

More Results on Dynamic Cumulative Inaccuracy Measure

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Abstract. In this paper, borrowing the intuition in Rao *et al.* (2004), we introduce a cumulative version of the inaccuracy measure (CIM). Also we obtain interesting and applicable properties of CIM for different cases based on the residual, past and interval lifetime random variables. Relying on various applications of stochastic classes in reliability and information theory fields, we study new classes of the lifetime in terms of the CIM along with their relations with other famous aging classes. Furthermore, some characterization results are obtained under the proportional reversed hazard rate model. Finally, considering that the time t changes in the range (t_1, t_2) , an extension of the CIM, called the interval cumulative residual (past) inaccuracy (ICR(P)I), is derived. We investigate the ICRI's relation with its analogous version based on Shannon entropy.

Keywords. Cumulative residual (past) inaccuracy, Interval cumulative residual (past) inaccuracy, Kerridge inaccuracy, Proportional (reversed) hazard model, Shannon entropy.

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1 Introduction and Preliminary Results

In the literature, the reliability and information theory are used to study the behaviour of a component or a system. For this reason, several information measures have been defined as practical tools. Let X be a continuous random variable representing the lifetime of a device or a system. Two dual random variables are known as the residual lifetime $X_t = (X-t|X \geq t)$ and the past lifetime $X_t^* = (t-X|X \leq t)$. These two fundamental concepts have been under attention simultaneously in different areas of statistics.

A prominent measure of uncertainty in terms of the continuous random variable X is the Shannon entropy (Shannon, 1948). This measure is called the differential entropy as well and is defined by

$$H(X) = - \int_0^{\infty} f(x) \log f(x) dx. \quad (1.1)$$

Note that, for a discrete random variable with occurrence probabilities $P = (p_1, \dots, p_n)$, this entropy is given by $H(P) = - \sum_{i=1}^n p_i \log p_i$.

Afterwards, numerous efforts have been made to enrich and extend the uncertainty measure in terms of underlying continuous density functions. An important attempt in this regard is the *inaccuracy measure*, Kerridge (1961), defined by

$$H(P, Q) = - \sum_{i=1}^n p_i \log q_i, \quad (1.2)$$

where $P = (p_1, \dots, p_n)$ refers to true probabilities and $Q = (q_1, \dots, q_n)$ refers to proposed experimenter probabilities.

Nath (1968) took one step further and extended this measure to the case when the lifetime random variable is continuous. Suppose that X and Y are two non-negative continuous random variables with probability density functions (PDFs) $f(t)$ and $g(t)$, respectively. Let $f(t)$ be the actual PDF corresponding to the observations and $g(t)$ be the estimated PDF applying by the experimenter. Then, as a generalization of the differential entropy, a useful tool for measuring the error in experimental outcomes is given by

$$\begin{aligned} H(X, Y) &= - \int_0^{\infty} f(x) \log g(x) dx \\ &= - \int_0^{\infty} f(x) \log f(x) dx + \int_0^{\infty} f(x) \log \frac{f(x)}{g(x)} dx. \end{aligned} \quad (1.3)$$

Noting here that the first item in (1.3) stands with Shannon entropy, $H(X)$ and the second term is the Kullback-Leibler measure of discrimination (see Kullback, 1959). Particularly, if $f(x) = g(x)$, then $H(X, Y)$ achieves its minimum value, *i.e.* Shannons entropy $H(X)$. The inaccuracy measure has applications in the statistical inference, estimation and coding theory. For more properties and applications of $H(X, Y)$, one can see Smitha (2010) and the references therein.

Under the application perspective, as long as the current age of a system is also taken into account, the two measures $H(X)$ and $H(X, Y)$ are not suitable to measure uncertainty. Therefore, in this sort of implementations, the researcher comes across with the residual measure. However, depending on whether before or after a certain age of the lifetime random variable is concerned, one may define the residual or past uncertainties (entropies). So, Ebrahimi and Pellerey (1995) and Ebrahimi (1996) defined the residual entropy (RE) based on the random variable $X_t = (X - t|X > t)$ by

$$\begin{aligned} RE(X; t) &= - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx, \\ &= 1 - \frac{1}{\bar{F}(t)} \int_t^\infty f(x) \log \lambda(x) dx \\ &= 1 - E(\log \lambda(X)|X \geq t), \end{aligned}$$

where $\lambda(x) = \frac{f(x)}{\bar{F}(x)}$ is the hazard rate function and $RE(X; 0) = H(X)$.

Accordingly, Nair and Gupta (2007) and Taneja *et al.* (2009) have extended the idea into the dynamic version as

$$H(X, Y; t) = - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx, \tag{1.4}$$

where $H(X, Y; t)$ tends to $H(X, Y)$ when $t \rightarrow 0$. It can be easily seen that the relation between hazard rates of X and Y and $H(X, Y; t)$ is established by

$$\lambda_X(t) = \frac{\frac{\partial H(X, Y; t)}{\partial t} + \lambda_Y(t)}{H(X, Y; t) + \log \lambda_Y(t)}.$$

Recently, Di Crescenzo and Longobardi (2002) introduced an updated form of the uncertainty measure in terms of the past lifetime, $X_t^* = (t - X|X \leq t)$, named the past

entropy (PE), given by

$$\begin{aligned} PE(X; t) &= - \int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} dx \\ &= 1 - \frac{1}{F(t)} \int_0^t f(x) \log \lambda^*(x) dx \\ &= 1 - E(\log \lambda^*(X) | X \leq t), \end{aligned}$$

where $\lambda^*(t) = \frac{f(x)}{F(x)}$ is the reversed hazard rate function and $PE(X; \infty) = H(X)$.

Furthermore, Nair and Gupta (2007) and Kumar *et al.* (2011), have studied a dynamic measure of the inaccuracy based on X_t^* as

$$H^*(X, Y; t) = - \int_0^t \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} dx, \quad (1.5)$$

where $H^*(X, Y; \infty) = H(X, Y)$.

Rao *et al.* (2004) defined an alternative measure of uncertainty, called "cumulative residual entropy" (CRE), through

$$CRE(X) = - \int_0^\infty \bar{F}(x) \log \bar{F}(x) dx.$$

CRE has many good properties. First, its definition is valid in the continuous and discrete cases which is more general than the Shannon entropy; second, it has more general mathematical properties than the Shannon entropy; and at last, it can be easily estimated from sample data and the estimator asymptotically converges to the true values. One can see many properties and applications of CRE in Rao (2005), Liu (2007) and Di Crescenzo and Longobardi (2009).

Many other extensions of Shannon entropy are redefined by the idea in Rao *et al.* (2004) via replacing $f(x)$ by $1 - F(x)$; see, for example, the works done by Drissi *et al.* (2008), Sunoj and Linu (2012), Psarrakos and Navarro (2013), Sati and Gupta (2015), Rajesh and Sunoj (2016) and Kundu *et al.* (2016).

Asadi and Zohrevand (2007) defined the dynamic version of the cumulative residual entropy (DCRE), which is the CRE of the residual random variable X_t , and it is given by

$$\begin{aligned} DCRE(X; t) &= - \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log \frac{\bar{F}(x)}{\bar{F}(t)} dx \\ &= - \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x) \log \bar{F}(x) dx + \mu(t) \log \bar{F}(t), \end{aligned}$$

where $\mu(t) = E(X - t|X \geq t) = \frac{\int_t^\infty F(x)dx}{\bar{F}(t)}$ is the mean residual life (MRL) function and $DCRE(X; 0) = CRE(X)$.

As a generalization of the CRE and DCRE, Taneja and Kumar (2012), Kumar and Taneja (2015) and Kumar (2016) have defined the cumulative residual inaccuracy (CRI) and the dynamic version of it by

$$\begin{aligned} CRI(X, Y) &= - \int_0^\infty \bar{F}(x) \log \bar{G}(x) dx \\ &= - \int_0^\infty \bar{F}(x) \log \bar{F}(x) dx + \int_0^\infty \bar{F}(x) \log \frac{\bar{F}(x)}{\bar{G}(x)} dx, \end{aligned}$$

and

$$DCRI(X, Y; t) = - \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log \frac{\bar{G}(x)}{\bar{G}(t)} dx.$$

The cumulative past entropy (CPE) has also introduced and studied by Di Crescenzo and Longobardi (2009) as

$$CPE(X) = - \int_0^\infty F(x) \log F(x) dx. \tag{1.6}$$

Furthermore, Navarro *et al.* (2010) studied many properties of DCRE and defined the dynamic version of the cumulative past entropy (DCPE) by

$$\begin{aligned} DCPE(X; t) &= - \int_0^t \frac{F(x)}{F(t)} \log \frac{F(x)}{F(t)} dx \\ &= \mu^*(t) \log F(t) - \frac{1}{F(t)} \int_0^t F(x) \log F(x) dx, \end{aligned} \tag{1.7}$$

where $\mu^*(t) = E(t - X|X \leq t) = \frac{\int_0^t F(x)dx}{F(t)}$ is the mean past life (MPL) function and $DCPE(X; \infty) = CPE(X)$.

Kumar and Taneja (2015) and Kundu *et al.* (2016) have introduced the cumulative past inaccuracy (CPI) and its dynamic version as a generalization of CPE (1.6) and DCPE (1.7), respectively, by

$$\begin{aligned} CPI(X, Y) &= - \int_0^\infty F(x) \log G(x) dx \\ &= - \int_0^\infty F(x) \log F(x) dx + \int_0^\infty F(x) \log \frac{F(x)}{G(x)} dx, \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} DCPI(X, Y; t) &= - \int_0^t F_t(x) \log G_t(x) dx \\ &= \mu^*(t) \log G(t) - \frac{1}{F(t)} \int_0^t F(x) \log G(x) dx, \end{aligned} \quad (1.9)$$

where $F_t(x) = \frac{F(x)}{F(t)}$ and $G_t(x) = \frac{G(x)}{G(t)}$ are the distribution functions of the past lifetime random variables $X_t^* = (t - X | X < t)$ and $Y_t^* = (t - Y | Y < t)$, respectively. They have studied some basic properties and applications of these measures. In this paper, we obtain some new results on these functions including the necessary and sufficient conditions for their monotonicity properties and some new characterization results. Also, some new properties based on double truncated (interval) lifetime random variables are obtained.

It should be noted that

$$CPI(X, Y) = CPE(X) + CKL(X, Y) + [E(Y) - E(X)],$$

where $CKL(X, Y) = \int_0^\infty F(t) \log \frac{F(t)}{G(t)} dt + [E(X) - E(Y)]$ is the cumulative Kullback-Leibler information introduced by Park *et al.* (2012) and Di Crescenzo and Longobardi (2015). Moreover, we have

$$DCPI(X, Y; t) = DCPE(X, Y; t) + DCKL(X, Y; t) + [\mu_X^*(t) - \mu_Y^*(t)],$$

where

$$DCKL(X, Y; t) = \int_0^t \frac{F(x)}{F(t)} \log \left(\frac{F(x)G(t)}{F(t)G(x)} \right) dx + [\mu_Y^*(t) - \mu_X^*(t)]$$

is the dynamic cumulative Kullback-Leibler information (Di Crescenzo and Longobardi, 2015).

Another reliability measure that has an important role in modeling and extracting the dependence structure between two distributions are proportional hazard model (Cox, 1959) and proportional reversed hazard model (Gupta *et al.*, 1998). If the random variables X and Y satisfy the proportional (reversed) hazard model denoted by PHM (PRHM), then we have $\bar{G}(x) = [\bar{F}(x)]^\beta$ for PHM, and $G(x) = [F(x)]^\beta$ for PRHM, where $\beta > 0$ is a constant.

Examples 1.1. Let a non-negative random variable X be uniformly distributed over $(a; b)$, $a < b$, with density and distribution functions, respectively, given by

$$f(x) = \frac{1}{b-a} \quad \text{and} \quad F(x) = \frac{x-a}{b-a}; \quad a < x < b.$$

Thus, by substitution, we obtain the CPI and DCPI under PHM, and CRI and DCRI under PRHM as follows.

$$\begin{aligned}
 CPI(X, Y) &= \beta CPE(X) = - \int_a^b \beta \left(\frac{u-a}{b-a} \right) \log \left(\frac{u-a}{b-a} \right) du = \frac{\beta(b-a)}{4}, \\
 DCPI(X, Y; t) &= \beta DCPE(X; t) = - \int_a^t \beta \frac{\left(\frac{u-a}{b-a} \right)}{\left(\frac{t-a}{b-a} \right)} \log \frac{\left(\frac{u-a}{b-a} \right)}{\left(\frac{t-a}{b-a} \right)} du = \frac{\beta(t-a)}{4}, \\
 CRI(X, Y) &= \beta CRE(X) = \frac{\beta(b-a)}{4}, \\
 DCRI(X, Y; t) &= \beta DCRE(X; t) = \frac{\beta(b-t)}{4}.
 \end{aligned}$$

In addition, it is easy to see that the cumulative inaccuracy measures are not symmetric by changing the X and Y ; for instance, $CPI(X, Y) \neq CPI(Y, X)$ or $DCRI(X, Y; t) \neq DCRI(X, Y; t)$. Since the properties of the CRI (or DCRI) and the CPI (or DCPI) are radically different (for example, we will show that there is no life distribution with decreasing DCPI), there is some scope for a separate study of the CPI and DCPI.

By using log-sum inequality, the following bounds for the CRI and CPI, which are related to CRE and CPE, can be easily obtained as

$$\begin{aligned}
 CRI(X, Y) &\geq CRE(X) + C, \\
 CPI(X, Y) &\geq CPE(X) + C,
 \end{aligned}$$

where $C = E(X) \ln \frac{E(X)}{E(Y)}$. The following theorem shows that the DCPI has a direct relation with the mean past lifetime.

Theorem 1.1. *Let X and Y be two non-negative continuous random variables satisfying the PRHM, such that $DCPI(X, Y; t) < \infty$ for all $t > 0$. Then,*

$$DCPI(X, Y; t) = \beta E(\mu^*(X) | X \leq t). \tag{1.10}$$

Proof. To prove the result, note that, under PRHM, we have

$$DCPI(X, Y; t) = \beta DCPE(X; t).$$

On the other hand,

$$\begin{aligned}
 E(\mu^*(X)|X \leq t) &= \int_0^t \mu^*(x) \frac{f(x)}{F(t)} dx \\
 &= \int_0^t \frac{f(x)}{F(t)F(x)} \left(\int_0^x F(u) du \right) dx \\
 &= \int_0^t \frac{F(u)}{F(t)} \left(\int_u^t \frac{f(x)}{F(x)} dx \right) du \\
 &= - \int_0^t \frac{F(u)}{F(t)} \log \frac{F(u)}{F(t)} du \\
 &= DCPE(X; t).
 \end{aligned}$$

Hence the proof is completed. \square

Corollary 1.1. In analogy of Theorem 1.1 for dynamic cumulative residual inaccuracy, we can prove that $DCRI(X, Y; t) = \beta E(\mu(X)|X \geq t)$.

For some recent results on the CPI and empirical CPI based on suitable stochastic orderings and also applications in failure of nanocomponents and image analysis, see Di Crescenzo and Longobardi (2013, 2015).

The rest of the paper is as follows. In Section 2, we define new class of life distributions based on the DCPI and DCRI, and their relation with other aging classes are implied. Also, the necessary and sufficient conditions for those classes are obtained. Furthermore an upper bound for $DCPI(X, Y; t)$ based on the mean past lifetime function under the PRHM is presented. We also study some characterization problems in Section 3. We show that the DCPI measure can uniquely determine the distribution function. We also characterize various distributions in terms of the relation between $DCPI(X, Y; t)$ and $\mu^*(t)$. Finally, in Section 4, some properties of the interval cumulative residual (past) inaccuracy measure is studied.

2 New Class of Life Distributions

In this section, we define a new class of life distributions based on $DCPI(X, Y; t)$. Similar definitions and results can be obtained for dynamic CRI.

Definition 2.1. The two distribution functions F and G are said to be increasing (decreasing) DCPI, denoted by IDCPI(DDCPI), if $DCPI(X, Y; t)$ is an increasing (decreasing) function of t .

The following theorem gives the necessary and sufficient condition for $DCPI(X, Y; t)$ to be increasing (decreasing).

Theorem 2.1. *The dynamic cumulative past inaccuracy is increasing (decreasing) with respect to t if and only if*

$$DCPI(X, Y; t) \leq (\geq) \mu_X^*(t) \frac{\lambda_Y^*(t)}{\lambda_X^*(t)}. \tag{2.1}$$

Proof. Differentiating (1.9) with respect to t , we easily get

$$\frac{\partial}{\partial t} DCPI(X, Y; t) = \lambda_Y^*(t) \mu_X^*(t) - \lambda_X^*(t) DCPI(X, Y; t), \tag{2.2}$$

which proves the inequality (2.1). □

Examples 2.1. Let X and Y be two nonnegative random variables having density functions, respectively, as

$$f(x) = \begin{cases} x & 0 < x < 1 \\ \frac{x}{3} & 1 \leq x < 2 \\ 0 & o.w. \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{2x+1}{4} & 0 < x < 1 \\ \frac{1}{2} & 1 \leq x < 2 \\ 0 & o.w. \end{cases}$$

Then, we have

$$DCPI(X, Y; t) = \begin{cases} \frac{1}{3t} (1 - \ln(t+1))^{1/t} + \frac{2t}{9} - \frac{1}{6} & 0 < t < 1 \\ \frac{2t^3 - 24 \ln(t) - 18 \ln(2) + 36t - 17}{18t^2 + 36} & 1 \leq t < 2 \\ \frac{71}{108} - \frac{7}{18} \ln(2) & t \geq 2 \end{cases}$$

$$\lambda_X^*(t) = \begin{cases} \frac{2}{t} & 0 < t < 1 \\ \frac{2t}{t^2 + 2} & 1 \leq t < 2 \\ 0 & o.w. \end{cases}$$

$$\lambda_Y^*(t) = \begin{cases} \frac{2t+1}{t^2+t} & 0 < t < 1 \\ \frac{1}{t} & 1 \leq t < 2 \\ 0 & o.w. \end{cases}$$

$$\mu_X^*(t) = \begin{cases} \frac{t}{3} & 0 < t < 1 \\ \frac{t^3+6t-4}{3t^2+6} & 1 \leq t < 2 \\ \frac{8}{9} & t \geq 2. \end{cases}$$

It is easy to see that the $DCPI(X, Y; t)$ is an increasing function in t . Figure 1 illustrates the results of Theorem 2.1.

Corollary 2.1. Under the satisfaction of the proportional reversed hazard model (PRHM) for X and Y , $DCPI(X, Y; t)$ is an increasing function of t , if and only if the inequality

$$DCPI(X, Y; t) \leq \beta \mu_X^*(t). \quad (2.3)$$

holds for all t .

Proof. Under the PRHM, we have

$$\frac{\partial}{\partial t} DCPI(X, Y; t) = \lambda_X^*(t)(\beta \mu_X^*(t) - DCPI(X, Y; t)), \quad (2.4)$$

which concludes the proof. \square

Corollary 2.2. In Corollary 2.1, equality holds in (2.3) if and only if X has a distribution function of the form $F_1(x)$ in Theorem 3.2, part (i).

The following theorem shows that no non-negative random variable exists with decreasing DCPI (DDCPI) over the domain $[0, \infty)$.

Theorem 2.2. *If X and Y are non-negative and non-degenerate random variables, then $DCPI(X, Y; t)$ cannot be a decreasing function of t for all values of t .*

Proof. On the contrary, we assume that $DCPI(X, Y; t)$ is a decreasing function of t . Then, for all $t > 0$, we should have $0 \leq DCPI(X, Y; t) \leq DCPI(X, Y; 0)$ and, since $\lim_{t \rightarrow 0} DCPI(X, Y; t) = 0$, the last inequality implies the contradiction $DCPI(X, Y; t) = 0$, for all t . \square

Corollary 2.3. Navarro *et al.* (2010) have shown that the class of decreasing DCPE is also empty, and hence in PRHM the class of DDCPI is empty too, that is, under PRHM, there exists no non-degenerate random variable with decreasing dynamic cumulative past inaccuracy.

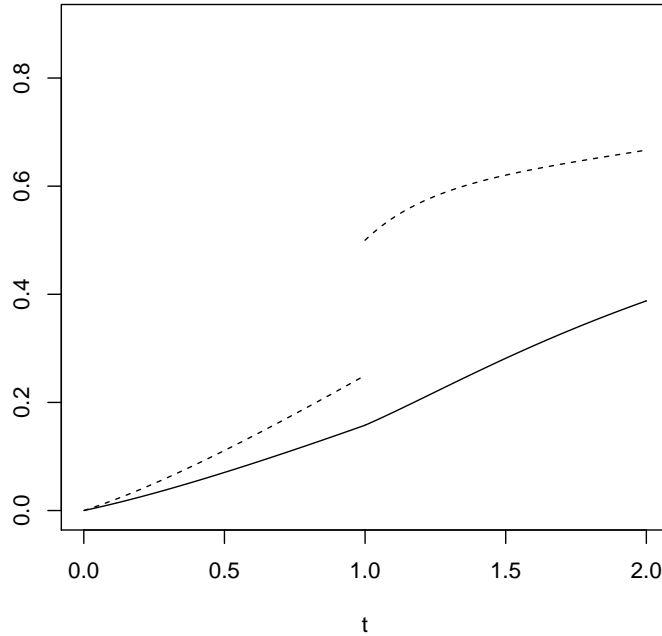


Figure 1: Dynamic cumulative past inaccuracy measure (continuous line) and $\mu_X^*(t) \frac{\lambda_Y^*(t)}{\lambda_X^*(t)}$ (dashed line) in Theorem 2.1 for Example 2.1.

Using the results of Nanda et al. (2003) and Navarro *et al.* (2010), we have the following relationships.

Remark 1. Under the assumption of the PRHM, we have

$$DRHR \implies IMPL \implies IDCPE \iff IDCPI,$$

where DRHR denotes for the decreasing reversed hazard rate, IMPL denotes for the increasing mean past lifetime and IDCPE denotes for the increasing dynamic cumulative past entropy. Similarly, under PHM,

$$DFR (IFR) \implies IMRL (DMRL) \implies IDCRE (DDCRE) \iff IDCRI (DDCRI).$$

3 Characterization Results

In this section, we study the characterization problem for the dynamic cumulative past inaccuracy measure (1.9). First, in the next theorem, we show that the DCPI is unique for any two distribution functions F and G .

Theorem 3.1. *If X and Y are two non-negative random variables and DCPI is an increasing function of t , then $DCPI(X, Y; t)$ is unique for the distribution functions F and G .*

Proof. Let X_1, Y_1 and X_2, Y_2 be two sets of random variables with distribution functions F_1, G_1 and F_2, G_2 , respectively. To prove the theorem, we will show that if, for all $t \geq 0$,

$$DCPI(X_1, Y_1; t) = DCPI(X_2, Y_2; t), \quad (3.1)$$

then we should have $\lambda_{X_1}^*(t) = \lambda_{X_2}^*(t)$ and $\lambda_{Y_1}^*(t) = \lambda_{Y_2}^*(t)$ which leads to $F_{X_1}(t) = F_{X_2}(t)$ and $G_{Y_1}(t) = G_{Y_2}(t)$. So, let

$$A_1 = \{t : t \geq 0 \text{ and } \lambda_{X_1}^*(t) \neq \lambda_{X_2}^*(t)\},$$

$$A_2 = \{t : t \geq 0 \text{ and } \lambda_{Y_1}^*(t) \neq \lambda_{Y_2}^*(t)\}.$$

If, for some $t \geq 0$, just one of A_1 or A_2 is empty (i.e. $\lambda_{X_1}^*(t) = \lambda_{X_2}^*(t)$ or $\lambda_{Y_1}^*(t) = \lambda_{Y_2}^*(t)$), it is easy to see that we have a contradiction. So, assume that, for some $t \geq 0$, the sets A_1 and A_2 are not empty, hence, without loss of generality, suppose that, for some $t_0 \in A_1 \cap A_2$,

$$\lambda_{X_1}^*(t_0) > \lambda_{X_2}^*(t_0), \quad (3.2)$$

and

$$\lambda_{Y_1}^*(t_0) < \lambda_{Y_2}^*(t_0) \text{ or } \lambda_{Y_1}^*(t_0) > \lambda_{Y_2}^*(t_0). \quad (3.3)$$

On the other hand, using (2.2), the Eq. (3.1) gives

$$\lambda_{Y_1}^*(t) \mu_{X_1}^*(t) - \lambda_{X_1}^*(t) DCPI(X_1, Y_1; t) = \lambda_{Y_2}^*(t) \mu_{X_2}^*(t) - \lambda_{X_2}^*(t) DCPI(X_2, Y_2; t),$$

which, after using the relations (3.2), (3.3) and (2.1) and some calculations, it leads to

$$\mu_{X_1}^*(t_0) > \mu_{X_2}^*(t_0).$$

Nanda et al. (2003) have shown that the reversed hazard rate ordering is stronger than the reversed mean residual life ordering, so the last inequality implies a contradiction to (3.2) and hence the sets A_1 and A_2 should be empty and then the proof is completed. \square

Remark 2. Results similar to that of Theorem 3.1 can be proved for the dynamic cumulative residual inaccuracy (DCRI).

Next, we consider the characterization of three distributions based on the DCPI.

Theorem 3.2. *Let X and Y be two absolutely continuous random variables and let $\mu = E(X)$ be finite and the support of them be (a, b) where $a = \inf\{t : F(t) > 0\}$ and $b = \sup\{t : F(t) < 1\}$. If a is taken to be zero, the random variables X and Y may be thought of as the random lifetime of a system or of a component. Further, suppose that X and Y are satisfying the PRHM. Then, the identity*

$$DCPI(X, Y; t) = c\mu^*(t),, \tag{3.4}$$

for every $t \geq 0$ and constant c , characterizes

(i) the distribution

$$F_1(x) = e^{\frac{x-b}{b-\mu}}; \quad x \in (-\infty, b] \quad \text{for } c = \beta,$$

(ii) the distribution

$$F_2(x) = \left[\frac{bc - \beta\mu + x(\beta - c)}{\beta(b - \mu)} \right]^{\frac{c}{\beta-c}}; \quad x \in \left[\frac{\beta\mu - bc}{\beta - c}, b \right] \quad \text{for } c < \beta,$$

which is a finite range distribution,

(iii) the distribution

$$F_3(x) = \left[\frac{bc - \beta\mu + x(\beta - c)}{\beta(b - \mu)} \right]^{\frac{c}{\beta-c}}; \quad x \in (-\infty, b] \quad \text{for } c > \beta,$$

where μ is the mean of distribution and is finite.

Proof. Suppose that (3.4) holds. Then we have

$$\beta\mu^*(t) \log F(t) - \frac{\beta}{F(t)} \int_0^t F(x) \log F(x) dx = c\mu^*(t).$$

Differentiating both sides with respect to t , we obtain

$$\begin{aligned} \frac{c}{\beta} \mu^{*\prime}(t) &= \mu^{*\prime}(t) \log F(t) + \mu^*(t) \lambda^*(t) - \log F(t) + \lambda^*(t) \frac{1}{F(t)} \int_0^t F(x) \log F(x) dx \\ &= \mu^{*\prime}(t) \log F(t) + \mu^*(t) \lambda^*(t) - \log F(t) + \lambda^*(t) [\mu^*(t) \log F(t) - \frac{c}{\beta} \mu^*(t)]. \end{aligned}$$

Using the relation $\mu^*(t) = 1 - \mu^*(t)\lambda^*(t)$ and simplifying,

$$\mu^*(t)\lambda^*(t) = \frac{c}{\beta}. \quad (3.5)$$

Chandra and Roy (2001) have characterized the distributions $F_1(x)$, $F_2(x)$ and $F_3(x)$ using the identity (3.5). \square

As a special case of Theorem 3.2, the following corollary gives a characterization of the power distribution. This distribution is a special case of $F_2(x)$ for $\mu = b\frac{c}{\beta}$.

Corollary 3.1. Let X and Y be two random variables with absolutely continuous distribution functions and the support $(0, b)$. If X and Y satisfy the PRHM, then X has a power distribution with $F(x) = \left(\frac{x}{b}\right)^{\frac{c}{\beta-c}}$, for $0 < x < b$ and $0 < c < \beta$, if and only if

$$DCPI(X, Y; t) = c\mu^*(t),$$

for $0 < t < b$.

Remark 3. Under the assumption of proportional hazard model (PHM), Kumar and Taneja (2015) have presented similar results to that of Theorem 3.2 based on the DCRI via the identity $DCRI(X, Y; t) = c\mu(t)$ for exponential distributions ($c = \beta$), Pareto distributions ($c > \beta$) and finite range distributions ($c < \beta$).

4 Interval Cumulative Inaccuracy

Another extension of information measures can be expressed via the doubly truncated random variables. If X denotes the lifetime of a unit, then the random variables $X_{t_1, t_2} = (X - t_1 | t_1 \leq X \leq t_2)$ and $X_{t_1, t_2}^* = (t_2 - X | t_1 \leq X \leq t_2)$ are called *doubly truncated (interval) residual lifetime* and *doubly truncated (interval) past lifetime*, respectively. Sunoj *et al.* (2009) and Misagh and Yari (2011, 2012) have introduced the interval Shannon entropy by

$$IH(X; t_1, t_2) = - \int_{t_1}^{t_2} \frac{f(x)}{F(t_2) - F(t_1)} \log \frac{f(x)}{F(t_2) - F(t_1)} dx, \quad (4.1)$$

where $(t_1, t_2) \in D = \{(t_1, t_2) : F(t_1) < F(t_2)\}$. It is clear that $IH(X; 0, t) = PE(X; t)$ and $IH(X; t, \infty) = RE(X; t)$. Similarly, the interval cumulative residual (past) entropy has been introduced by Khorashadizadeh *et al.* (2013). Moreover, Kundu and Nanda (2015) defined and studied the interval inaccuracy measure. For more motivation

and applications of the doubly truncated information measure, see the recent works such as Yasaei Sekeh *et al.* (2015), Kundu *et al.* (2016), Kundu (2017) and Jalayeri and Khorashadizadeh (2017). The interval cumulative residual inaccuracy (ICRI) can be defined by

$$ICRI(X, Y; t_1, t_2) = - \int_{t_1}^{t_2} \frac{\bar{F}(x)}{\bar{F}(t_1) - \bar{F}(t_2)} \log \frac{\bar{G}(x)}{\bar{G}(t_1) - \bar{G}(t_2)} dx, \tag{4.2}$$

and, for the interval cumulative past inaccuracy (ICPI), we have

$$ICPI(X, Y; t_1, t_2) = - \int_{t_1}^{t_2} \frac{F(x)}{F(t_2) - F(t_1)} \log \frac{G(x)}{G(t_2) - G(t_1)} dx, \tag{4.3}$$

where $(t_1, t_2) \in D$. In the special case of $t_1 \rightarrow 0$, $ICRI(X, Y; 0, t_2)$ reduces to the DCPI and, when $t_2 \rightarrow \infty$, $ICPI(X, Y; t_1, \infty)$ reduces to the DCRI. Kundu *et al.* (2016) have obtained some properties for ICRI and ICPI.

The next theorem shows the relationship between the ICPI and the interval Shannon entropy. Similar results for the ICRI can be obtained.

Theorem 4.1. *Let X and Y be two non-negative random variables, then*

$$ICPI(X, Y; t_1, t_2) \geq C. \exp \{IH(X; t_1, t_2) + IH(Y; t_1, t_2)\}, \tag{4.4}$$

where $C = \exp \left\{ \int_{q_1}^{q_2} \log \left(x \log \frac{G(F^{-1}(x[F(t_2)-F(t_1)]))}{G(t_2)-G(t_1)} \right) dx \right\}$ and $q_i = \frac{F(t_i)}{F(t_2)-F(t_1)}$ for $i = 1, 2$.

Proof. If we let $P(x, y) = \frac{f(x)}{F(t_2)-F(t_1)} \cdot \frac{g(y)}{G(t_2)-G(t_1)}$ and $Q(x, y) = \frac{F(x)}{F(t_2)-F(t_1)} \log \frac{G(x)}{G(t_2)-G(t_1)}$, then, using log-sum inequality as

$$\int \int P(x, y) \log \frac{P(x, y)}{Q(x, y)} dx dy \geq \left(\int \int P(x, y) dx dy \right) \log \frac{\int \int P(x, y) dx dy}{\int \int Q(x, y) dx dy},$$

the required results would be obtained. □

Remark 4. In Theorem 4.1, by putting $t_1 \rightarrow 0$ or $t_2 \rightarrow \infty$, similar relationships would be obtained for the DCPI and DCRI, respectively.

In the following, based on the ICRI function, an upper bound for the difference of two independent random variables is obtained. Analogous results based on the ICPI can be obtained in a similar way.

Theorem 4.2. Let X and Y be two independent non-negative continuous random variables. Then,

$$\begin{aligned} E(|X - E(X)| | X \in I) &\leq E(|X - Y| | (X, Y) \in I^2) \\ &\leq ICRI(X, Y; t_1, t_2) + ICRI(Y, X; t_1, t_2), \end{aligned}$$

where $I = (t_1, t_2) \in D$.

Proof. For two independent random variables,

$$\begin{aligned} &P(\max(X, Y) > z | (X, Y) \in I^2) - P(\min(X, Y) > z | (X, Y) \in I^2) \\ &= \frac{\bar{F}(z)}{\bar{F}(t_1) - \bar{F}(t_2)} + \frac{\bar{G}(z)}{\bar{G}(t_1) - \bar{G}(t_2)} - 2 \frac{\bar{F}(z)\bar{G}(z)}{[\bar{F}(t_1) - \bar{F}(t_2)][\bar{G}(t_1) - \bar{G}(t_2)]}. \end{aligned}$$

So, by integrating both sides from t_1 to t_2 , we can write

$$\begin{aligned} E(|X - Y| | (X, Y) \in I^2) &= \int_{t_1}^{t_2} \frac{\bar{F}(z)}{\bar{F}(t_1) - \bar{F}(t_2)} \left(1 - \frac{\bar{G}(z)}{\bar{G}(t_1) - \bar{G}(t_2)}\right) dz \\ &\quad + \int_{t_1}^{t_2} \frac{\bar{G}(z)}{\bar{G}(t_1) - \bar{G}(t_2)} \left(1 - \frac{\bar{F}(z)}{\bar{F}(t_1) - \bar{F}(t_2)}\right) dz. \end{aligned}$$

On the other hand, using the inequality $1 - x \leq |\log x|$ for all $x > 0$,

$$\begin{aligned} E(|X - Y| | (X, Y) \in I^2) &\leq \int_{t_1}^{t_2} \frac{\bar{F}(z)}{\bar{F}(t_1) - \bar{F}(t_2)} \left| \log \frac{\bar{G}(z)}{\bar{G}(t_1) - \bar{G}(t_2)} \right| dz \\ &\quad + \int_{t_1}^{t_2} \frac{\bar{G}(z)}{\bar{G}(t_1) - \bar{G}(t_2)} \left| \log \frac{\bar{F}(z)}{\bar{F}(t_1) - \bar{F}(t_2)} \right| dz \\ &= ICRI(X, Y; t_1, t_2) + ICRI(Y, X; t_1, t_2). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} E(|X - Y| | (X, Y) \in I^2) &= \int_{t_1}^{t_2} E(|X - u| | X \in I) f(u) du \\ &\geq \int_{t_1}^{t_2} |u - E(X)| f(u) du \\ &= E(|X - E(X)| | X \in I), \end{aligned}$$

and the proof is completed. \square

Corollary 4.1. In the special cases, when $t_1 = 0$ and $t_2 = \infty$ in Theorem 4.2, then $E(|X - E(X)|) \leq E(|X - Y|) \leq CRI(X, Y) + CRI(Y, X)$.

5 Conclusion

When we estimate the probability density function based on lifetime observations, our estimation results have some precision, one is due to the lack of enough information or vagueness in our estimation (*e.g.* missing observation or insufficient data) and the other is due to incorrect information (*e.g.* miss-specifying the model). It is known that the error due to vagueness can be explained by Shannon entropy and the error due to wrongly specifying the distribution can be measured by the Kullback-Liebler measure of discrimination. So, the inaccuracy measure is a good tool for considering these two errors together.

Motivated by this, in the present paper, we considered some extensions of cumulative Kerridge inaccuracy measure due to the residual, past and interval lifetime random variables. Among our results, we have presented some characterization results for three certain specific lifetime distributions. Also, the relationships with other well-known aging classes such as the DRHR, IMPL and IDCPE are obtained. Of course, other properties of these functions are still waiting to be discovered.

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