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A Few Characterizations of the Univariate Continuous Distributions

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Abstract. Various characterizations of distributions, in their generality, are presented in terms of the conditional expectations. Some special examples are given as well.

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1 Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. The present work deals with various characterizations of distributions in terms of the conditional expectations of certain functions of random variables.

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2 Characterization Results

In this section we provide ten characterizations of distributions based on the conditional expectations of functions of random variables. The proofs follow similar arguments, however, we give all the proofs for the sack of self-containment. We like to point out that Propositions 2.1 and 2.2 have already appeared in our previous published and unpublished works, Hamedani, G.G. (2013) and Hamedani, G.G. and Mameli, V. (2016) respectively.

Proposition 2.1. *Let $X : \Omega \to (a, b)$ be a continuous random variable with cdf (cumulative distribution function) F and corresponding pdf (probability density function) f. Let ψ be a differentiable function with $\psi(x) < 1$ on (a, b), such that $\lim_{x\to a+} \psi(x) = 1 - c$, $0 < c < 1$, and $\lim_{x\to b^-} \psi(x) = 1$. Then,*

$$E[\psi(X) \mid X \geq x] = c + (1-c)\,\psi(x), \quad x \in (a, b), \tag{1}$$

if and only if

$$F(x) = 1 - \left(\frac{1-\psi(x)}{c}\right)^{\frac{1-c}{c}}. \tag{2}$$

Proof. If (1) holds, then

$$\int_x^b \psi(u)\, f(u)\, du = \{c + (1-c)\,\psi(x)\}\,(1 - F(x)).$$

Differentiating both sides of the above equation with respect to x and rearranging the terms, we arrive at

$$\frac{f(x)}{1-F(x)} = \frac{1-c}{c}\left(\frac{\psi'(x)}{1-\psi(x)}\right). \tag{3}$$

Integrating both sides of (3) from a to x and using the condition $\lim_{x\to a+} \psi(x) = 1 - c$, we arrive at (2).

Conversely, if (2) holds, then $\psi(x) = 1 - c\,(1-F(x))^{\frac{c}{1-c}}$ and

$$\begin{aligned}
E[\psi(X) \mid X \geq x] &= \frac{\int_x^b \left\{1 - c\,(1-F(u))^{\frac{c}{1-c}}\right\} f(u)\, du}{1-F(x)} \\
&= \frac{(1-F(x)) - c\,(1-c)\,(1-F(x))^{\frac{1}{1-c}}}{1-F(x)} \\
&= 1 - c\,(1-c)\,(1-F(x))^{\frac{c}{1-c}} \\
&= 1 - (1-c)\,(1-\psi(x)) \\
&= c + (1-c)\,\psi(x),
\end{aligned}$$

which is (1). □

Proposition 2.2. *Let $X : \Omega \to (a,b)$ be a continuous random variable with cdf F and corresponding pdf f. Let ψ be a differentiable function with $\psi(x) > 1$ on (a,b), such that* $\lim_{x\to a+} \psi(x) = 1$ *and* $\lim_{x\to b^-} \psi(x) = 1 + c$, $0 < c < 1$. *Then,*

$$E[\psi(X) \mid X \leq x] = c + (1-c)\psi(x), \quad x \in (a,b), \tag{4}$$

if and only if

$$F(x) = \left(\frac{\psi(x)-1}{c}\right)^{\frac{1-c}{c}}. \tag{5}$$

Proof. If (4) holds, then

$$\int_a^x \psi(u) f(u)\, du = \{c + (1-c)\psi(x)\} F(x).$$

Differentiating both sides of the above equation with respect to x and rearranging the terms, we arrive at

$$\frac{f(x)}{F(x)} = \frac{1-c}{c}\left(\frac{\psi'(x)}{\psi(x)-1}\right). \tag{6}$$

Integrating both sides of (6) from x to b and using the condition $\lim_{x\to b^-} \psi(x) = 1 + c$, we arrive at (5).

Conversely, if (5) holds, then $\psi(x) = 1 + c(F(x))^{\frac{c}{1-c}}$ and

$$\begin{aligned} E[\psi(X) \mid X \leq x] &= \frac{\int_a^x \left\{1 + c(F(u))^{\frac{c}{1-c}}\right\} f(u)\, du}{F(x)} \\ &= \frac{F(x) + c(1-c)(F(x))^{\frac{1}{1-c}}}{F(x)} \\ &= 1 + c(1-c)(F(x))^{\frac{c}{1-c}} \\ &= 1 + (1-c)(\psi(x)-1) \\ &= c + (1-c)\psi(x), \end{aligned}$$

which is (4). □

Proposition 2.3. *Let $X : \Omega \to (a,b)$ be a continuous random variable with cdf F and corresponding pdf f. Let ψ be a positive and differentiable function on (a,b) such that* $\lim_{x\to a+} \psi(x) = \gamma > 0$ *and* $\lim_{x\to b^-} \psi(x) = 0$. *Then, for $\delta > 1$ and $c \neq 0$,*

$$E\left[(\psi(X))^{\delta} \mid X \geq x\right] = (\psi(x))^{\delta} + c(\psi(x))^{\delta-1}, \quad x \in (a,b), \tag{7}$$

implies

$$F(x) = 1 - \left(\frac{\psi(x)}{\gamma}\right)^{1-\delta} e^{\frac{\delta}{c}(\gamma - \psi(x))}.$$

Proof. From (7), we have

$$\int_x^b (\psi(u))^{\delta} f(u)\,du = \left\{(\psi(x))^{\delta} + c(\psi(x))^{\delta-1}\right\}(1-F(x)).$$

Differentiating both sides of the above equation with respect to x and rearranging the terms, we arrive at

$$\frac{f(x)}{1-F(x)} = \frac{\delta}{c}\psi'(x) + (\delta-1)\frac{\psi'(x)}{\psi(x)}. \tag{8}$$

Integrating both sides of (8) from a to x and using the condition $\lim_{x\to a+}\psi(x) = \gamma$, we arrive at

$$F(x) = 1 - e^{\frac{\gamma\delta}{c}}\left(\frac{\psi(x)}{\gamma}\right)^{1-\delta} e^{-\frac{\delta}{c}\psi(x)}.$$

□

Examples 2.1. (i) Taking $(a,b) = (1,\infty)$, $\gamma = 1$ and $\psi(x) = 1/x$, from Proposition 2.3 we have $F(x) = 1 - x^{\delta-1}e^{\frac{\delta}{c}(1-x^{-1})}$. (ii) Taking $(a,b) = (0,\infty)$, $\gamma = 1$ and $\psi(x) = e^{-x}$, from Proposition 2.3 we have $F(x) = 1 - \exp\left\{(\delta-1)x + \frac{\delta}{c}(1-e^{-x})\right\}$.

Proposition 2.4. *Let $X : \Omega \to (a,b)$ be a continuous random variable with cdf F and corresponding pdf f. Let ψ be a positive and differentiable function on (a,b) such that $\lim_{x\to a+}\psi(x) = \infty$ and $\lim_{x\to b^-}\psi(x) = \gamma > 0$. Then, for $\delta > 1$ and $c > 0$,*

$$E\left[(\psi(X))^{\delta} \mid X \le x\right] = (\psi(x))^{\delta} + c(\psi(x))^{\delta-1}, \quad x \in (a,b), \tag{9}$$

implies

$$F(x) = \left(\frac{\psi(x)}{\gamma}\right)^{1-\delta} e^{\frac{\delta}{c}(\gamma-\psi(x))}.$$

Proof. From (9), we have

$$\int_a^x (\psi(u))^{\delta} f(u)\,du = \left\{(\psi(x))^{\delta} + c(\psi(x))^{\delta-1}\right\}F(x).$$

Differentiating both sides of the above equation with respect to x and rearranging the terms, we arrive at

$$\frac{f(x)}{F(x)} = -\frac{\delta}{c}\psi'(x) - (\delta-1)\frac{\psi'(x)}{\psi(x)}. \tag{10}$$

Integrating both sides of (10) from x to b and using the condition $\lim_{x\to b^-}\psi(x)=\gamma$, we arrive at

$$F(x)=\left(\frac{\psi(x)}{\gamma}\right)^{1-\delta}e^{\frac{\delta}{c}(\gamma-\psi(x))}.$$

□

Examples 2.2. Taking $(a,b)=(0,1)$, $\gamma=1$ and $\psi(x)=1/x$, from Proposition 2.4 we have $F(x)=x^{\delta-1}e^{\frac{\delta}{c}(1-x^{-1})}$.

Proposition 2.5. *Let $X:\Omega\to(a,b)$ be a continuous random variable with cdf F and corresponding pdf f. Let ψ be a positive and differentiable function on (a,b) such that $\lim_{x\to a+}\psi(x)=\gamma>0$ and $\lim_{x\to b^-}\psi(x)=\infty$. Then, for $\delta>1$ and $0<c<1$,*

$$E\left[(\psi(X))^{\delta}\mid X\geq x\right]=(\psi(x))^{\delta}+(1-c)(\psi(x))^{\delta-1},\ x\in(a,b),\quad (11)$$

implies

$$F(x)=1-\left(\frac{\psi(x)}{\gamma}\right)^{1-\delta}e^{\frac{\delta}{1-c}(\gamma-\psi(x))}.$$

Proof. From (11), we have

$$\int_x^b(\psi(u))^{\delta}f(u)\,du=\left\{(\psi(x))^{\delta}+(1-c)(\psi(x))^{\delta-1}\right\}(1-F(x)).$$

Differentiating both sides of the above equation with respect to x and rearranging the terms, we arrive at

$$\frac{f(x)}{1-F(x)}=\frac{\delta}{1-c}\psi'(x)+(\delta-1)\frac{\psi'(x)}{\psi(x)}.\quad (12)$$

Integrating both sides of (12) from a to x and using the condition $\lim_{x\to a+}\psi(x)=\gamma$, we arrive at

$$F(x)=1-\left(\frac{\psi(x)}{\gamma}\right)^{1-\delta}e^{\frac{\delta}{1-c}(\gamma-\psi(x))}.$$

□

Examples 2.3. (i) Taking $(a,b)=(1,\infty)$, $\gamma=1$ and $\psi(x)=x$, from Proposition 2.5 we have $F(x)=1-x^{1-\delta}e^{\frac{\delta}{1-c}(1-x)}$. (ii) Taking $(a,b)=(0,\infty)$, $\gamma=1$ and $\psi(x)=e^x$, from Proposition 2.5 we have $F(x)=1-\exp\left\{(1-\delta)x+\frac{\delta}{1-c}(1-e^x)\right\}$.

Proposition 2.6. *Let* $X : \Omega \to (a,b)$ *be a continuous random variable with cdf* F *and corresponding pdf* f. *Let* ψ *be a positive and differentiable function on* (a,b) *such that* $\lim_{x\to a+} \psi(x) = \infty$ *and* $\lim_{x\to b^-} \psi(x) = \gamma > 0$. *Then, for* $\delta > 1$ *and* $0 < c < 1$,

$$E\left[(\psi(X))^{\delta} \mid X \le x\right] = (\psi(x))^{\delta} + (1-c)(\psi(x))^{\delta-1}, \; x \in (a,b), \quad (13)$$

implies

$$F(x) = \left(\frac{\psi(x)}{\gamma}\right)^{1-\delta} e^{\frac{\delta}{1-c}(\gamma-\psi(x))}.$$

Proof. From (13), we have

$$\int_a^x (\psi(u))^{\delta} f(u)\,du = \left\{(\psi(x))^{\delta} + (1-c)(\psi(x))^{\delta-1}\right\} F(x).$$

Differentiating both sides of the above equation with respect to x and rearranging the terms, we arrive at

$$\frac{f(x)}{F(x)} = -\frac{\delta}{1-c}\psi'(x) - (\delta-1)\frac{\psi'(x)}{\psi(x)}. \quad (14)$$

Integrating both sides of (14) from x to b and using the condition $\lim_{x\to b^-} \psi(x) = \gamma$, we arrive at

$$F(x) = \left(\frac{\psi(x)}{\gamma}\right)^{1-\delta} e^{\frac{\delta}{1-c}(\gamma-\psi(x))}.$$

□

Examples 2.4. Taking $(a,b) = (0,1)$, $\gamma = 1$ and $\psi(x) = 1/x$, from Proposition 2.6 we have $F(x) = x^{\delta-1} e^{\frac{\delta}{1-c}(1-x^{-1})}$.

Proposition 2.7. *Let* $X : \Omega \to (a,b)$ *be a continuous random variable with cdf* F *and corresponding pdf* f. *Let* ψ *be a differentiable function on* (a,b) *such that* $\lim_{x\to a+} \psi(x) = \gamma > 0$ *and* $\lim_{x\to b^-} \psi(x) = \infty$. *Then,*

$$E[\psi(X) \mid X \ge x] = (\psi(x))^2 + \psi(x), \quad x \in (a,b), \quad (15)$$

implies

$$F(x) = 1 - \left(\frac{\gamma}{\psi(x)}\right)^2 e^{(\gamma-\psi(x))/\gamma\psi(x)}.$$

Proof. If (15) holds, then

$$\int_x^b \psi(u) f(u)\, du = \left\{(\psi(x))^2 + \psi(x)\right\}(1 - F(x)).$$

Differentiating both sides of the above equation with respect to x and rearranging the terms, we arrive at

$$\frac{f(x)}{1 - F(x)} = \frac{2\psi'(x)}{\psi(x)} + \frac{\psi'(x)}{(\psi(x))^2}. \tag{16}$$

Integrating both sides of (16) from a to x and using the condition $\lim_{x\to a+} \psi(x) = \gamma$, we arrive at $F(x) = 1 - \left(\frac{\gamma}{\psi(x)}\right)^2 e^{(\gamma - \psi(x))/\gamma\psi(x)}$. □

Examples 2.5. Taking $(a, b) = (1, \infty)$, $\gamma = 1$ and $\psi(x) = x$, from Proposition 2.7 we have $F(x) = 1 - x^{-2} e^{\left(\frac{1}{x} - 1\right)}$.

Proposition 2.8. *Let* $X : \Omega \to (a, b)$ *be a continuous random variable with cdf* F *and corresponding pdf* f*. Let* ψ *be a differentiable function on* (a, b) *such that* $\lim_{x\to a+} \psi(x) = \infty$ *and* $\lim_{x\to b^-} \psi(x) = \gamma > 0$*. Then,*

$$E[\psi(X) \mid X \le x] = (\psi(x))^2 + \psi(x), \quad x \in (a, b), \tag{17}$$

implies

$$F(x) = \left(\frac{\gamma}{\psi(x)}\right)^2 e^{(\gamma - \psi(x))/\gamma\psi(x)}.$$

Proof. If (17) holds, then

$$\int_a^x \psi(u) f(u)\, du = \left\{(\psi(x))^2 + \psi(x)\right\} F(x).$$

Differentiating both sides of the above equation with respect to x and rearranging the terms, we arrive at

$$\frac{f(x)}{F(x)} = -\frac{2\psi'(x)}{\psi(x)} - \frac{\psi'(x)}{(\psi(x))^2}. \tag{18}$$

Integrating both sides of (18) from x to b and using the condition $\lim_{x\to b^-} \psi(x) = \gamma$, we arrive at $F(x) = \left(\frac{\gamma}{\psi(x)}\right)^2 e^{(\gamma - \psi(x))/\gamma\psi(x)}$. □

Examples 2.6. Taking $(a, b) = (0, 1)$, $\gamma = 1$ and $\psi(x) = 1/x$, from Proposition 2.8 we have $F(x) = x^2 e^{(x-1)}$.

Proposition 2.9. *Let* $X : \Omega \to (a,b)$ *be a continuous random variable with cdf* F *and corresponding pdf* f*. Let* ψ *be a positive and differentiable function on* (a,b) *such that* $\lim_{x\to a+} \psi(x) = 0$ *and* $\lim_{x\to b-} \psi(x) = \infty$*. Then,* $n \in \mathbb{N}$*,*

$$E\left[(\psi(X))^{2n} \mid X \geq x\right] = (1+\psi(x))^{2n}, \quad x \in (a,b), \tag{19}$$

implies

$$F(x) = 1 - \left\{(1+\psi(x))^{2n} - (\psi(x))^{2n}\right\}^{-1}.$$

Proof. If (19) holds, then

$$\int_x^b (\psi(u))^{2n} f(u)\,du = (1+\psi(x))^{2n}(1-F(x)).$$

Differentiating both sides of the above equation with respect to x and rearranging the terms, we arrive at

$$\frac{f(x)}{1-F(x)} = \frac{n\psi'(x)(1+\psi(x))^{n-1} - n\psi'(x)(\psi(x))^{n-1}}{(1+\psi(x))^n - (\psi(x))^n} + \frac{n\psi'(x)(1+\psi(x))^{n-1} + n\psi'(x)(\psi(x))^{n-1}}{(1+\psi(x))^n + (\psi(x))^n}. \tag{20}$$

Integrating both sides of (20) from a to x and using the condition $\lim_{x\to a+} \psi(x) = 0$, we arrive at $F(x) = 1 - \left\{(1+\psi(x))^{2n} - (\psi(x))^{2n}\right\}^{-1}$. □

Examples 2.7. Taking $(a,b) = (0,\infty)$ and $\psi(x) = x$, from Proposition 2.9 we have $F(x) = 1 - \left\{(1+x)^{2n} - x^{2n}\right\}^{-1}$.

Proposition 2.10. *Let* $X : \Omega \to (a,b)$ *be a continuous random variable with cdf* F *and corresponding pdf* f*. Let* ψ *be a negative differentiable function on* (a,b) *such that* $\lim_{x\to a+} \psi(x) = -\infty$ *and* $\lim_{x\to b-} \psi(x) = -1$*. Then,* $n \in \mathbb{N}$*,*

$$E\left[(\psi(X))^{2n} \mid X \leq x\right] = (1+\psi(x))^{2n}, \quad x \in (a,b), \tag{21}$$

implies

$$F(x) = \frac{(\psi(x))^{2n} - (1+\psi(x))^{2n}}{\left\{(\psi(x))^n + (1+\psi(x))^n\right\}^2}.$$

Proof. If (21) holds, then

$$\int_a^x (\psi(u))^{2n} f(u)\,du = (1+\psi(x))^{2n} F(x).$$

Differentiating both sides of the above equation with respect to x and rearranging the terms, we arrive at

$$\frac{f(x)}{F(x)} = -\frac{n\psi'(x)(\psi(x))^{n-1} + n\psi'(x)(1+\psi(x))^{n-1}}{(\psi(x))^n + (1+\psi(x))^n} + \frac{n\psi'(x)(1+\psi(x))^{n-1} - n\psi'(x)(1+\psi(x))^{n-1}}{(\psi(x))^n - (1+\psi(x))^n}. \tag{22}$$

Integrating both sides of (22) from x to b and using the condition $\lim_{x\to b^-} \psi(x) = -1$, we arrive at $F(x) = \frac{(\psi(x))^n - (1+\psi(x))^n}{(\psi(x))^n + (1+\psi(x))^n} = \frac{(\psi(x))^{2n} - (1+\psi(x))^{2n}}{\{(\psi(x))^n + (1+\psi(x))^n\}^2}$. □

Examples 2.8. Taking $(a,b) = (-\infty, -1)$ and $\psi(x) = x$, from Proposition 2.10 we have $F(x) = \frac{x^{2n} - (1+x)^{2n}}{\{x^n + (1+x)^n\}^2}$.

3 Concluding Remarks

The problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. The goal of this paper is to provide various characterizations of distributions in their generality, with the hope that they will be useful for the investigators who are vitally interested to know if their model fits the requirements of a specific underlying distribution.

References

Hamedani, G.G. (2013), *On generalized gamma convolution distributions II*, Technical Report, No. 484, MSCS, MU.

Hamedani, G.G. and Mameli, V. (2016), *Characterizations of the generalized beta-generated family of distributions*, to appear in JSTA, 2016.