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Stochastic Comparisons of Series and Parallel Systems with Heterogeneous Extended Generalized Exponential Components

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Abstract. In this paper, we discuss the usual stochastic, likelihood ratio, dispersive and convex transform order between two parallel systems with independent heterogeneous extended generalized exponential components. We also establish the usual stochastic order between series systems from two independent heterogeneous extended generalized exponential samples. Finally, we find lower and upper bounds for the Renyi entropy and cumulative residual entropy of series and parallel systems.

Keywords. Cumulative Residual Entropy, Extended Generalized Exponential Distribution, Parallel Systems, Series Systems, Stochastic Orderings, Renyi Entropy.

MSC: Primary xx; Secondary xx.

1 Introduction

The two-parameter generalized exponential (GE) distribution has been introduced by Gupta and Kundu (1999) and it has the following cumulative distribution, and density

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functions:

$$F(x;\alpha,\lambda) = \left(1 - e^{-\lambda x}\right)^{\alpha},$$
$$f(x;\alpha,\lambda) = \alpha\lambda\, e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha-1}, \qquad x > 0, \alpha > 0, \lambda > 0,$$

respectively. Note that α is the shape parameter, and λ is the scale parameter. We denote this distribution by $GE(\alpha,\lambda)$. The two-parameter GE distribution has been used quite effectively for analysing lifetime data. The readers are referred to the recent review article by Gupta and Kundu (2007) for a current account on the generalized exponential distribution.

Although the GE distribution can be used quite effectively to analyse a data set which has a monotone hazard function, unfortunately it cannot be used if the hazard rate function is unimodal or bathtub shaped, similar to the cases for the Weibull and gamma distributions. The extended generalized exponential distribution is a three-parameter distribution, with an additional shape parameter. This new family of distribution functions is always positively skewed, and the skewness decreases as both of the shape parameters increase to infinity. Interestingly, the new three-parameter distribution has increasing, decreasing, unimodal and bathtub shaped hazard rate functions. Therefore, it can be used quite effectively for analysing lifetime data of different types.

A random variable X is said to have the extended generalized exponential distribution with shape parameters $\alpha > 0$, $-\infty < \beta < \infty$ and scale parameter $\lambda > 0$ (denoted by $X \sim EGE(\alpha,\beta,\lambda)$) if its cumulative distribution function is given by

$$F(x;\alpha,\beta,\lambda) = \begin{cases} \left(1 - (1-\beta\,\lambda\,x)^{1/\beta}\right)^{\alpha}, & \beta \neq 0, \\ \\ \left(1 - e^{-\lambda\,x}\right)^{\alpha}, & \beta = 0. \end{cases} \tag{1.1}$$

The support of the random variable X in (1.1) is $(0,\infty)$ if $\beta \leq 0$ and $(0, 1/(\beta\,\lambda))$ if $\beta > 0$. The density function of $EGE(\alpha,\beta,\lambda)$ is

$$f(x;\alpha,\beta,\lambda) = \begin{cases} \alpha\,\lambda\left(1 - (1-\beta\,\lambda\,x)^{1/\beta}\right)^{\alpha-1}(1-\beta\,\lambda\,x)^{(1/\beta)-1}, & \beta \neq 0, \\ \\ \alpha\,\lambda\left(1 - e^{-\lambda\,x}\right)^{\alpha-1} e^{-\lambda\,x}, & \beta = 0, \end{cases} \tag{1.2}$$

Kundu and Gupta (2011) showed that the hazard rate function of the EGE distribution is (i) decreasing for $\alpha < 1$ and $\beta < 0$, (ii) increasing for $\alpha > 1$ and $\beta > 1$, (iii) bathtub shaped if $\alpha > 1$ and $\beta > 1$, and (iv) unimodal if $\alpha > 1$ and $\beta < 0$. One may refer to **?** for comprehensive discussions on various properties and applications of the extended generalized exponential distribution.

A system, consisting of n components, is called a k-out-of-n system if it works when at least k of its components work. The well-known parallel, and series systems are special cases of k-out-of-n systems corresponding to the cases $k = 1$, and $k = n$, respectively. Let $X_1, \cdots, X_n$ denote the lifetimes of components of the system, and $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the corresponding order statistics. Then, $X_{n-k+1:n}$ corresponds to the lifetime of a k-out-of-n system. The comparison of important characteristics associated with lifetimes of technical systems is an interesting topic in reliability theory, because it usually enables us to approximate complex systems with those systems consisting of simple components, through the obtaining various bounds for their important aging characteristics. A convenient tool to achieve such a purpose is the theory of stochastic orderings. Stochastic comparisons of series and parallel systems with heterogeneous components have been studied extensively in the literature for the exponential case. We refer the readers to Dykstra et al. (1997); Khaledi and Kochar (2000); Joo and Mi (2010) and Zhao and Balakrishnan (2011a,b) for detailed discussions on this topic. Some of these results have been extended to the case when the lifetimes of components follow Weibull (Khaledi and Kochar , 2006; Kochar and Xu , 2007a,b; Fang and Zhang , 2013), proportional reversed hazard model (Dolati et al. , 2011), and gamma (Zhao and Balakrishnan , 2011c), and (Balakrishnan and Zhao , 2013)), generalized exponential (Balakrishnan et al. , 2015) distributions, while the general case is discussed by Misra and Misra (2012), and Ding et al. (2013).

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and notation pertinent to stochastic orders and vector majorization order. Section 3 concerns stochastic comparisons of parallel systems with independent heterogeneous EGE components. Finally, in Section 4, some ordering results between series systems with independent heterogeneous EGE components, under the vector majorization are discussed.

2 Definitions and notation

In this section, we first recall the definitions and notion of some well-known concepts on stochastic orderings and majorization that are most pertinent to the discussion here. Suppose X and Y are two non-negative random variables with density function f and g, distribution functions F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, right continuous inverses (quantile functions) F^{-1} and G^{-1}, hazard rates $r_X = f/\bar{F}$ and

$r_Y = g/\bar{G}$, respectively. Throughout this article, we use 'increasing' to mean 'non-decreasing' and similarly 'decreasing' to mean 'non-increasing'.

The following definition introduces some well-known orders that compare skewness of probability distributions..

Definition 1. (i) X is said to be larger than Y in the convex transform order (denoted by $X \geq_c Y$) if $F^{-1}G(x)$ is convex in $x \geq 0$. Equivalently, $X \geq_c Y$ if and only if $G^{-1}F(x)$ is concave in $x \geq 0$;

(ii) X is said to be larger than Y in the Lorenz order order (denoted by $X \geq_{Lorenz} Y$) if

$$\frac{1}{E(X)}\int_0^{F^{-1}(u)} xdx \leq \frac{1}{E(Y)}\int_0^{G^{-1}(u)} xdx \qquad for\ all\ u \in (0,1]. \tag{2.1}$$

The following implications between these orderings are well-known:

$$X \geq_c Y \Longrightarrow X \geq_{Lorenz} Y \Longrightarrow cv(X) \geq cv(Y),$$

where $cv(X) = \sqrt{Var(X)}/E(X)$ and $cv(Y) = \sqrt{Var(Y)}/E(Y)$ denote the coefficients of variation of X and Y, respectively. The Lorenz order is an important criterion in economics to compare income distributions and the risks associated with different prospects (see Marshall and Olkin (2007, p. 68)).

The following definition introduces some well-known orders that compare the dispersion of two random variables.

Definition 2. (i) X is said to be larger than Y in the dispersive order (denoted by $X \geq_{disp} Y$) if

$$F^{-1}(\beta) - F^{-1}(\alpha) \geq G^{-1}(\beta) - G^{-1}(\alpha) \qquad \text{for} \quad 0 \leq \alpha < \beta \leq 1; \tag{2.2}$$

(ii) X is said to be larger than Y in the right-spread order (denoted by $X \geq_{RS} Y$) if

$$\int_{F^{-1}(u)}^{\infty} \bar{F}(x)\,dx \geq \int_{G^{-1}(u)}^{\infty} \bar{G}(x)\,dx \qquad for\ all\ u \in (0,1). \tag{2.3}$$

It is well-known that the dispersive order implies the right-spread order which, in turn, implies the order between the corresponding variances. One may refer to Barmalzan and Payandeh Najafabadi (2015) and Barmalzan et al. (2016) for discussions on the right-spread and dispersive order and its applications.

The next definition introduces some well-known orders that compare the magnitude of two random variables.

Definition 3. (i) X is said to be larger than Y in the usual stochastic order (denoted by $X \geq_{st} Y$) if $\bar{F}(x) \geq \bar{G}(x)$;

(ii) X is said to be larger than Y in the hazard rate order (denoted by $X \geq_{hr} Y$) if $\bar{F}(x)/\bar{G}(x)$ is increasing in x. In other words, $X \geq_{hr} Y$ if and only if $r_Y(x) \geq r_X(x)$;

(iii) X is said to be larger than Y in the likelihood ratio order (denoted by $X \geq_{lr} Y$) if $f(x)/g(x)$ is increasing in x.

Note that the likelihood ratio order implies the hazard rate order, and the hazard rate order implies the usual stochastic order. Moreover, for non-negative random variables, the dispersive order implies the usual stochastic order. For a comprehensive discussion on various stochastic orderings, one may refer to Müller and Stoyan (2002) and Shaked and Shanthikumar (2007).

The Shannon entropy of a non-negative absolutely continuous random variable X with probability density function $f(x)$ is defined by

$$H(X) = -\int_0^\infty f(x) \ln f(x)\, dx.$$

Renyi (1961) defined the entropy of order γ as

$$H_\gamma(X) = -\frac{1}{\gamma - 1} \ln \int_0^\infty f^\gamma(x)\, dx, \qquad \gamma > 0\, (\gamma \neq 1).$$

In particular, $H(X) = \lim_{\gamma\to 1} H_\gamma(X)$. Rao et al. (2004) and Wang et al. (2003) defined the cumulative residual entropy (CRE) as follows:

$$\mathcal{E}(X) = -\int_0^\infty \bar{F}(x) \ln \bar{F}(x)\, dx.$$

Definition 4. (i) X is said to be larger than Y in the Shannon entropy order (denoted by $X \geq_{SE} Y$) if $H(X) \geq H(Y)$;

(ii) X is said to be larger than Y in the Renyi entropy order (denoted by $X \geq_{Re} Y$) if $H_\gamma(X) \geq H_\gamma(Y)$, for all $\gamma > 0$;

(iii) X is said to be larger than Y in the cumulative residual entropy order (denoted by $X \geq_{CRE} Y$) if $\mathcal{E}(X) \geq \mathcal{E}(Y)$.

Obviously, we can observe that the Renyi entropy order implies Shannon entropy order, but in general there is no any relationship between the cumulative residual entropy and Renyi entropy orders. It is important to note that the dispersive order implies the Renyi entropy order Abbasnejad and Arghami (2011, see). Also, Zardasht (2015) showed that the usual stochastic order implies the cumulative residual entropy order.

In the next definition, we present some basic majorization and related orders, which become quite useful in establishing the main results in the subsequent sections.

Definition 5. For two vectors $\boldsymbol{a} = (a_1, \cdots, a_n)$ and $\boldsymbol{b} = (b_1, \cdots, b_n)$, let $\{a_{(1)}, \cdots, a_{(n)}\}$ and $\{b_{(1)}, \cdots, b_{(n)}\}$ denote the increasing arrangements of their components, respectively. A vector $\boldsymbol{a}$ is said to majorize another vector $\boldsymbol{b}$ (written $\boldsymbol{a} \overset{m}{\succ} \boldsymbol{b}$) if $\sum_{j=1}^{i} a_{(j)} \leq \sum_{j=1}^{i} b_{(j)}$ for $i = 1, \cdots, n-1$, and $\sum_{j=1}^{n} a_{(j)} = \sum_{j=1}^{n} b_{(j)}$.

The inequality $\mathbf{a} \overset{m}{\succeq} \mathbf{b}$ implies that, for a fixed sum, the a_i's are more diverse than the b_i's. To illustrate this point, note that $\mathbf{b} \overset{m}{\succeq} \bar{\mathbf{b}}$ always holds, where $\bar{\mathbf{b}} = (\bar{b}, \cdots, \bar{b})$ with $\bar{b} = n^{-1} \sum_{i=1}^{n} b_i$.

The concept of majorization, enabling the comparison of dispersion of the components of two vectors of real numbers, and the closely related Schur-convexity, turns out to be the useful tools to deal with this problem.

Definition 6. A real-valued function ϕ, defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$, is said to be Schur-convex (Schur-concave) on $\mathbb{A}$ if $\mathbf{a} \overset{m}{\succeq} \mathbf{b}$ implies $\phi(\mathbf{a}) \geq (\leq) \phi(\mathbf{b})$ for any $\mathbf{a}, \mathbf{b} \in \mathbb{A}$.

In other words, functions which preserve majorization are said to be Schur-convex. Note that a Schur-convex (Schur-concave) function must be a symmetric function in its arguments. For example, $\phi_1(\mathbf{a}) = \prod_{i=1}^{n} a_i$ $(a_i > 0)$ and $\phi_2(\mathbf{a}) = \max_{1 \leq i \leq n} a_i$, are Schur-concave and Schur-convex, respectively.

3 Stochastic comparisons of parallel systems

Let us consider a parallel system with the lifetimes of its components following the extended generalized exponential distribution. In this section, we consider the stochastic comparisons of parallel systems with respect to the usual stochastic, likelihood ratio, dispersive and convex transform orders.

Theorem 3.1. *Suppose $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are independent random variables with $X_i \sim EGE(\alpha, \beta, \lambda_i)$, $Y_i \sim EGE(\alpha, \beta, \mu_i)$, $i = 1, \cdots, n$. Then for $\beta \leq 0$, we have*

$$(\lambda_1, \cdots, \lambda_n) \overset{m}{\succeq} (\mu_1, \cdots, \mu_n) \Longrightarrow X_{n:n} \geq_{st} Y_{n:n}.$$

Proof. The proof for the case $\beta = 0$ has been shown in Theorem 8 in (Balakrishnan et al. , 2015). Now, we present the proof only for the case when $\beta < 0$. For the case $\beta < 0$, the distribution function of $X_{n:n}$ is

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n} \left(1 - (1 - \beta\,\lambda_i\,x)^{1/\beta}\right)^{\alpha}, \qquad x > 0.$$

To establish the desired result, we must show that for fixed $x > 0$, $F_{X_{n:n}}(x)$ is Schur-concave in $(\lambda_1, \cdots, \lambda_n)$. To this end, for fixed $x > 0$ and $i \neq j$, let us define the function ϕ as

$$\phi(\boldsymbol{\lambda}) = (\lambda_i - \lambda_j)\left(\frac{\partial F_{X_{n:n}}(x)}{\partial \lambda_i} - \frac{\partial F_{X_{n:n}}(x)}{\partial \lambda_j}\right) \tag{3.1}$$

The partial derivatives of $F_{X_{n:n}}(x)$ with respect to λ_i is

$$\frac{\partial F_{X_{n:n}}(x)}{\partial \lambda_i} = \alpha\,x\,F_{X_{n:n}}(x)\,\frac{(1 - \beta\,\lambda_i\,x)^{\frac{1}{\beta}-1}}{1 - (1 - \beta\,\lambda_i\,x)^{1/\beta}}.$$

Now, by substituting for these derivatives in (3.1), we get

$$\phi(\boldsymbol{\lambda}) = \alpha\,x\,F_{X_{n:n}}(x)\,(\lambda_i - \lambda_j)\left(\frac{(1 - \beta\,\lambda_i\,x)^{\frac{1}{\beta}-1}}{1 - (1 - \beta\,\lambda_i\,x)^{1/\beta}} - \frac{(1 - \beta\,\lambda_j\,x)^{\frac{1}{\beta}-1}}{1 - (1 - \beta\,\lambda_j\,x)^{1/\beta}}\right). \tag{3.2}$$

Note that for $\beta < 0$, the function $(1 - \beta\,\lambda\,x)^{\frac{1}{\beta}-1}/1 - (1 - \beta\,\lambda\,x)^{1/\beta}$ is decreasing with respect to λ, which, together with the assumptions $\lambda_j \geq \lambda_i$, implies that the right hand side of (3.2) is non-positive. Thus, the proof of theorem is completed. □

Theorem 3.1 can be used to compute the lower bound for the survival function of a parallel system consisting of independent heterogeneous EGE components, in terms of the corresponding functions of a parallel system comprising independent homogeneous EGE components. More precisely, because $(\lambda_1, \cdots, \lambda_n) \overset{m}{\geq} (\bar{\lambda}, \cdots, \bar{\lambda})$ where $\bar{\lambda} = n^{-1} \sum_{i=1}^{n} \lambda_i$, then we derive the following lower bound for the survival function of $X_{n:n}$ based on Theorem 3.1, for the case $\beta < 0$

$$\bar{F}_{X_{n:n}}(x) \geq 1 - \left(1 - (1 - \beta\,\bar{\lambda}\,x)^{1/\beta}\right)^{n\,\alpha}, \qquad x > 0.$$

The following corollary, which is a direct consequence of Theorem 3.1, enables us to compare the two parallel systems with heterogeneous EGE components in the sense of the cumulative residual entropy order.

Corollary 3.1. Suppose $X_1,\cdots,X_n$ and $Y_1,\cdots,Y_n$ are independent random variables with $X_i \sim EGE(\alpha,\beta,\lambda_i)$, $Y_i \sim EGE(\alpha,\beta,\mu_i)$, $i=1,\cdots,n$. Then for $\beta \le 0$, we have

$$(\lambda_1,\cdots,\lambda_n) \overset{m}{\ge} (\mu_1,\cdots,\mu_n) \Longrightarrow X_{n:n} \ge_{CRE} Y_{n:n}.$$

Corollary 3.1 can be used to compute the lower bound for the cumulative residual entropy of a parallel system consisting of independent heterogeneous EGE components, in terms of the corresponding functions of a parallel system comprising independent homogeneous EGE components. Since $(\lambda_1,\cdots,\lambda_n) \overset{m}{\ge} (\bar{\lambda},\cdots,\bar{\lambda})$, then a lower bound for the cumulative residual entropy of $X_{n:n}$ is given by

$$\mathcal{E}(X_{n:n}) \ge -\int_0^\infty \left(1-\left(1-(1-\beta\,\bar{\lambda}\,x)^{1/\beta}\right)^{n\,\alpha}\right)\ln\left(1-\left(1-(1-\beta\,\bar{\lambda}\,x)^{1/\beta}\right)^{n\,\alpha}\right)dx.$$

Theorem 3.2. *Suppose $X_1,\cdots,X_n$ and $Y_1,\cdots,Y_n$ are independent random variables with $X_i \sim EGE(\alpha_i,\beta,\lambda)$, $Y_i \sim EGE(\nu_i,\beta,\lambda)$, $i=1,\cdots,n$. Then for $\beta \ne 0$, we have*

$$\sum_{i=1}^n \nu_i \le \sum_{i=1}^n \alpha_i \Longrightarrow X_{n:n} \ge_{lr} Y_{n:n}.$$

Proof. Suppose $\beta \ne 0$. It is easy to show that the ratio of the density functions of $X_{n:n}$ and $Y_{n:n}$ can be rewritten as

$$\frac{f_{X_{n:n}}(x)}{g_{Y_{n:n}}(x)} = \frac{\sum_{i=1}^n \alpha_i}{\sum_{i=1}^n \nu_i}\left(1-(1-\beta\,\lambda\,x)^{1/\beta}\right)^{\sum_{i=1}^n \alpha_i - \sum_{i=1}^n \nu_i}.$$

Thus, if $\sum_{i=1}^n \nu_i \le \sum_{i=1}^n \alpha_i$, then the desired result follows. □

We say that X has a decreasing hazard rate (DHR), if $r(x) = f(x)/\bar{F}(x)$ is decreasing function in x. Next, we will prove the dispersive order between parallel systems with heterogeneous EGE components. For this purpose, we first need the following lemma.

Lemma 3.1. *Shaked and Shanthikumar (2007, p. 156) Let X and Y be two non-negative random variables. If $X \ge_{hr} Y$ and X or Y has a DHR, the $X \ge_{disp} Y$.*

Theorem 3.3. *Suppose $X_1,\cdots,X_n$ and $Y_1,\cdots,Y_n$ are independent random variables with $X_i \sim EGE(\alpha_i,\beta,\lambda)$, $Y_i \sim EGE(\nu_i,\beta,\lambda)$, $i=1,\cdots,n$. Then for $\beta < 0$, we have*

$$\sum_{i=1}^n \nu_i \le \sum_{i=1}^n \alpha_i < 1 \Longrightarrow X_{n:n} \ge_{disp} Y_{n:n}.$$

Proof. From Kundu and Gupta (2011), it is easy to show that $X_{n:n}$ has DHR property if $\sum_{i=1}^{n} \alpha_i < 1$ and $\beta < 0$. Now, the desired result follows from Lemma 3.1 and the fact likelihood ratio order implies hazard rate order. □

The following corollary, which is a direct consequence of Theorem 3.3, enables us to compare the two parallel systems with heterogeneous EGE components in the sense of the Renyi entropy order.

Corollary 3.2. Suppose $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are independent random variables with $X_i \sim EGE(\alpha_i, \beta, \lambda)$, $Y_i \sim EGE(\nu_i, \beta, \lambda)$, $i = 1, \cdots, n$. Then for $\beta < 0$, we have

$$\sum_{i=1}^{n} \nu_i \leq \sum_{i=1}^{n} \alpha_i < 1 \Longrightarrow X_{n:n} \geq_{Re} Y_{n:n}.$$

From Corollary 3.2, we can derive a lower bound for the Renyi entropy of $X_{n:n}$ in terms of the arithmetic mean of α_i and the assumption $\bar{\alpha} < 1/n$ for $\gamma > 0, \quad (\gamma \neq 1)$ as follows:

$$H_\gamma(X_{n:n}) \geq -\frac{1}{\gamma - 1} \ln \int_0^\infty \left(n\lambda\bar{\alpha} \left(1 - (1 - \beta\,\lambda\,x)^{1/\beta}\right)^{n\bar{\alpha}-1} (1 - \beta\,\lambda\,x)^{(1/\beta)-1} \right)^{\gamma-1} dx.$$

Theorem 3.4. *Suppose $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are independent random variables with $X_i \sim EGE(\alpha_i, \beta_1, \lambda_1)$, $Y_i \sim EGE(\nu_i, \beta_2, \lambda_2)$, $i = 1, \cdots, n$. Then for $\beta_1 \neq 0, \beta_2 \neq 0$ and $\beta_1 \leq \beta_2$, we have*

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \nu_i \Longrightarrow X_{n:n} \geq_c Y_{n:n}.$$

Proof. Assume $\beta_1 \neq 0, \beta_2 \neq 0$. The distribution functions $X_{n:n}$ and $Y_{n:n}$, respectively, as follow:

$$F_{X_{n:n}}(x) = \left(1 - (1 - \beta_1\,\lambda_1\,x)^{1/\beta_1}\right)^{\sum_{i=1}^{n} \alpha_i}$$
$$F_{Y_{n:n}}(x) = \left(1 - (1 - \beta_2\,\lambda_2\,x)^{1/\beta_2}\right)^{\sum_{i=1}^{n} \nu_i}.$$

Note that

$$F^{-1}_{X_{n:n}}(x) = \frac{1}{\beta_1\,\lambda_1} \left(1 - \left(1 - x^{1/\sum_{i=1}^{n} \alpha_i}\right)^{\beta_1}\right),$$

and if $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \nu_i$, then

$$F^{-1}_{X_{n:n}}(F_{Y_{n:n}}(x)) = \frac{1}{\beta_1 \lambda_1} \left(1 - (1 - \beta_2 \lambda_2 x)^{\beta_1/\beta_2}\right).$$

In order to obtain the required result, in view of Part (i) of Definition 2.1, it suffices to show that $F^{-1}_{X_{n:n}}(F_{Y_{n:n}}(x))$ is convex in x. The partial derivatives $F^{-1}_{X_{n:n}}(F_{Y_{n:n}}(x))$ with respect to x, respectively, are

$$\frac{\partial F^{-1}_{X_{n:n}}(F_{Y_{n:n}}(x))}{\partial x} = \frac{\lambda_2}{\lambda_1} (1 - \beta_2 \lambda_2 x)^{\frac{\beta_1}{\beta_2} - 1}$$

and

$$\frac{\partial^2 F^{-1}_{X_{n:n}}(F_{Y_{n:n}}(x))}{\partial x^2} = \frac{\lambda_2^2}{\lambda_1} (\beta_2 - \beta_1)\,(1 - \beta_2 \lambda_2 x)^{\frac{\beta_1}{\beta_2} - 2}.$$

Thus, for any $\beta_1 \leq \beta_2$ we readily observe that $\partial F^{-1}_{X_{n:n}}(F_{Y_{n:n}}(x))/\partial x^2$ is positive, which completes the proof of the theorem. □

The following corollary, which is of independent interest in economics, is a direct consequence of Theorem 3.4. This theorem enables us to compare the largest order statistics in the sense of the Lorenz ordering.

Corollary 3.3. Suppose $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are independent random variables with $X_i \sim EGE(\alpha_i, \beta_1, \lambda_1)$, $Y_i \sim EGE(\nu_i, \beta_2, \lambda_2)$, $i = 1, \cdots, n$. Then for $\beta_1 \neq 0, \beta_2 \neq 0$ and $\beta_1 \leq \beta_2$, we have

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \nu_i \Longrightarrow X_{n:n} \geq_{Lorenz} Y_{n:n}.$$

The following corollary is another direct consequence of Theorem 3.4.

Corollary 3.4. Suppose $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are independent random variables with $X_i \sim EGE(\alpha_i, \beta_1, \lambda_1)$, $Y_i \sim EGE(\nu_i, \beta_2, \lambda_2)$, $i = 1, \cdots, n$. Then for $\beta_1 \neq 0, \beta_2 \neq 0$ and $\beta_1 \leq \beta_2$, we have

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \nu_i \Longrightarrow cv(X_{n:n}) \geq cv(Y_{n:n}).$$

4 Stochastic comparisons of series systems

In this section, we discuss the usual stochastic order between series systems from two heterogeneous samples of EGE random variables.

Theorem 4.1. *Suppose $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are independent random variables with $X_i \sim EGE(\alpha_i, \beta, \lambda)$, $Y_i \sim EGE(\nu_i, \beta, \lambda)$, $i = 1, \cdots, n$. Then for $\lambda > 0$ and $\beta \neq 0$, we have*

$$(\alpha_1, \cdots, \alpha_n) \overset{m}{\geq} (\nu_1, \cdots, \nu_n) \Longrightarrow Y_{1:n} \geq_{st} X_{1:n}.$$

Proof. For $\beta \neq 0$, the survival function of $X_{1:n}$ is

$$\bar{F}_{X_{1:n}}(x) = \prod_{i=1}^{n} \left(1 - \left(1 - (1 - \beta\,\lambda\,x)^{1/\beta}\right)^{\alpha_i}\right), \qquad x > 0.$$

To establish the desired result, we must show that for fixed $x > 0$, $\bar{F}_{X_{1:n}}(x)$ is Schur-concave in $(\alpha_1, \cdots, \alpha_n)$. To this end, for fixed $x > 0$ and $i \neq j$, let us define the function ϕ as

$$\phi(\boldsymbol{\alpha}) = (\alpha_i - \alpha_j)\left(\frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial \alpha_i} - \frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial \alpha_j}\right). \tag{4.1}$$

The partial derivatives of $\bar{F}_{X_{1:n}}(x)$ with respect to α_i is

$$\begin{aligned}\frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial \alpha_i} &= -\ln\left(1 - (1 - \beta\,\lambda\,x)^{1/\beta}\right) \bar{F}_{X_{1:n}}(x) \frac{\left(1 - (1 - \beta\,\lambda\,x)^{1/\beta}\right)^{\alpha_i}}{1 - \left(1 - (1 - \beta\,\lambda\,x)^{1/\beta}\right)^{\alpha_i}} \\ &= -\ln\left(1 - (1 - \beta\,\lambda\,x)^{1/\beta}\right) \bar{F}_{X_{1:n}}(x)\ \psi\left(\alpha_i, 1 - (1 - \beta\,\lambda\,x)^{1/\beta}\right),\end{aligned}$$

where $\psi(\alpha, z) = z^{\alpha}/1 - z^{\alpha}$, $0 < z < 1$. Now, by substituting for these derivatives in (**??**), we get

$$\begin{aligned}\phi(\boldsymbol{\alpha}) &= -\ln\left(1 - (1 - \beta\,\lambda\,x)^{1/\beta}\right) \bar{F}_{X_{1:n}}(x)\,(\alpha_i - \alpha_j) \\ &\quad \times \left(\psi\left(\alpha_i, 1 - (1 - \beta\,\lambda\,x)^{1/\beta}\right) - \psi\left(\alpha_j, 1 - (1 - \beta\,\lambda\,x)^{1/\beta}\right)\right).\end{aligned}$$

Note that for $\alpha > 0$, the function $\psi(\alpha, z)$ is decreasing with respect to α, for all $z \in (0, 1)$. By combining these observations, we conclude that $\phi(\boldsymbol{\alpha}) \leq 0$ and then the proof of theorem is completed. □

The following corollary, which is a direct consequence of Theorem 4.1, enables us to compare the two series systems with heterogeneous EGE components in the sense of the cumulative residual entropy order.

Corollary 4.1. Suppose $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are independent random variables with $X_i \sim EGE(\alpha_i, \beta, \lambda)$, $Y_i \sim EGE(\nu_i, \beta, \lambda)$, $i = 1, \cdots, n$. Then for $\lambda > 0$ and $\beta \neq 0$, we have

$$(\alpha_1, \cdots, \alpha_n) \overset{m}{\succeq} (\nu_1, \cdots, \nu_n) \Longrightarrow Y_{1:n} \geq_{CRE} X_{1:n}.$$

Since $(\alpha_1, \cdots, \alpha_n) \overset{m}{\succeq} (\bar{\alpha}, \cdots, \bar{\alpha})$, then a upper bound for the cumulative residual entropy of $X_{1:n}$ is given by

$$\mathcal{E}(X_{1:n}) \leq -n \int_0^\infty \left(1 - \left(1 - (1 - \beta\,\lambda\,x)^{1/\beta}\right)^{\bar{\alpha}}\right)^n \ln\left(1 - \left(1 - (1 - \beta\,\lambda\,x)^{1/\beta}\right)dx.$$

References

Abbasnejad, M., Arghami, N.R. (2011). Renyi entropy properties of order statistics. *Communications in Statistics-Theory and Methods*, **40**, 40-52.

Balakrishnan, N., Haidari, A., Masoumifard, K. (2015). Stochastic comparisons of series and parallel systems with generalized exponential components. *IEEE Transactions on Reliability*, **64**, 333-348.

Balakrishnan, N., Zhao, P. (2013). Hazard rate comparison of parallel systems with heterogeneous gamma components. *Journal of Multivariate Analysis,* **113**, 153-160.

Barmalzan, G., Payandeh Najafabadi, A.T. (2015). On the convex transform and right-spread orders of smallest claim amounts. *Insurance: Mathematics and Economics*, **64**, 380-384.

Barmalzan, G., Payandeh Najafabadi, A.T., Balakrishnan, N. (2016). Likelihood ratio and dispersive orders for smallest order statistics and smallest claim amounts from heterogeneous Weibull sample. *Statistics and Probability Letters*, **110**, 1-7.

Ding, W., Y. Zhang, Y., Zhao, P. (2013). Comparisons of k-out-of-n systems with heterogeneous components. *Statistics and Probability Letters*, **83**, 493-502.

Dolati, A., Towhidi, M., Shekari, M. (2011). Stochastic and dependence comparisons between extreme order statistics in the case of proportional reversed hazard model. *Journal of Iranian Statistical Society*, **10** , 29-43.

Dykstra, R.A., Kochar, S.C., Rojo, J. (1997). Stochastic comparisons of parallel systems of heterogeneous exponential components. *Journal of Statistical Planning and Inference*, **65**, 203-211.

Fang, L., Zhang, X. (2013). Stochastic comparison of series systems with heterogeneous Weibull components. *Statistics and Probability Letters*, **83**, 1649-1653.

Gupta, R.D., Kundu, D. (1999). Generalized exponential distributions. *Australian and New Zealand Journal of Statistics*, **41**, 173-188.

Gupta, R.D., Kundu, D. (2007). Generalized exponential distribution: existing methods and some recent developments. *Journal of Statistical Planning and Inference*, **137**, 3537-3547.

Joo, S., Mi, J. (2010). Some properties of hazard rate functions of systems with two components. *Journal of Statistical Planning and Inference*, **140**, 444-453.

Khaledi B.E., Kochar, S.C. (2000). Some new results on stochastic comparisons of parallel systems. *Journal of Applied Probability*, **37**, 1123-1128.

Khaledi, B.E., Kochar, S.C. (2006). Weibull distribution: some stochastic comparisons results. *Journal of Statistical Planning and Inference*, **136**, 3121-3129.

Kochar, S.C., Xu, M. (2007a). Some recent results on stochastic comparisons and dependence among order statistics in the case of PHR model. *Journal of Iranian Statistical Society*, **6**, 125-140.

Kochar, S.C., Xu, M. (2007b). Stochastic comparisons of parallel systems when components have proportional hazard rates. *Probability in the Engineering and Informational Sciences*, **21**, 597-609.

Kundu, D., Gupta, R.D. (2011). An extension of the generalized exponential distribution. *Statistical Methodology*, **8**, 485-496.

Marshall A.W., Olkin, I. (2007). *Life Distributions*, Springer-Verlag, New York.

Misra, N., Misra A.K. (2012). New results on stochastic comparisons of two-component series and parallel systems. *Statistics and Probability Letters*, **82**, 283-290.

Müller, A., Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks*. John Wiley and Sons, New York.

Rao, M., Chen, Y., Vemuri, B.C., Wang, F. (2004). Cumulative residual entropy: a new measure of information. *Information Theory, IEEE Transactions on Information Theory*, **50**, 1220-1228.

Renyi, A. (1961). On measures of entropy and information. *Fourth Berkeley symposium on mathematical statistics and probability*, **1**, 547-561.

Shaked, M., Shanthikumar, J.G. (2007). *Stochastic Orders*. Springer, New York.

Wang, F., Vemuri, B.C., Rao, M., Chen, Y. (2003). A new and robust information theoretic measure and its applications to image alignment. *In: International Conference on Information Processing in Medical Imaging*. 3880-4000.

Zardasht, V. (2015). A test for the increasing convex order based on the cumulative residual entropy. *Journal of the Korean Statistical Society*, In Press.

Zhao, P., Balakrishnan, N. (2011a). Some characterization results for parallel systems with two heterogeneous exponential components. *Statistics*, **65**, 593-604.

Zhao, P., Balakrishnan, N. (2011b). MRL ordering of parallel systems with two heterogeneous components. *Journal of Statistical Planning and Inference*, **141**, 631-638.

Zhao, P., Balakrishnan, N. (2011c) New results on comparisons of parallel systems with heterogeneous gamma components. *Statistics and Probability Letters*, **81**, 36-44.