

## Asymmetric Uniform-Laplace Distribution: Properties and Applications

Fatemeh Arezoomand<sup>1</sup>, Masoud Yarmohammadi<sup>2</sup> and Rahim Mahmoudvand<sup>3</sup>

<sup>1,2</sup>Department of Statistics, Payame Noor University, Tehran, Iran.

<sup>3</sup>Department of Statistics, Bu-Ali Sina University, Hamedan, Iran.

Received: 03/07/2017, Revision received: 04/04/2018, Published online: 06/08/2018

**Abstract.** The goal of this study is to introduce an Asymmetric Uniform-Laplace (AUL) distribution. We present a detailed theoretical description of this distribution. We try to estimate the parameters of AUL distribution using the maximum likelihood method. Since the likelihood approach results in complicated forms, we suggest a bootstrap-based approach for estimating the parameters. The proposed method is mainly based on the shape of the empirical density. We conduct a simulation study to assess the performance of the proposed procedure. We also fit the AUL distribution to real data sets: daily working time and Pontius data sets. The results show that AUL distribution is a more appropriate choice than the Skew-Normal, Skew t, Asymmetric Laplace and Uniform-Normal distributions.

**Keywords.** Estimation, Fitting, Uniform-Laplace distribution.

**MSC:** 62E15; 62E10.

---

Fatemeh Arezoomand (f\_arezoomand2007@yahoo.com)

Masoud Yarmohammadi (masyar@pnu.ac.ir)

Corresponding Author: Rahim Mahmoudvand (r.mahmodvand@gmail.com)

## 1 Introduction

The Laplace distribution, one of the oldest distributions in probability theory, is among the best choices whenever the distribution of the data reveals heavier than Gaussian tails. Laplace distribution has been advocated for use in many problems. We encourage the reader to review the bibliography of Kotz *et al.* (2001) which is impressive and contains over 100 references. In addition to the classical Laplace distribution, in recent years, several extensions have been proposed to adapt it and to improve its performance for different applications. Extensions to a skewed model can be found, for example, in Balakrishnan and Ambagaspitiya (1996), Yu and Moyeed (2001), Huang *et al.* (2003), Yu and Zhang (2005) and Harandi and Alamatsaz (2013). Nevertheless, the current forms of the Laplace distribution (both classical and generalized forms) have a sharp peak in the middle, which potentially restricts their usefulness. In the light of this issue, Mahmoudvand *et al.* (2015) introduced a modified classical Laplace distribution ( $MC\mathcal{L}$ ) with the probability density function (pdf)

$$f(x, \theta, \sigma) = \frac{1}{3\sigma} \begin{cases} \exp\left\{\frac{x-\theta}{\sigma}\right\} & x < \theta \\ 1 & \theta \leq x < \theta + \sigma \\ \exp\left\{-\frac{x-\theta-\sigma}{\sigma}\right\} & \theta + \sigma \leq x, \end{cases} \quad (1.1)$$

where  $\theta \in R$  and  $\sigma > 0$ . Equation (1.1) indicates that the  $MC\mathcal{L}$  distribution is symmetric. It also shows that the flatness parameter strongly depends on the scale parameter, which potentially restricts its usefulness. We must mention that  $MC\mathcal{L}$  might be considered as a different parametrization of the Uniform-Laplace distribution, briefly described by Scott (2010). Our objective in this research is to extend the  $MC\mathcal{L}$  distribution to a new modified asymmetric Laplace distribution.

Flat peaked densities have been used in sciences, engineering and economics. For instance, Arie *et al.* (1991) showed that the pdf of the fluctuating output intensity of a two-beam interferometer can be a highly peaked pdf. Another example, provided by Drop *et al.* (2007), assumes that a decision maker wishes to estimate his market share of a new clothing line in the upcoming season. He is convinced that it will most likely fall between 30% and 50%. Moreover, he believes that it is about twice as likely for this market share to fall within the range [30%, 50%] as compared to being either less than 30% or more than 50%. Figure 1 depicts a density that is consistent with his degree of beliefs.

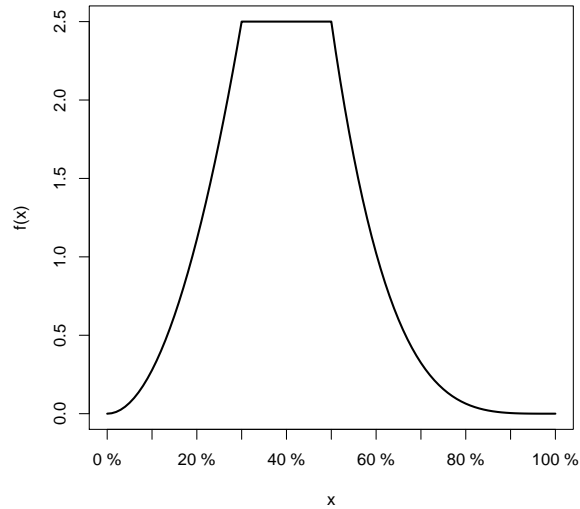


Figure 1: Graph of the pdf of the market share example

For a broader discussion of the earlier effort in this area, see Drop *et al.* (2007), Hamdan (2010), Scott (2010) and Mahmoudvand *et al.* (2015). In this paper we extend the distribution proposed by Mahmoudvand *et al.* (2015) to handle both asymmetry and a flat parameter.

The paper is organized as follows. In Section 2, we give a short illustration of the asymmetric Uniform-Laplace distribution. In Section 3, we provide the maximum likelihood estimates of the parameters. We compare the fits of two models, symmetric and asymmetric Uniform-Laplace distributions, to several real data sets in Section 4. In the last section, we summarize the contents of the paper and present the new ideas for further research in this direction.

## 2 Asymmetric Uniform-Laplace distribution

Using the suggestion by Mahmoudvand *et al.* (2015), a possible form for asymmetric Uniform-Laplace distribution is provided via the pdf

$$f(x; \theta, \mu, \delta, \sigma) = \frac{1}{\mu + \delta + \sigma} \begin{cases} \exp\left\{\frac{x-\theta}{\delta}\right\} & x < \theta \\ 1 & \theta \leq x < \theta + \mu \\ \exp\left\{-\frac{x-\theta-\mu}{\sigma}\right\} & \theta + \mu \leq x, \end{cases} \quad (2.1)$$

where  $\theta \in R$  is the location parameter,  $\mu > 0$  is the flatness parameter and  $\delta > 0$  and  $\sigma > 0$  are the left and right tail parameters, respectively. Here we use the notation  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$  for the asymmetric Uniform-Laplace distribution with the above probability density function. The probability density function  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$  may be expressed as a mixture involving three densities  $f_1(\cdot)$ ,  $f_2(\cdot)$  and  $f_3(\cdot)$  with bounded support, such that

$$f(x; \theta, \mu, \delta, \sigma) = \sum_{i=1}^3 \pi_i f_i(x; \theta, \mu, \delta, \sigma), \quad (2.2)$$

where

$$\begin{aligned} f_1(x; \theta, \mu, \delta, \sigma) &= \frac{1}{\delta} e^{\frac{x-\theta}{\delta}} I_{(-\infty, \theta)}(x) & \pi_1 &= \frac{\delta}{\mu + \delta + \sigma} \\ f_2(x; \theta, \mu, \delta, \sigma) &= \frac{1}{\mu} I_{(\theta, \theta + \mu)}(x) & \pi_2 &= \frac{\mu}{\mu + \delta + \sigma} \\ f_3(x; \theta, \mu, \delta, \sigma) &= \frac{1}{\sigma} e^{-\frac{x-\theta-\mu}{\sigma}} I_{(\theta + \mu, \infty)}(x) & \pi_3 &= \frac{\sigma}{\mu + \delta + \sigma}. \end{aligned}$$

We observe that the mixture weight to the first part,  $\pi_1$ , decreases as its tail parameter  $\delta$  increases. A similar observation can be made for the second and the third part with obvious modifications. It is worth noting that Equation (2.1) can be represented as

$$\frac{1}{\mu + \delta + \sigma} \exp\left\{-\frac{|x - \theta| - (x - \theta)}{2\delta} - \frac{|x - \theta - \mu| + (x - \theta - \mu)}{2\sigma}\right\}. \quad (2.3)$$

Allowing additional parameters offers the potential to fit more subtle features of the distribution than it is possible with two parameters, and describes the shape of tails more accurately. Figure 2 shows that the shape of the density (2.2) takes on various forms based on different values of  $\theta$ ,  $\mu$ ,  $\delta$  and  $\sigma$ . Positive skewness is produced when  $\delta < \sigma$  and negative skewness for  $\delta > \sigma$ .

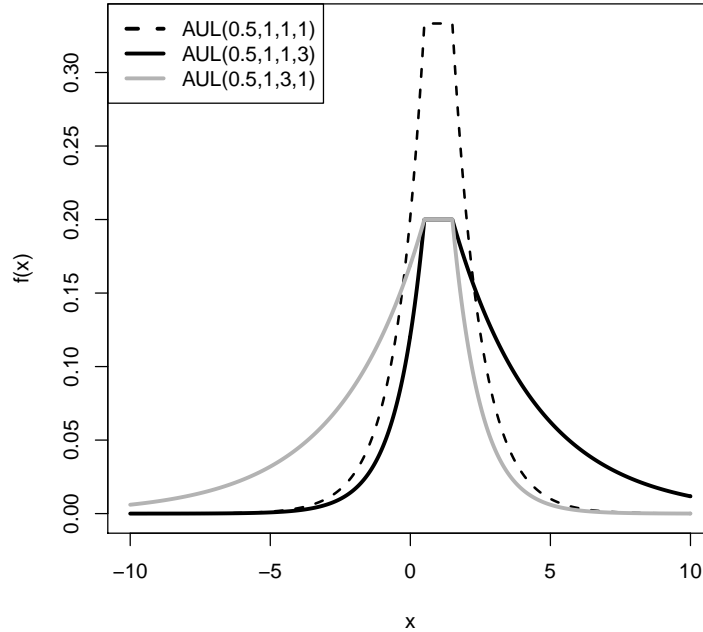


Figure 2: Graph of the pdf of the asymmetric Uniform-Laplace distribution for selected parameter values

The cumulative distribution function (cdf) of  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$  is given by

$$\frac{1}{\mu + \delta + \sigma} \begin{cases} \delta \exp\left\{\frac{x-\theta}{\delta}\right\} & x < \theta \\ x - \theta + \delta & \theta \leq x < \theta + \mu \\ \mu + \delta + \sigma \left(1 - \exp\left\{-\frac{x-\theta-\mu}{\sigma}\right\}\right) & \theta + \mu \leq x. \end{cases} \quad (2.4)$$

Figure 3, shows the plots of the cdf for the fixed values of  $\theta$  and  $\mu$  and different values of the scale parameters  $\delta$  and  $\sigma$ . Regarding the second piece of cdf, one can observe that for  $\sigma > \delta$  (right-skewed density) the slope of the graph is much steeper than the cases of  $\sigma < \delta$  and  $\sigma = \delta$  (symmetric and left-skewed densities). Thus, the area under the curve of the density in the second interval  $(\theta, \theta + \mu)$ , for  $\sigma > \delta$ , is less than two other graphs.

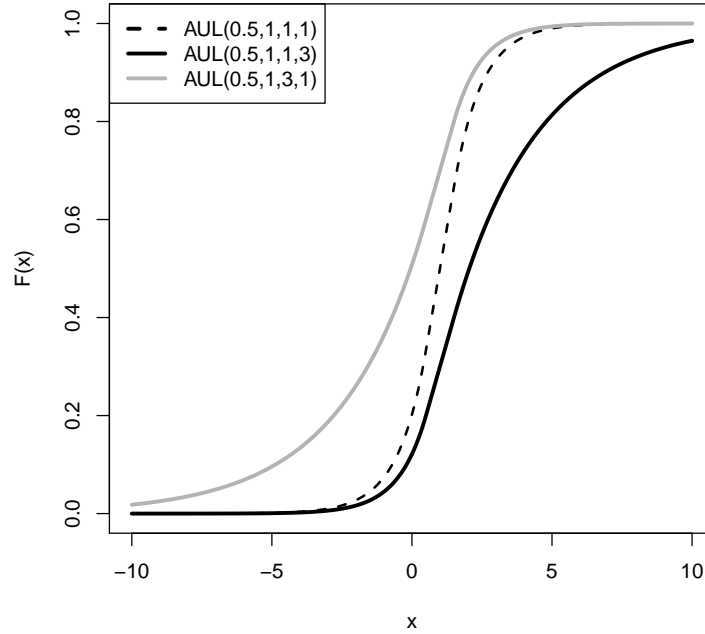


Figure 3: Graph of the cdf of the asymmetric Uniform-Laplace distribution for selected values of the parameters

The moment generating function of  $\mathcal{AUL}$  distribution is given by

$$M(t) = \frac{\exp\{\theta t\}}{\mu + \delta + \sigma} \left( \delta 1 + \delta t + \frac{\exp\{\mu t\} - 1}{t} + \frac{\sigma \exp\{\mu t\}}{1 - \sigma t} \right), \quad (2.5)$$

provided that  $-\frac{1}{\delta} \leq t \leq \frac{1}{\sigma}$ .

**Proposition 2.1.** Let  $X \sim \mathcal{AUL}(\theta, \mu, \delta, \sigma)$ . The  $k^{\text{th}}$  raw moment for  $X$  is given by

$$\begin{aligned} E(X^k) &= \frac{\delta}{\mu + \delta + \sigma} \sum_{t=0}^k \binom{k}{t} t! (-1)^t \theta^{k-t} \delta^t \\ &\quad + \frac{1}{\mu + \delta + \sigma} \left[ \frac{(\theta + \mu)^{k+1} - \theta^{k+1}}{k+1} \right] \\ &\quad + \frac{\sigma}{\mu + \delta + \sigma} \sum_{t=0}^k \binom{k}{t} t! \sigma^t (\theta + \mu)^{k-t}. \end{aligned} \quad (2.6)$$

**Proposition 2.2.** Let  $X \sim \mathcal{AUL}(\theta, \mu, \delta, \sigma)$ . Then, the positive skewness is produced when  $\delta < \sigma$  and negative skewness for  $\delta > \sigma$ .

*Proof.* Without loss of generality, we can assume that  $\theta = 0$ . Then, using Observation 1, we get

$$\begin{aligned} E(X) &= \frac{-2\delta^2 + \mu^2 + 2\mu\sigma + 2\sigma^2}{2(\mu + \sigma + \delta)} \\ E(X^2) &= \frac{6\delta^3 + \mu^3 + 3\mu^2\sigma + 6\sigma^2\mu + 6\sigma^3}{3(\mu + \sigma + \delta)} \\ E(X^3) &= \frac{-24\delta^4 + \mu^4 + 4\mu^3\sigma + 12\sigma^2\mu^2 + 24\sigma^3\mu + 24\sigma^4}{4(\mu + \sigma + \delta)}. \end{aligned} \quad (2.7)$$

Denoting the third central moment by  $M_3$ , we get

$$\begin{aligned} M_3 = E(X - E(X))^3 &= E(X^3) - 3E(X^2)E(X) + 2E^3(X) \\ &= \frac{(\sigma - \delta)B}{4(\mu + \delta + \sigma)^3}, \end{aligned} \quad (2.8)$$

where  $B$  is given as

$$\begin{aligned} B &= 8\sigma^5 + 32\delta\sigma^4 + 24\mu\sigma^4 + 24\mu^2\sigma^3 + 72\delta\mu\sigma^3 + 56\sigma^3\delta^2 + 8\mu^3\sigma^2 \\ &\quad + 48\mu^2\delta\sigma^2 + 96\delta^2\mu\sigma^2 + 56\delta^3\sigma^2 + 72\delta^3\mu\sigma + 32\delta^4\sigma + \mu^4\sigma + 12\mu^3\delta\sigma \\ &\quad + 48\mu^2\delta^2\sigma + 8\delta^5 + 24\delta^3\mu^2 + 8\mu^3\delta^2 + 24\delta^4\mu + \mu^4\delta. \end{aligned}$$

Since  $\mu, \delta$  and  $\sigma$  are positive, we have

$$M_3 = \begin{cases} > 0, & \text{if } \sigma > \delta \\ = 0, & \text{if } \sigma = \delta \\ < 0, & \text{if } \sigma < \delta, \end{cases}$$

which completes the proof.  $\square$

**Proposition 2.3.** Let  $X \sim \mathcal{AUL}(\theta, \mu, \delta, \sigma)$ . The quantile  $Q_p$  of  $X$  is written as

$$Q_p = \begin{cases} \delta \ln\left(\frac{p(\mu+\delta+\sigma)}{\delta}\right) + \theta & , p < \frac{\delta}{\mu+\delta+\sigma} \\ p(\mu + \delta + \sigma) + \theta - \delta & , \frac{\delta}{\mu+\delta+\sigma} \leq p < \frac{\mu+\delta}{\mu+\delta+\sigma} \\ -\sigma \ln\left(\frac{(\mu+\delta+\sigma)(1-p)}{\sigma}\right) + \theta + \mu & , p \geq \frac{\mu+\delta}{\mu+\delta+\sigma}. \end{cases} \quad (2.9)$$

Here we use the inverse transformation method to generate random sample from  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$ . It suffices to generate random samples from a uniform distribution. Then Equation (2.9) can be used to obtain quantiles.

**Proposition 2.4.** Rényi entropy (see Rényi , 1961) of  $\mathcal{AUL}$  is given by

$$\begin{aligned} H_\alpha(X) &= \frac{1}{1-\alpha} \log \left\{ \int f^\alpha(x) dx \right\} \\ &= \frac{1}{1-\alpha} \log \left\{ \left( \frac{1}{\mu + \delta + \sigma} \right)^\alpha \left( \frac{\delta}{\alpha} + \mu + \frac{\sigma}{\alpha} \right) \right\}, \end{aligned} \quad (2.10)$$

where  $\alpha > 0$ .

*Remark 1.* Shannon's entropy of  $\mathcal{AUL}$ , which is the limiting case of (2.7) for  $\alpha \rightarrow 1$  (see Shannon , 1948), is given by

$$H(X) = E(-\log(f(X))) = \log((\mu + \delta + \sigma)e^{\frac{\delta+\sigma}{\mu+\delta+\sigma}}). \quad (2.11)$$

Shanon's entropies of the classical and modified classical Laplace distribution  $\mathcal{CL}(\theta, \sigma)$  and  $\mathcal{MCL}(\theta, \sigma)$  are  $\log(2\sigma e)$  and  $\log(3\sigma e^{\frac{2}{3}})$ , respectively (Mahmoudvand et al. , 2015). We can easily conclude that if  $\mu + \delta > 2\sigma$  then the entropy of  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$  is larger than the entropy of both  $\mathcal{MCL}(\theta, \sigma)$  and  $\mathcal{CL}(\theta, \sigma)$ .

### 3 Parameter Estimation

Equation (2.7) and the general Equation (2.6) show that the method of moments involves solving large degree polynomials, without any guarantee that the produced solution will be numerically stable. Hence, we check the maximum likelihood approach. Let  $x_1, x_2, \dots, x_n$  be a random sample from  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$  and  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$  be



the corresponding ordered sample. Using Equation (2.3) the log-likelihood function is given by

$$l(\theta, \mu, \delta, \sigma) = -n \ln(\mu + \delta + \sigma) - \frac{1}{2\delta} \sum_{i=1}^n \{|x_i - \theta| - (x_i - \theta)\} \\ - \frac{1}{2\sigma} \sum_{i=1}^n \{|x_i - \theta - \mu| + (x_i - \theta - \mu)\}.$$

We consider several cases. For simplicity, we denote by  $\ell$  the log-likelihood function  $l(\theta, \mu, \delta, \sigma)$  and by  $\ell'_\theta, \ell'_\mu, \ell'_\delta$  and  $\ell'_\sigma$  the partial derivatives  $\frac{d}{d\theta}l(\theta, \mu, \delta, \sigma)$ ,  $\frac{d}{d\mu}l(\theta, \mu, \delta, \sigma)$ ,  $\frac{d}{d\delta}l(\theta, \mu, \delta, \sigma)$  and  $\frac{d}{d\sigma}l(\theta, \mu, \delta, \sigma)$ , respectively.

**Case 1:  $\mu, \sigma$  and  $\delta$  are known.**

Using notations  $y_i$  and  $w_i$  as

$$y_i = \begin{cases} x_i - \mu, & i = 1, \dots, n, \\ x_{i-n}, & i = n+1, \dots, 2n. \end{cases}, \quad w_i = \begin{cases} \frac{1}{\sigma}, & i = 1, \dots, n, \\ \frac{1}{\delta}, & i = n+1, \dots, 2n, \end{cases}$$

we can see that the log-likelihood function is proportional to

$$-n \ln(\mu + \delta + \sigma) - \frac{1}{2} \sum_{i=1}^{2n} w_{(i)} |y_{(i)} - \theta| + \frac{n\theta}{2} \left( \frac{1}{\sigma} - \frac{1}{\delta} \right),$$

where  $y_{(i)}$  are ordered values and  $w_{(i)}$  are corresponding coefficients. Then, we get

$$\ell'_\theta = \begin{cases} \frac{n}{\sigma}, & \theta < y_{(1)}, \\ \frac{n}{\sigma} - \sum_{i=1}^m w_{(i)} & y_{(m)} < \theta < y_{(m+1)}, \quad m = 1, \dots, 2n-1, \\ -\frac{n}{\delta}, & \theta > y_{(2n)}. \end{cases}$$

Therefore, assuming that  $p = \min\{m : \frac{n}{\sigma} - \sum_{i=1}^m w_{(i)} < 0\}$ , any statistic of the form

$$\lambda y_{(p)} + (1 - \lambda) y_{(p+1)}, \quad \lambda \in [0, 1] \quad (3.1)$$

may be taken as an MLE of the parameter  $\theta$  in this case.

**Case 2:  $\theta, \sigma$  and  $\delta$  are known.**

In this case, the log-likelihood function is equal to

$$\ell \propto -n \ln(\mu + \sigma + \delta) - \frac{1}{2\sigma} \sum_{i=1}^n |x_i - \theta - \mu| - \frac{1}{2\sigma} \sum_{i=1}^n (x_i - \theta) + \frac{n\mu}{2\sigma}.$$

Using this equation, we get

$$\ell'_\mu = -\frac{n}{\mu + \sigma + \delta} + \frac{n-m}{\sigma}, \quad x_{(m)} - \theta < \mu < x_{(m+1)} - \theta, \quad m = 0, \dots, n,$$

where, we set  $x_{(0)} = -\infty$  and  $x_{(n+1)} = \infty$ . Since  $\mu > 0$  and  $\ell'_\mu$  takes positive values  $-\frac{n}{\mu + \sigma + \delta} + \frac{n}{\sigma}$  for  $\mu < x_{(1)} - \theta$  and decreases with  $\mu$  and finally takes negative values on the interval  $(x_{(n)}, \infty)$ , then the MLE for  $\mu$  exists and equals to

$$\hat{\mu} = \frac{d}{n-d} \sigma - \delta, \quad (3.2)$$

in which

$$d = \min \left\{ m : -\frac{n}{\mu + \sigma + \delta} + \frac{n-m}{\sigma} \leq 0 \right\}.$$

**Case 3:  $\theta$  and  $\mu$  are known.**

In this case, we have

$$\begin{aligned} \ell'_\delta &= -\frac{n}{\mu + \delta + \sigma} + \frac{1}{2\delta^2} \sum_{i=1}^n \{|x_i - \theta| - (x_i - \theta)\} \\ \ell'_\sigma &= -\frac{n}{\mu + \delta + \sigma} + \frac{1}{2\sigma^2} \sum_{i=1}^n \{|x_i - \theta - \mu| + (x_i - \theta - \mu)\}. \end{aligned}$$

It is easy to see that

$$\ell'_\delta = \begin{cases} > 0 & , \quad \delta < \frac{\delta_0 + \sqrt{\delta_0^2 + 4\delta_0(\mu + \sigma)}}{2} \\ = 0 & , \quad \delta = \frac{\delta_0 + \sqrt{\delta_0^2 + 4\delta_0(\mu + \sigma)}}{2} \\ < 0 & , \quad \delta > \frac{\delta_0 + \sqrt{\delta_0^2 + 4\delta_0(\mu + \sigma)}}{2}, \end{cases}$$

where  $\delta_0 = \sum_{i=1}^n \{|x_i - \theta| - (x_i - \theta)\} / 2n$ . Similarly, we have

$$\ell'_\sigma = \begin{cases} > 0 & , \sigma < \frac{\sigma_0 + \sqrt{\sigma_0^2 + 4\sigma_0(\mu + \delta)}}{2} \\ = 0 & , \sigma = \frac{\sigma_0 + \sqrt{\sigma_0^2 + 4\sigma_0(\mu + \delta)}}{2} \\ < 0 & , \sigma > \frac{\sigma_0 + \sqrt{\sigma_0^2 + 4\sigma_0(\mu + \delta)}}{2} \end{cases},$$

where  $\sigma_0 = \sum_{i=1}^n \{|x_i - \theta - \mu| + (x_i - \theta - \mu)\} / 2n$ . The above results show that maximum likelihood estimates of  $\delta$  and  $\sigma$  exist and can be obtained via numerical methods. For instance, setting  $\ell'_\sigma$  equal to zero, substituting in the MLE  $\hat{\delta}_{MLE} = \frac{\delta_0 + \sqrt{\delta_0^2 + 4\delta_0(\mu + \sigma)}}{2}$  for  $\delta$  and solving by the numerical method give the MLE for  $\sigma$ . It is worth mentioning that

$$\begin{aligned} \hat{\delta}_{MLE} &\geq \delta_0 = \frac{\sum_{i=1}^n \{|x_i - \theta| - (x_i - \theta)\}}{2n}, \\ \hat{\sigma}_{MLE} &\geq \sigma_0 = \frac{\sum_{i=1}^n \{|x_i - \theta - \mu| + (x_i - \theta - \mu)\}}{2n}. \end{aligned}$$

#### Case 4: All parameters are unknown.

In this case, we have:

$$\begin{aligned} \ell'_\theta &= \frac{n}{2} \left( \frac{1}{\sigma} - \frac{1}{\delta} \right) + \frac{1}{2\delta} \sum_{i=1}^n \frac{|x_i - \theta|}{x_i - \theta} + \frac{1}{2\sigma} \sum_{i=1}^n \frac{|x_i - \theta - \mu|}{x_i - \theta - \mu} \\ \ell'_\mu &= -\frac{n}{\mu + \delta + \sigma} + \frac{1}{2\sigma} \sum_{i=1}^n \frac{|x_i - \theta - \mu|}{x_i - \theta - \mu} + \frac{n}{2\sigma} \\ \ell'_\delta &= -\frac{n}{\mu + \delta + \sigma} + \frac{1}{2\delta^2} \sum_{i=1}^n \{|x_i - \theta| - (x_i - \theta)\} \\ \ell'_\sigma &= -\frac{n}{\mu + \delta + \sigma} + \frac{1}{2\sigma^2} \sum_{i=1}^n \{|x_i - \theta - \mu| + (x_i - \theta - \mu)\}. \end{aligned}$$

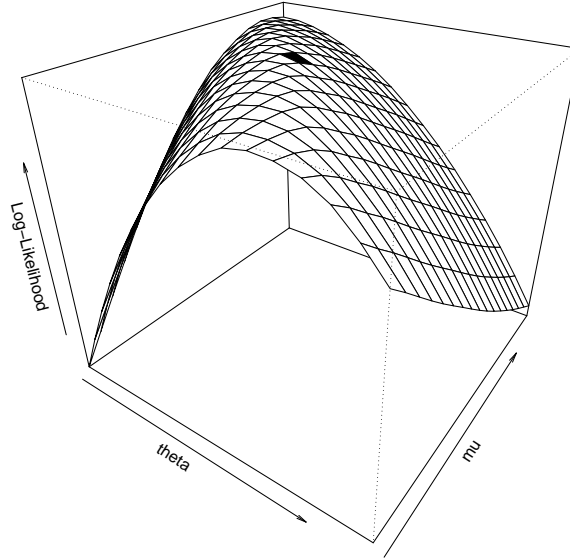


Figure 4: Graph of the function  $l(\theta, \mu, \delta, \sigma)$  for a fixed  $\delta$  and  $\sigma$  based on a simulated data set of size  $n = 200$  from  $AUL(0.4, 1.3, 3, 1.5)$

Using cases 1 and 2,

$$\ell'_\theta = \begin{cases} \text{positive,} & \theta + \mu < x_{(1)}, \\ \text{decreasing,} & y_{(2)} < \theta < y_{(2n)}, \\ \text{negative,} & \theta > x_{(n)}; \end{cases}$$

$$\ell'_\mu = \begin{cases} \text{positive,} & \theta + \mu < x_{(1)}, \\ \text{decreasing,} & x_{(1)} < \theta + \mu < x_{(n)}, \\ \text{negative,} & \theta + \mu > x_{(n)}. \end{cases}$$

Figure 3 shows a sample of the logarithm of the likelihood function for fixed  $\delta$  and  $\sigma$  with respect to  $\theta$  and  $\mu$ . This figure shows that the function has a maximum. We highlighted the maximum point on the surface by a black colour. We can conclude that  $\hat{\theta}_{MLE}$  and  $\hat{\mu}_{MLE}$  exist. In addition, using Case 3 we can find a solution for  $\ell'_\delta = 0$  and  $\ell'_\sigma = 0$  for each values of  $\theta$  and  $\mu$ . Thus,  $\hat{\delta}_{MLE}$  and  $\hat{\sigma}_{MLE}$  exist and can be obtained via numerical methods.

Considering the above inequalities, we should use numerical methods to produce the MLE for the parameters. However, there is no guarantee that the produced solution will be numerically stable. Therefore, we suggest the following approximation for obtaining estimators. Since the mode of  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$  can be any number between  $\theta$  and  $\theta + \mu$ , we suggest a bootstrap approach to find an estimator for the parameters  $\theta$  and  $\mu$ .

### 3.1 Bootstrap-based Estimators

Using the probability density function of  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$  it is obvious that the mode is  $M = \{x : \theta \leq x \leq \theta + \mu\}$ . Thus, we get

$$\theta = \min M, \quad \mu = \max M - \min M. \quad (3.3)$$

Intuitively speaking,  $\theta$  and  $\mu$  could be estimated by substituting  $M$  by the sample mode. We use a Bootstrap-based algorithm to find these estimates. Assume that  $x_1, \dots, x_n$  is a sample from an underlying cdf  $F(x)$ , and  $\hat{F}(x)$  is the corresponding empirical cdf. Repeat the following steps for  $b = 1, \dots, B$ :

- (i) Generate a copy  $x_1^{(b)}, \dots, x_n^{(b)}$  from original observations  $x_1, \dots, x_n$ ,
- (ii) Compute the sample mode of the bootstrap sample  $x_1^{(b)}, \dots, x_n^{(b)}$  and denote it by  $\tilde{x}^{(b)}$ .

Then, we have

$$\hat{\theta} = \min_{b=1, \dots, B} \{\tilde{x}^{(b)}\}, \quad \hat{\mu} = \max_{b=1, \dots, B} \{\tilde{x}^{(b)}\} - \hat{\theta}. \quad (3.4)$$

Using these estimations, we get

$$\widehat{F(\theta)} = \hat{F}(\hat{\theta}), \quad 1 - \widehat{F(\theta + \mu)} = \hat{F}(\hat{\theta} + \hat{\mu}). \quad (3.5)$$

Now, we are able to estimate  $\delta$  and  $\sigma$  by the method of maximum likelihood. Note that

$$\begin{aligned} \ell'_\delta &= -\frac{n}{\delta}F(\theta) + \frac{1}{2\delta^2} \sum_{i=1}^n \{|x_i - \theta| - (x_i - \theta)\} \\ \ell'_\sigma &= -\frac{n}{\sigma}(1 - F(\theta + \mu)) + \frac{1}{2\sigma^2} \sum_{i=1}^n \{|x_i - \theta - \mu| + (x_i - \theta - \mu)\}. \end{aligned}$$

Substituting  $F(\theta)$  and  $F(\theta + \mu)$  by  $\hat{F}(\hat{\theta})$  and  $\hat{F}(\hat{\theta} + \hat{\mu})$  we get

$$\hat{\delta} = \frac{1}{2n\hat{F}(\hat{\theta})} \sum_{i=1}^n \{|x_i - \hat{\theta}| - (x_i - \hat{\theta})\} \quad (3.6)$$

$$\hat{\sigma} = \frac{1}{2n(1 - \hat{F}(\hat{\theta} + \hat{\mu}))} \sum_{i=1}^n \{|x_i - \hat{\theta} - \hat{\mu}| + (x_i - \hat{\theta} - \hat{\mu})\}. \quad (3.7)$$

It is worth mentioning that there are several different ways to calculate the sample mode, but none is superior to others. The simplest mode estimator is the midpoint of the estimated modal interval, the interval of some fixed width  $w$  that includes the maximum number of data points of a sample (Chernoff, 1964). This estimator is called *Chernoff* or *Navie* estimator. Wegman (1971) proved that this simple estimator of the mode has a strong consistency. More complicated estimators have been proposed by Venter (1967) and Lientz (1970). In the Venter method, the modal interval, i.e. the shortest interval among intervals containing  $k+1$  observations, is first computed. Then, the median of the modal interval is returned as the mode. The Lientz mode estimator is the value minimizing the Lientz function estimate. The Lientz function is the smallest non-negative quantity  $S(x, \beta)$ , such that

$$F(x + S(x, \beta)) - F(x - S(x, \beta)) \geq \beta,$$

where  $\beta$  is equal to the the bandwidth.

Two other relatively new estimators are the Half-Sample Mode and Half-Range Mode. Although there are some comparisons between these methods in the literature, these are not systematic or extensive, see for example Bickel (2002), Blair Hedges and Shah (2003) and Bickel and Fruehwirth (2006).

R package *modeest* provides different estimators of the mode of univariate distributions, see Poncet (2012). We perform a simulation study in order to see the efficiency of the bootstrap-based method for  $\mathcal{AUL}$  distribution. We generate 1000 samples of size  $n = 50, 100$  and  $200$  from  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$  with  $\theta = 5, \mu = 3, \delta = 1$  and  $\sigma = 0.5$ . We have reported the results for three methods of the mode estimation: *Naive*, *Lientz* and *Venter*. The results are presented in Table 1. Of the three methods tested, the Naive estimator is preferred for large  $n$  because it performed better than the Lientz and Venter estimators in terms of lowering bias. However, the Lientz estimator is preferred for small  $n$  in terms of lowering bias. The results show that the Venter estimator produced minimum MSE in 7 out of 12 cases versus 4 with Lientz and 2 with Naive.

Table 1: Bootstrap-based estimates of the parameters of the  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$  distribution for simulated data when  $\theta = 5$ ,  $\mu = 3$ ,  $\delta = 1$  and  $\sigma = 0.5$  and for  $n = 50, 100$  and  $200$ .

Parameter	$n$	Method					
		Naive		Lientz		Venter	
		Estimate	MSE	Estimate	MSE	Estimate	MSE
$\theta$	50	4.42	0.61	<b>4.67</b>	0.31	4.66	<b>0.29</b>
	100	4.69	0.23	4.93	0.13	<b>4.95</b>	<b>0.11</b>
	200	<b>4.88</b>	0.10	5.18	<b>0.09</b>	5.21	0.11
$\mu$	50	3.69	0.90	<b>3.22</b>	<b>0.09</b>	3.24	0.31
	100	3.32	0.33	<b>2.75</b>	<b>0.22</b>	2.73	0.24
	200	<b>3.01</b>	<b>0.15</b>	2.31	0.56	2.28	0.60
$\delta$	50	0.89	0.18	<b>0.96</b>	0.14	0.94	<b>0.13</b>
	100	0.97	0.07	<b>0.98</b>	0.06	<b>0.98</b>	<b>0.05</b>
	200	0.99	0.03	<b>1.00</b>	<b>0.02</b>	1.02	<b>0.02</b>
$\sigma$	50	<b>0.48</b>	0.06	0.53	0.05	<b>0.52</b>	<b>0.04</b>
	100	<b>0.49</b>	0.03	0.58	0.03	0.57	<b>0.02</b>
	200	<b>0.50</b>	<b>0.01</b>	0.62	0.03	0.63	0.03

### 3.2 Asymptotic Normality

Let us start with the derivation of the Fisher information matrix,  $\mathbf{I}(\theta, \mu, \delta, \sigma)$ , corresponding to an  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$  distribution. We may note that the  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$  does not completely satisfy the standard differentiability assumptions required for the computation of the Fisher information matrix, since its density is not differentiable with respect to  $\theta$  and  $\mu$  at some points. However, the following relation is valid under a weaker assumption that the density is absolutely continuous, which is the case for the  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$ , i.e.,

$$\mathbf{I}(\theta, \mu, \delta, \sigma) = nE(DD^T),$$

where  $D^T = \left[ \frac{d}{d\theta} \log f, \frac{d}{d\mu} \log f, \frac{d}{d\delta} \log f, \frac{d}{d\sigma} \log f \right]$  is the vector of the partial derivative of density  $f$  with respect to the parameters  $\theta, \mu, \delta$  and  $\sigma$ . One can see easily that

$$\begin{aligned} E\left(\frac{|X - \theta|}{X - \theta}\right) &= \frac{-\delta + \mu + \sigma}{\delta + \mu + \sigma} \\ E\left(\frac{|X - \theta - \mu|}{X - \theta - \mu}\right) &= \frac{-\delta - \mu + \sigma}{\delta + \mu + \sigma} \\ E\left(\frac{|X - \theta|}{X - \theta} \times \frac{|X - \theta - \mu|}{X - \theta - \mu}\right) &= \frac{\delta - \mu + \sigma}{\delta + \mu + \sigma}. \end{aligned}$$

Using the above equations after routine calculations we obtain

$$\mathbf{I}(\theta, \mu, \delta, \sigma) = \begin{bmatrix} \frac{n(\sigma+\delta)}{\sigma\delta(\mu+\delta+\sigma)} & \frac{n}{\sigma(\mu+\delta+\sigma)} & -\frac{n}{\delta(\mu+\delta+\sigma)} & +\frac{n}{\sigma(\mu+\delta+\sigma)} \\ \frac{n}{\sigma(\mu+\delta+\sigma)} & \frac{n(\mu+\delta)}{\sigma(\mu+\delta+\sigma)^2} & -\frac{n}{(\mu+\delta+\sigma)^2} & \frac{n}{\sigma} \frac{\mu+\delta}{(\mu+\delta+\sigma)^2} \\ -\frac{n}{\delta(\mu+\delta+\sigma)} & -\frac{n}{(\mu+\delta+\sigma)^2} & \frac{n\delta+2n\mu+2n\sigma}{\delta(\mu+\delta+\sigma)^2} & -\frac{n}{(\mu+\delta+\sigma)^2} \\ \frac{n}{\sigma(\mu+\delta+\sigma)} & \frac{n}{\sigma} \frac{\mu+\delta}{(\mu+\delta+\sigma)^2} & -\frac{n}{(\mu+\delta+\sigma)^2} & \frac{n\sigma+2n\mu+2n\delta}{\sigma(\mu+\delta+\sigma)^2} \end{bmatrix}. \quad (3.8)$$

Denoting the asymptotic covariance matrix of  $(\hat{\theta}, \hat{\mu}, \hat{\delta}, \hat{\sigma})$  by  $\Sigma$  and taking the inverse of the above Fisher information matrix, we get

$$\Sigma = \left( \frac{\mu + \sigma + \delta}{n} \right) \begin{bmatrix} \frac{\delta(\delta+2\mu)}{\mu} & -\frac{\delta(\sigma+\delta+2\mu)}{\mu} & \delta & 0 \\ -\frac{\delta(\sigma+\delta+2\mu)}{\mu} & \frac{(\sigma+\delta)(\sigma+\delta+2\mu)}{\mu} & -\delta & -\sigma \\ \delta & -\delta & \delta & 0 \\ 0 & -\sigma & 0 & \sigma \end{bmatrix}.$$

We conduct a simulation study in order to calculate the coverage probabilities and confidence lengths of the confidence intervals obtained via the asymptotic normal distribution assumed for parameters estimations according to the sample size. The simulation size was 1000 and the Naive method was used to obtain the estimates for the parameters.



Table 2: Coverage probability (C.P) and Confidence length (C.L) of the confidence interval 95% for the parameters of the  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$  distribution using simulated data when  $\theta = 5, \mu = 3, \delta = 1$  and  $\sigma = 0.5$  and for  $n = 50, 100$  and  $200$ .

$n$	Parameter							
	$\theta$		$\mu$		$\delta$		$\sigma$	
	C.P	C.L	C.P	C.L	C.P	C.L	C.P	C.L
50	0.845	1.72	0.934	2.22	0.824	1.13	0.942	0.82
100	0.907	1.22	0.941	1.59	0.884	0.70	0.948	0.59
200	0.928	0.87	0.943	1.14	0.911	0.57	0.951	0.42

Table 2 shows the results. This Table indicates that the coverage probability increases with sample size and approaches to 0.95 for all estimates. It shows also that the confidence length decreases with the sample size in all cases.

## 4 Real data

In this section we discuss a comparison of the asymmetric Uniform-Laplace distribution with other distributions via real data sets. Kolmogorov-Smirnov (K-S) test has been employed to evaluate the efficiency of the new proposal.

### Daily Working Time

The OECD Jobs Strategy recommends that the governments take measures aimed at increasing working-time flexibility (see OECD , 2004). A common work pattern in Iranian offices is to begin between 7:30 or 8:00 AM and end at 3:30 PM. We consider daily working time (in hours) for 374 samples in Bu-Ali Sina University, Iran. The MLE and associated MSEs of the parameters for  $\mathcal{AUL}$  are given in Table 3. Since the estimated  $\delta$  is less than  $\sigma$ , we decided to fit and compare the other skew models to this data set. We considered Skew-Normal (Azzalini and Capitanio , 2014), Skew-t (Azzalini and Capitanio , 2014) and Asymmetric Laplace distributions (Yu and Zhang , 2005). Computations for these alternatives were done by R packages *sn* and *ald*. For this data set, the K-S statistic and its P-value are presented in Table 4. This table shows a considerable improvement in terms of the K-S test. We have also checked graphical fitting of the  $\mathcal{AUL}$  to this data set. Figure 5 indicates that the  $\mathcal{AUL}$  fits to this data set very well.

Table 3: MLE of the parameters of the  $\mathcal{AUL}(\theta, \mu, \delta, \sigma)$  distribution for the daily working time data

$\theta$		$\mu$		$\delta$		$\sigma$	
Estimate	MSE	Estimate	MSE	Estimate	MSE	Estimate	MSE
7.50123	0.0205	0.85231	0.014	0.23022	0.0011	0.33124	0.0063

Table 4: K-S test for fitting distributions to daily wage

	$\mathcal{AUL}$	Skew-Normal	Skew-t	Asymmetric Laplace
K-S	<b>0.0602</b>	0.2127	0.0752	0.1252
P-value	<b>0.1276</b>	1.1e-15	0.0276	1.4e-05

### Pontius Data Set

The Pontius data set is from the National Institute of Standards and Technology. We chose Pontius data set because Hamdan (2010) investigated the possibility of fitting a flat density to this data set. He used the density

$$f(x; \theta, \sigma) = \frac{1}{\sqrt{2\pi\sigma} + 2\theta} \begin{cases} e^{-\frac{(x+\theta)^2}{2\sigma^2}} & x < \theta \\ 1 & |x| \leq \theta \\ e^{-\frac{(x-\theta)^2}{2\sigma^2}} & x > \theta. \end{cases} \quad (4.1)$$

Since the data seems to be symmetric, we fitted a special case of  $\mathcal{AUL}$  with  $\sigma = \delta$ . In addition, we compared fitting with the flat density  $\mathcal{FD}$  given by Equation (4.1). For this density we used the estimate that Hamdan (2010) produced. The Kolmogorov-Smirnov statistic for  $\mathcal{AUL}$  is 0.04 with a P-value equal to 0.999. Hamdan (2010) reported the Kolmogorov-Smirnov statistic with  $\mathcal{FD}$  as 0.12 with a P-value equal to 0.541. Therefore, we find an improvement by  $\mathcal{AUL}$  when it is compared with  $\mathcal{FD}$ , in terms of the Kolmogorov-Smirnov statistic. Additionally, the graphical comparison is provided by Figure 6. As it is pointed out,  $\mathcal{AUL}$  fits quite well.

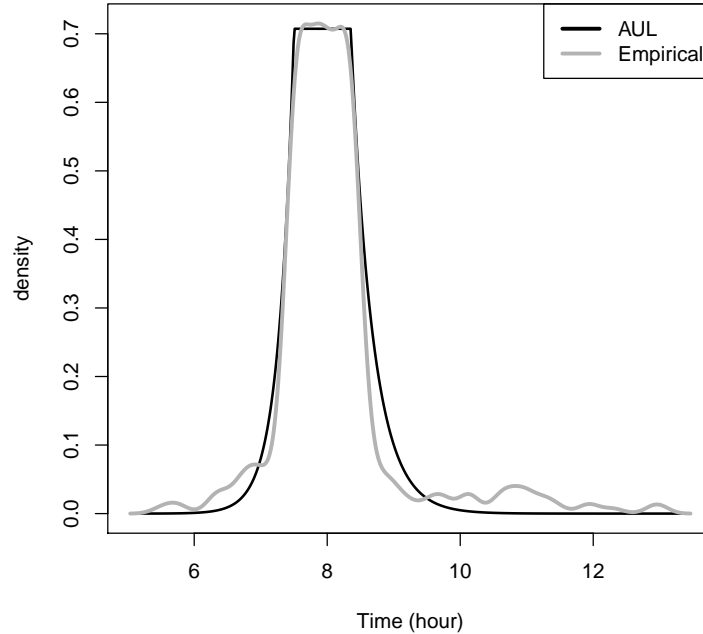


Figure 5: Graphical fitting of  $AUL$  to Daily working time data set

## 5 Conclusion

This study is an attempt aimed at investigating a new form of the modified Laplace distribution which is asymmetric and flat in the middle. We have derived the analytical forms of moments, the moment generating function, quantiles and Shannon's entropy for this distribution. Additionally, the condition in which the new proposed distribution is more informative than the modified symmetric and classical Laplace distributions has been provided.

We have presented statistical comparisons of asymmetric Uniform-Laplace distributions via two real data sets namely daily working time in Iran and Pontius data set. Application of the new distribution has proved that it can be used quite effectively to provide a better fit than other distributions when there is a flatness in the middle. It might be interesting to extend the results of this paper to other distributions.

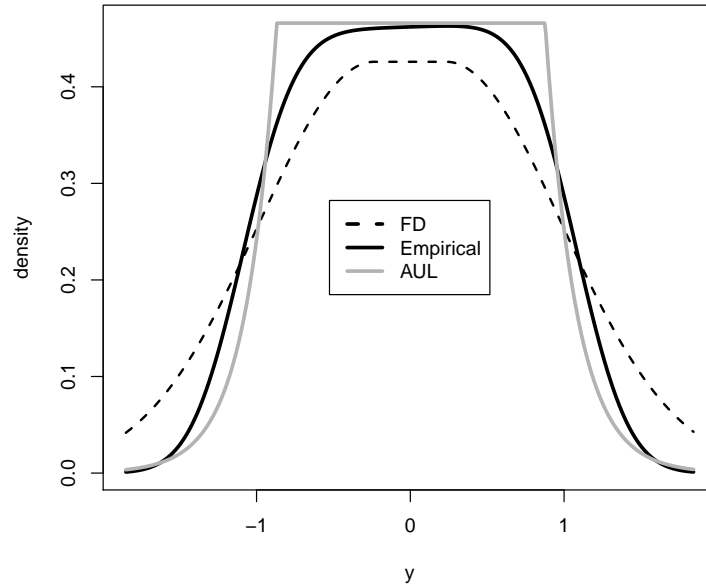


Figure 6: Graphical comparison of  $\mathcal{AUL}$  and  $\mathcal{FD}$  to Pontius data set

## References

- Arie, A., Tur, M., Goldstein, E. L. (1991). Probability-density function of noise at output of a two-beam interferometer. *JOSA*, **A 8**, 1936–1942.
- Azzalini, A., Capitanio, A. (2014). *The Skew-normal and Related Families*. Cambridge University Press, IMS Monographs Series.
- Balakrishnan, N. and Ambagaspitiya, R. S. (1994). *On skew-Laplace distributions*. Technical Report, Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada.
- Bickel DR. (2002). Robust estimators of the mode and skewness of continuous data. *Computational Statistics and Data Analysis*, **39**, 153–163.

- Bickel D.R. and Fruehwirth R. (2006). On a Fast, Robust Estimator of the Mode: Comparisons to Other Robust Estimators with Applications. *Computational Statistics and Data Analysis*, **50**(12), 3500–3530.
- Blair Hedges, S., and Shah, P. (2003). Comparison of mode estimation methods and application in molecular clock analysis. *BMC Bioinformatics*, **4**, 31–41.
- Chernoff H. (1964). Estimation of the mode. *Annals of the Institute of Statistical Mathematics*, **16**, 31–41.
- Dorp, J. R. V., Rambaud, S. C., Pérez, J. G. and Pleguezuelo, R. H. (2007). An Elicitation Procedure for the Generalized Trapezoidal Distribution with a Uniform Central Stage. *Decision Analysis Journal*, **4**, 156–166.
- Harandi, S. S., & Alamatsaz, M. H. (2013). Alpha-Skew-Laplace distribution. *Statistics & Probability Letters*, **83**(3), 774-782.
- Huang, H., Jiang, Z., Yu, K., & Lu, Z. (2003). A skewed Laplace distribution with financial application. *Financial Systems Engineering*, **19**(2), 39-52.
- Hamdan, H. (2010). Modeling data with flat densities. *Communications in Statistics Theory and Methods*, **39**, 722–728.
- Kotz, S., Kozubowski, T.J. and Podgi, K., (2001). Asymmetric multivariate Laplace distribution. In *The Laplace Distribution and Generalizations* (pp. 239-272). Birkher, Boston, MA.
- Lientz B.P. (1970). Results on nonparametric modal intervals. *SIAM Journal on Applied Mathematics*, **19**, 356–366.
- Mahmoudvand, R.; Faradmal, J.; Abbasi, N.; and Lurz, K. (2015). A New Modification of the Classical Laplace distribution. *Journal of The Iranian Statistical Society*, **14**(2), 93–110.
- OECD: Clocking in and clocking out: Recent trends in working hours (OECD, 2004). Available at <http://www.oecd.org/dataoecd/42/49/33821328.pdf>.
- Poncet, P. (2012). modeest: Mode Estimation. R package version 2.1. <http://CRAN.R-project.org/package=modeest>

- Rényi, A. (1961). On measures of entropy and information, Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, I: 547–561, University of California Press. Berkeley.
- Shannon, C. E. (1948). A mathematical theory of communication. *Bell system technical journal*, **27**(3), 379–423.
- Scott, W.F. (2010). The uniform-Laplace distribution and the D-test. *Communications in Statistics Theory and Methods*, **39**, 249–255.
- Wegman E.J. (1971). A note on the estimation of the mode. *The Annals of Mathematical Statistics*, **42**(6), 1909- 1915.
- Venter J.H. (1967). On estimation of the mode. *The Annals of Mathematical Statistics*, **38**(5), 1446-1455.
- Yu and Moyeed (2001) Yu, K. and Moyeed, R. A. (2001), Bayesian quantile regression. *Statistics and Probability Letters*, **54**(4), 437-447.
- Yu, K. and Zhang, J. (2005). A Three-parameter asymmetric Laplace distribution and its extension. *Communications in Statistics-Theory and Methods*, **34**(9-10), 1867–1879.