lambda t}, & \text{if } 0 \leq t < d_1, \\ 1 - (d_1\lambda + 1)e^{-\lambda t} - \lambda(t - d_1)e^{-\lambda(t+l')}, & \text{if } d_1 \leq t < d_2, \\ 1 - (\lambda(d_1 + d_2 - t) + 1)e^{-\lambda t} - \lambda(2t - d_1 - d_2)e^{-\lambda(t+l')}, & \text{if } d_2 \leq t < d_1 + d_2, \\ 1 - (\lambda d_1 + \lambda d_2 + 2)e^{-\lambda(t+l')} - (\lambda(t - d_1 - d_2) - 1)e^{-\lambda(t+2l')}, & \text{if } d_1 + d_2 \leq t. \end{cases}$$

It is easy to see that

$$F_{X_{d_1, d_2}^{l', l'}}(t) - F_{X_{d_1^*, d_2^*}^{l', l'}}(t) = \begin{cases} 0, & 0 \leq t < d_1, \\ \lambda(t - d_1)e^{-\lambda t} - \lambda(t - d_1)e^{-\lambda(l'+t)} \geq 0, & d_1 \leq t < d_1^*, \\ \lambda(d_1^* - d_1)e^{-\lambda t} - \lambda(d_1^* - d_1)e^{-\lambda(l'+t)} \geq 0, & d_1^* \leq t < d_2^*, \\ \lambda(d_1^* + d_2^* - t - d_1)e^{-\lambda t} \\ -\lambda(d_1^* + d_2^* - t - d_1)e^{-\lambda(l'+t)} \geq 0, & d_2^* \leq t < d_2, \\ 0, & d_2 \leq t < d_1 + d_2, \\ 0, & d_1 + d_2 \leq t \end{cases}$$

and the required result. Now, to prove Part (b), We need to show that  $F_{X_{d', d'}^{l_1^*, l_2^*}}(t) \geq F_{X_{d', d'}^{l_1, l_2}}(t)$  for all  $t$ . From Lemma 2.2,

$$F_{X_{d', d'}^{l_1, l_2}}(t) = \begin{cases} 1 - (t\lambda + 1)e^{-\lambda t}, & \text{if } 0 \leq t < d', \\ 1 - \lambda(t - d')(e^{-\lambda(t+l_2)} + e^{-\lambda(t+l_1)}) \\ -(\lambda(2d' - t) + 1)e^{-\lambda t}, & \text{if } d' \leq t < 2d', \\ 1 - (\lambda d' + 1)(e^{-\lambda(t+l_1)} + e^{-\lambda(t+l_2)}) \\ -(\lambda(t - 2d') - 1)e^{-\lambda(t+l_1+l_2)}, & \text{if } 2d' \leq t, \end{cases}$$

It is easy to see that

$$F_{X_{d', d'}^{l_1^*, l_2^*}}(t) - F_{X_{d', d'}^{l_1, l_2}}(t) = \begin{cases} 0, & 0 \leq t < d', \\ \lambda(t - d')(e^{-\lambda(t+l_2)} + e^{-\lambda(t+l_1)}) \\ -e^{-\lambda(t+l_2^*)} - e^{-\lambda(t+l_1^*)}, & d' \leq t < 2d', \\ (\lambda d' + 1)(e^{-\lambda(t+l_1)} + e^{-\lambda(t+l_2)}) \\ -e^{-\lambda(t+l_1^*)} - e^{-\lambda(t+l_2^*)}, & 2d' \leq t. \end{cases}$$

Now, by the assumption  $(l_1^*, l_2^*) \leq_m (l_1, l_2)$  and the fact that  $g(x_1, x_2) = \sum_{i=1}^2 e^{-\lambda x_i}$  is a Schur-convex function, the required result follows.  $\square$