

## Shrinkage Preliminary Test Estimation under a Precautionary Loss Function with Applications on Records and Censored Data

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**Abstract.** Shrinkage preliminary test estimation in exponential distribution under a precautionary loss function is considered. The minimum risk-unbiased estimator is derived and some shrinkage preliminary test estimators are proposed. We apply our results on censored data and records. The relative efficiencies of proposed estimators with respect to the minimum risk-unbiased estimator based on record data under the considered loss function are computed for evaluating the performance of these estimators.

**Keywords.** Censored data, Exponential distribution, Precautionary loss function, Records, Shrinkage preliminary test estimators.

**MSC:** Primary 62F03; Secondary 62F10.

### 1 Introduction

Suppose that  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  be the sample of size  $n$  coming from the exponential distribution with probability density function (p.d.f.) given by

$$f(x|\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0, \quad \theta > 0. \quad (1.1)$$

It is well-known that the maximum likelihood estimator (MLE) of  $\theta$  based on  $\mathbf{X}$  is  $\bar{X} = \sum_{i=1}^n X_i/n$ . In some situations, the experimenter has some prior information

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about the parameter  $\theta$  in the form of a point guess value. To utilize this guess value, Thompson (1968) proposed some shrunken techniques for estimating the mean. According to Thompson (1968), a linear point shrinkage estimator for the parameter  $\theta$  is given by

$$S = k\bar{X} + (1 - k)\theta_0, \quad 0 \leq k \leq 1, \quad (1.2)$$

where  $k$  is the shrinkage factor and  $\theta_0$  is a prior point guess value of  $\theta$ . The value of  $k$  near to zero (one) implies strong belief in the guess value  $\theta_0$  (sample values). It seems that, for the values of  $\theta$  near to  $\theta_0$ , the shrinkage estimators should have performance better than the estimator  $\bar{X}$ . Then, a preliminary test as

$$\begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta \neq \theta_0 \end{cases} \quad (1.3)$$

is performed for that  $\theta_0$  is near to  $\theta$  or not. For testing (1.3), a test statistic is  $2n\bar{X}/\theta \sim \chi_{2n}^2$ .

One can construct shrinkage preliminary test estimators for the parameter  $\theta$  based on the acceptance or rejection of  $H_0$ . Pandey and Singh (1980), Prakash and Singh (2008), Kibria et al. (2010), Ahmed et al. (2012), Mirfarah and Ahmadi (2014), Arabi Belaghi et al. (2014, 2015a,b), Naghizadeh Qomi and Barmoodah (2015) and Hossain and Howlader (2016) considered the problem of shrinkage estimation. The aim of this paper is constructing shrinkage preliminary test estimators in exponential distribution under a precautionary loss function. Norstrom (1996) defined the concept of precautionary loss functions and introduced a class of precautionary loss functions of the form

$$L(\theta, \hat{\theta}) = w(\theta) \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}^a}, \quad 0 \leq a \leq 2, \quad w(\theta) > 0, \quad (1.4)$$

where  $a$  is a precautionary index. For the case  $a = 1$  and  $w(\theta) = 1/\theta$  in (1.4), we get the following asymmetric scale invariant loss function

$$L(\theta, \hat{\theta}) = \left( \sqrt{\frac{\hat{\theta}}{\theta}} - \sqrt{\frac{\theta}{\hat{\theta}}} \right)^2 = \frac{\hat{\theta}}{\theta} + \frac{\theta}{\hat{\theta}} - 2, \quad (1.5)$$

which is strictly convex and asymmetric in  $\hat{\theta}$  and as a function of  $\hat{\theta}$  has a unique minimum at  $\hat{\theta} = \theta$ . This loss is useful in situations where underestimation is more serious than overestimation. Naghizadeh Qomi et al. (2010) considered this loss for estimation of the scale parameter of the selected gamma population. Karimnezhad et al. (2014) dealt with Bayes and robust Bayes prediction under the loss (1.4) with an application to a rainfall prediction problem. Al-Mosavi and

Khan (2016) considered estimating moments of a selected Pareto population under this loss.

In the rest of the paper, we obtain the minimum risk-unbiased estimator of the form  $c\bar{X}$  in Section 2. We propose some shrinkage preliminary test estimators and derive their risks in Section 3. Section 4 belongs to the application of type-II censored data and extension to the Weibull distribution. An illustrative example is also presented. In Section 5, we apply our results for record data and perform a comparison between the proposed test estimators and the minimum risk-unbiased estimator via the relative efficiency of them under the loss function (1.5). Finally, we end the paper with some remarks.

## 2 Minimum risk-unbiased estimator $c\bar{X}$

Consider a class of estimators for  $\theta$  of the form  $c\bar{X}$ . The risk of  $c\bar{X}$  under the loss function (1.5) is given by

$$\begin{aligned} R(\theta, c\bar{X}) &= E\left(\frac{c\bar{X}}{\theta}\right) + E\left(\frac{\theta}{c\bar{X}}\right) - 2 \\ &= \frac{c}{2n}E\left(\frac{2n\bar{X}}{\theta}\right) + \frac{2n}{c}E\left(\frac{1}{2n\bar{X}/\theta}\right) - 2 = c + \frac{n}{c(n-1)} - 2, \end{aligned}$$

which is a strictly convex function of  $c$  and minimizes at  $c = c^* = \sqrt{n/(n-1)}$ ,  $n > 1$ . Also, the risk of the minimum risk estimator  $\hat{\theta} = c^*\bar{X}$  is given by

$$R(\theta, c^*\bar{X}) = c^* + \frac{n}{c^*(n-1)} - 2 = 2\left(\sqrt{\frac{n}{n-1}} - 1\right), \quad n > 1. \tag{2.1}$$

Following the definition of Lehmann (1951), an estimator  $\hat{\theta}$  of  $\theta$  is said to be risk-unbiased if it satisfies

$$E[L(\theta, \hat{\theta})] \leq E[L(\theta', \hat{\theta})], \quad \forall \theta' \neq \theta. \tag{2.2}$$

Under the loss (1.5), we have

$$\begin{aligned} E[L(\theta, \hat{\theta})] - E[L(\theta', \hat{\theta})] &= E\left[\frac{\hat{\theta}}{\theta} + \frac{\theta}{\hat{\theta}} - 2\right] - E\left[\frac{\hat{\theta}}{\theta'} + \frac{\theta'}{\hat{\theta}} - 2\right] \\ &= \frac{\theta' - \theta}{\theta\theta'}E[\hat{\theta}] + (\theta - \theta')E[\hat{\theta}^{-1}]. \end{aligned}$$

If we consider  $E[\hat{\theta}^{-1}] = E[\hat{\theta}]/\theta^2$ , we get

$$E[L(\theta, \hat{\theta})] - E[L(\theta', \hat{\theta})] = -\frac{(\theta - \theta')^2}{\theta^2\theta'}E[\hat{\theta}] < 0.$$

Therefore, an estimator  $\hat{\theta}$  of  $\theta$  is risk-unbiased under the loss (1.5) if it satisfies in the condition

$$\sqrt{\frac{E(\hat{\theta})}{E(\hat{\theta}^{-1})}} = \theta.$$

Now, using the fact that  $E[c^* \bar{X}] = c^* \theta$  and  $E[(c^* \bar{X})^{-1}] = n((n-1)c^* \theta)^{-1}$ , it is easy to check that the minimum risk estimator  $c^* \bar{X}$  is risk-unbiased and then it is the minimum risk-unbiased estimator for  $\theta$  under the class  $c\bar{X}$ .

### 3 Shrinkage preliminary test estimators

In this section, we propose three shrinkage preliminary test estimators and calculate their risks under the loss function (1.5). The proposed shrinkage test estimator is  $k\bar{X} + (1-k)\theta_0$ , if  $H_0 : \theta = \theta_0$  is accepted or  $c^* \bar{X}$ , otherwise. If  $H_0 : \theta = \theta_0$  is accepted at the level of  $\alpha$ , then we have

$$\Pr\left(q_1 \leq \frac{2n\bar{X}}{\theta_0} \leq q_2\right) = 1 - \alpha,$$

where  $q_1 = \chi_{\alpha/2, 2n}^2$  and  $q_2 = \chi_{1-\alpha/2, 2n}^2$  are left quantiles of a chi-square distribution with  $2n$  degrees of freedom. Therefore, the proposed test estimators can be written as

$$\hat{\theta}_{st}^i = \begin{cases} k_i \bar{X} + (1 - k_i) \theta_0 & r_1 \leq Y_n \leq r_2 \\ c^* \bar{X} & Y_n < r_1 \text{ or } Y_n > r_2, \end{cases} \quad (3.1)$$

where  $Y_n = \sum_{i=1}^n X_i = n\bar{X}$ ,  $r_1 = q_1 \theta_0 / 2$ ,  $r_2 = q_2 \theta_0 / 2$  and  $k_i$ ,  $i = 1, 2, 3$ , are shrinkage factors which are defined in the sequel:

- The risk of the point shrinkage estimator  $S$  defined in (1.2) under the loss function (1.5) is given by

$$\begin{aligned} R(\theta, S) &= E\left(\frac{S}{\theta}\right) + E\left(\frac{\theta}{S}\right) - 2 \\ &= E\left(\frac{k\bar{X} + (1-k)\theta_0}{\theta}\right) + E\left(\frac{\theta}{k\bar{X} + (1-k)\theta_0}\right) - 2 \\ &= \delta(1-k) + k + \int_0^\infty \frac{1}{\delta(1-k) + \frac{k}{n}w} g(w) dw - 2, \end{aligned} \quad (3.2)$$

where  $\delta = \theta_0/\theta$  and  $W = Y_n/\theta$  has a *Gamma*( $n, 1$ ) distribution with p.d.f.  $g(w) = w^{n-1}e^{-w}/\Gamma(n)$ ,  $w > 0$ . The value of  $k_1 = k_{min}$  which minimizes (3.2),

constructing the shrinkage preliminary test estimator  $\hat{\theta}_{st}^1$ , can be obtained numerically.

- If  $H_0 : \theta = \theta_0$  is accepted, then the inequality  $q_1 \leq 2Y_n/\theta_0 \leq q_2$  (Waikar et al. , 1984) implies that

$$0 \leq k_2 = \frac{1}{q_2 - q_1} \left( \frac{2Y_n}{\theta_0} - q_1 \right) \leq 1.$$

The value of  $k_2$  can be used for constructing the shrinkage preliminary test estimators  $\hat{\theta}_{st}^2$ .

- If  $H_0 : \theta = \theta_0$  is accepted, then the inequality  $q_1 \leq 2n \leq q_2$  (Prakash and Singh , 2008) implies that  $q_1/(2n) \leq 1$ . For small values of shrinkage factor, we can take  $q_1/(2n) \approx 1$ . Hence,

$$\frac{2n}{q_2 - q_1} \left( \frac{Y_n/\theta_0}{n} - \frac{q_1}{2n} \right) \approx \frac{2n}{q_2 - q_1} \left( \frac{Y_n}{n\theta_0} - 1 \right).$$

Therefore, the shrinkage factor  $k_3$  for constructing the shrinkage preliminary test estimator  $\hat{\theta}_{st}^3$  is given by

$$k_3 = \frac{2n}{q_2 - q_1} \left| \frac{Y_n}{n\theta_0} - 1 \right|,$$

where the absolute value is for avoiding negative values.

Now, we can calculate the risk of  $\hat{\theta}_{st}^i$ ,  $i = 1, 2, 3$ , under the loss function (1.5) which is given by

$$\begin{aligned} R(\theta, \hat{\theta}_{st}^i) &= E\left(\frac{\hat{\theta}_{st}^i}{\theta}\right) + E\left(\frac{\theta}{\hat{\theta}_{st}^i}\right) - 2 \\ &= E\left[\left(\frac{k_i \bar{X} + (1 - k_i)\theta_0}{\theta} + \frac{\theta}{k_i \bar{X} + (1 - k_i)\theta_0} - 2\right)I(r_1 \leq Y_n \leq r_2)\right] \\ &\quad + E\left[\left(\frac{c^* \bar{X}}{\theta} + \frac{\theta}{c^* \bar{X}} - 2\right)I(Y_n < r_1 \text{ or } Y_n > r_2)\right], \quad i = 1, 2, 3. \end{aligned}$$

After some algebraic computations, we get

$$R(\theta, \hat{\theta}_{st}^i) = J(w_1, w_2; f_i) + J(w_1, w_2; \frac{1}{f_i}) - J(w_1, w_2; f_0)$$

$$- J(w_1, w_2; \frac{1}{f_0}) + 2\left(\sqrt{\frac{n}{n-1}} - 1\right), \quad i = 1, 2, 3, \quad (3.3)$$

where  $w_1 = q_1\delta/2, w_2 = q_2\delta/2, f_0 = c^*w/n, f_i = k_iw/n + (1 - k_i)\delta, i = 1, 2, 3$  and  $J(w_1, w_2; u(w)) = \int_{w_1}^{w_2} u(w)g(w)dw$ , where  $u(w)$  is a function of  $w$  such as  $f_0, f_0^{-1}, f_i$  or  $f_i^{-1}$ . The risk of test estimators  $\hat{\theta}_{st}^i$  given in (3.3) can be computed numerically using statistical package R version 3.1.2.

### 4 Shrinkage preliminary test estimators based on type-II censored data

Let  $X$  denote the time-to-failure of a specific device with the mean lifetime  $\theta$  has an exponential distribution with p.d.f. given in (1.1). Assume that  $n$  randomly selected devices are placed on test simultaneously and the experiment is finished when a specified number of units (say,  $r < n$ ) have failed. It is then well-known that, if  $\mathbf{X} = (X_{(1)}, \dots, X_{(r)})$  be the observed type-II censored sample, the MLE of  $\theta$  based on  $\mathbf{X}$  is  $T_r/r$  where

$$T_r = \sum_{i=1}^r X_{(i)} + (n - r)X_{(r)} \quad (4.1)$$

is the total test time which is the complete sufficient statistic for  $\theta$  and  $2T_r/\theta \sim \chi_{2r}^2$ .

Considering the class of estimators as  $dT_r/r$  for  $\theta$ , and using the method of Section 2, It is easy to show that the minimum risk-unbiaesd estimator under the loss (1.5) is  $d^*T_r/r$  where  $d^* = \sqrt{r/(r-1)}, r > 1$  and  $R(\theta, d^*T_r/r) = 2(\sqrt{r/(r-1)} - 1), r > 1$ . Moreover, we can define the same shrinkage preliminary test estimators for  $\theta$  as given in (3.1) by replacing  $n, c^*, \bar{X}, Y_n$  with  $r, d^*, T_r/r, T_r$  respectively.

#### 4.1 Extension to Weibull distribution

Suppose that  $Y$  represents the time-to-failure of a device and consider that  $\mathbf{Y} = (Y_{(1)}, \dots, Y_{(r)})$  be the available censored sample. Provided that  $Y$  is Weibull distributed under the model

$$f(y|\alpha, \theta) = \frac{\alpha}{\theta} y^{\alpha-1} e^{-\frac{y^\alpha}{\theta}}, \quad y, \alpha, \theta > 0, \quad (4.2)$$

where  $\alpha$  and  $\theta$  are the given shape parameter and the unknown scale parameter, respectively, then  $X = Y^\alpha$  has an exponential distribution with p.d.f. given in (1.1). For  $\alpha = 2$ , the Weibull distribution is Rayleigh distribution. Sinha (1986)

showed that  $S_r/r$  is UMVU estimator of  $\theta$ , where  $S_r$  is the total time on the test given by

$$S_r = \sum_{i=1}^r y_{(i)}^\alpha + (n-r)y_{(r)}^\alpha, \quad n > r,$$

and  $2S_r/\theta \sim \chi_{2r}^2$ . Therefore, it is immediately deduced that the study of shrinkage preliminary test estimation under the Weibull censored data is the same as given in the previous section by replacing  $T_r$  with  $S_r$ .

### 4.2 A real example

Consider the following data set which are the failure times (in minutes) for a sample of fifteen electronic components in an accelerated life test as (Lawless , 2003)

1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2  
23, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2.

For checking the adequacy of the fitness of the Rayleigh distribution with  $\theta = 1161.195$ , we use the Kolmogorov-Smirnov (K-S) test with the test statistic  $D = 0.2341$  and a corresponding p-value= 0.3304. Hence, we claim that the Rayleigh distribution is not an inadequate distribution for modeling these data.

Assume that we have failed to observe the last 9 ordered data so that  $r = 6$  and  $n = 15$ . The MLE of  $\theta$  is  $S^* = S_r/r = 3753.21/6 = 625.535$  and its risk is  $R(\theta, S^*) = 0.2$ . Moreover,  $d^* = \sqrt{r/(r-1)} = 1.0954$  and then the minimum risk-unbiased estimator is  $d^*S^* = (1.0954)(625.535) = 685.211$  with the corresponding risk  $R(\theta, d^*S^*) = 0.1909$ . If we consider the point guess value  $\theta_0 = 700$  for

Table 1: The values of estimators and their risks

estimator	$S^*$	$d^*S^*$	$\hat{\theta}_{st}^1$	$\hat{\theta}_{st}^2$	$\hat{\theta}_{st}^3$	$\hat{\theta}_{st}^{0.5}$	$\hat{\theta}_{st}^{0.7}$	$\hat{\theta}_{st}^{0.9}$
value	625.535	685.211	699.8511	675.4256	694.7875	662.7675	647.8745	632.9815
risk	0.2	0.1909	0.0587	0.0707	0.0592	0.0871	0.1175	0.1643

true value  $\theta$ , then using  $\hat{\theta} = d^*S^*$  for estimating  $\theta$ , the corresponding value of  $\hat{\delta} = \theta_0/\hat{\theta}$  is 1.02. Therefore, the value of shrinkage factor  $k_1$ , founded by minimizing the risk of shrinkage estimator  $S$  and given in (1.2), is 0.002. The test statistic for testing the null hypothesis  $H_0 : \theta = 700$  is  $\chi^2 = 10.72$ . If we consider  $\alpha = 0.05$ , then the left quantiles of a chi-square distribution with 12 degrees of freedom are  $q_1 = 4.40$  and  $q_2 = 23.34$ . This implies that the null hypothesis is accepted at the level of significance 0.05. Then the values of shrinkage factors  $k_2$  and  $k_3$  are as

$$k_2 = \frac{1}{q_2 - q_1} \left( \frac{2S_r}{\theta_0} - q_1 \right) = \frac{1}{23.34 - 4.40} \left( \frac{2(3753.21)}{700} - 4.40 \right) = 0.33,$$

$$k_3 = \frac{2r}{q_2 - q_1} \left| \frac{S_r}{r\theta_0} - 1 \right| = \frac{2(6)}{23.34 - 4.40} \left| \frac{3753.21}{6(700)} - 1 \right| = 0.07.$$

Using these values of shrinkage factors and selected values of  $k = 0.5, 0.7, 0.9$ , we summarize the values of MLE, minimum risk-unbiased estimate, shrinkage preliminary test estimates and their risks in Table 1 for comparison purposes. We observe from Table 1 that all of the shrinkage test estimators are better than the MLE and the minimum risk-unbiased estimator. Also, the shrinkage test estimator  $\hat{\theta}_{st}^1$  has smaller risk than other shrinkage test estimators. This implies that the shrinkage test estimator with small  $k$  is more efficient when the value of  $\delta = \theta_0/\theta$  is close to one ( $\theta_0$  is in the vicinity of  $\theta$ ).

## 5 Shrinkage preliminary test estimators based on records

Consider a sequence of independent and identically distributed (i.i.d.) absolutely continuous random variables  $X_1, X_2, X_3, \dots$  according to the cumulative distribution function (c.d.f.)  $F$  and p.d.f.  $f$ . An observation  $x_k$  will be called an upper record value if its value is larger than all previous values  $x_1, x_2, x_{k-1}$ . An analogous definition can be given for lower record values. We denote the  $m$ th upper record value by  $R_m$ . Interested readers are referred to Arnold et al. (1998) for more details about record values. The joint density of the first  $m$ -records  $\mathbf{R} = (R_1, \dots, R_m)$  is given as

$$f_{R_1, \dots, R_m}(r_1, \dots, r_m) = f(r_m) \prod_{i=1}^{m-1} \frac{f(r_i)}{1 - F(r_i)}, \quad r_1 < r_2 < \dots < r_m. \quad (5.1)$$

Moreover, the p.d.f. of  $s$ th record,  $R_s$ , is given by

$$f_{R_s}(x) = \frac{[-\log(1 - F(x))]^{s-1}}{(s-1)!} f(x). \quad (5.2)$$

If  $\mathbf{R} = (R_1, \dots, R_m)$  be the first  $m$ -records samples from the exponential distribution with p.d.f. (1.1), then from (5.1), the likelihood function of  $\theta$  based on  $\mathbf{R} = (R_1, \dots, R_m)$  at  $\mathbf{r} = (r_1, \dots, r_m)$  is given by

$$L(\theta|\mathbf{r}) = \frac{1}{\theta^m} \exp\left(-\frac{r_m}{\theta}\right), \quad \theta > 0.$$

Therefore, the MLE of  $\theta$ , denoted by  $\hat{\theta}_{ml}$ , can be derived from the equation  $\frac{\partial L(\theta|\mathbf{r})}{\partial \theta} = 0$  which is given by  $\hat{\theta}_{ml} = R_m/m$ . Upon substituting the p.d.f. and c.d.f.



of exponential distribution into (5.2), the p.d.f. of the  $m$ th record,  $R_m$ , is given by

$$f_{R_m}(x) = \frac{x^{m-1} \exp(-\frac{x}{\theta})}{\Gamma(m)\theta^m}, \quad \theta > 0.$$

This leads us to  $2m\hat{\theta}_{ml}/\theta = 2R_m/\theta \sim \chi_{2m}^2$ .

Consider the class of estimators for  $\theta$  as  $lR_m/m$ . It is easy to check that the minimum risk estimator under the loss (1.5) is  $l^*R_m/m$  where  $l^* = \sqrt{m/(m-1)}$ ,  $m > 1$ , and  $R(\theta, l^*R_m/m) = 2(\sqrt{m/(m-1)} - 1)$ ,  $m > 1$ . Moreover, we can define the same shrinkage preliminary test estimators for  $\theta$  as given in (3.1) by replacing  $n, c^*, \bar{X}, Y_n$  with  $m, l^*, R_m/m, R_m$ , respectively.

### 5.1 Performance of shrinkage preliminary test estimators

We now evaluate the performance of shrinkage preliminary test estimators  $\hat{\theta}_{st}^i$ ,  $i = 1, 2, 3$  based on record data. Relative efficiency of  $\hat{\theta}_{st}^i$ ,  $i = 1, 2, 3$  with respect to the minimum risk-unbiased estimator  $\hat{\theta} = l^*R_m/m$  is calculated as

$$RE(\hat{\theta}_{st}^i, \hat{\theta}) = \frac{R(\theta, \hat{\theta})}{R(\theta, \hat{\theta}_{st}^i)}, \quad i = 1, 2, 3. \tag{5.3}$$

Tables 2-4 give the relative efficiency (5.3) for the selected values of  $m = 2(1)6$ ,  $\delta = 0.4(0.2)1.8$  and  $\alpha = 0.01, 0.05, 0.1$ . From these tables, we observe that no test estimator performs uniformly better than the minimum risk-unbiased estimator  $\hat{\theta}$ . The test estimator  $\hat{\theta}_{st}^1$  performs better than  $\hat{\theta}$  in  $0.8 \leq \delta \leq 1.8$ . Moreover, the test estimators  $\hat{\theta}_{st}^2$  and  $\hat{\theta}_{st}^3$  have good performance for  $0.6 \leq \delta \leq 1.4$  and  $0.8 \leq \delta \leq 1.4$ , respectively. All test estimators attain maximum efficiency at the point  $\delta = 1$ . For fixed  $m$ , as the value of  $\alpha$  increases, the relative efficiency decreases in  $0.6 \leq \delta \leq 1.8$  and  $0.6 \leq \delta \leq 1.4$  for the test estimators  $\hat{\theta}_{st}^1$  and  $\hat{\theta}_{st}^2$  and, for test estimator  $\hat{\theta}_{st}^3$ , it decreases for  $0.6 \leq \delta \leq 1.6$ .

One may compare shrinkage test estimators themselves. The shrinkage test estimator  $\hat{\theta}_{st}^1$  are of good performance for  $0.8 \leq \delta \leq 1.8$ . The test estimator  $\hat{\theta}_{st}^2$  performs well when  $\delta = 0.4$  for large  $\alpha = 0.05, 0.1$ , and also when  $\delta = 0.6$ .

## 6 Concluding remarks

In this paper, we considered the problem of constructing shrinkage preliminary test estimators under a precautionary loss function in exponential models based on censored data and records. The minimum risk-unbiased estimator is derived, three shrinkage preliminary test estimators are proposed and their risks are

Table 2: Relative efficiency between  $\hat{\theta}_{st}^1$  and  $\hat{\theta}$ 

$m$	$\alpha$	$\delta$							
		0.4	0.6	0.8	1	1.2	1.4	1.6	1.8
2	0.01	1.0373	1.9404	3.9275	5.5100	4.3996	3.2634	2.6702	2.3315
	0.05	0.9194	1.3594	2.0472	2.5009	2.3248	1.9973	1.7625	1.6085
	0.1	0.8845	1.1728	1.5698	1.8302	1.7914	1.6392	1.5106	1.4191
3	0.01	0.8446	1.5056	3.8550	7.6084	4.7800	3.0043	2.3111	1.9586
	0.05	0.8150	1.1772	2.0685	2.8910	2.4317	1.8744	1.5721	1.4012
	0.1	0.8177	1.0638	1.5818	1.9923	1.8279	1.5466	1.3679	1.2617
4	0.01	0.7847	1.2506	3.4074	8.7129	4.4961	2.6212	1.9830	1.6711
	0.05	0.7876	1.0458	1.9625	3.0788	2.3680	1.7124	1.4105	1.2538
	0.1	0.8052	0.9790	1.5321	2.0695	1.7927	1.4424	1.2562	1.1579
5	0.01	0.7734	1.1013	3.0050	9.3726	4.1216	2.3196	1.7498	1.4764
	0.05	0.7932	0.9614	1.8414	3.1875	2.2661	1.5762	1.2927	1.1547
	0.1	0.8175	0.9222	1.4707	2.1140	1.7394	1.3545	1.1756	1.0900
6	0.01	0.7816	1.0077	2.6845	9.8065	3.7771	2.0915	1.5814	1.3397
	0.05	0.8119	0.9069	1.7293	3.2580	2.1605	1.4674	1.2064	1.0866
	0.1	0.8390	0.8852	1.4107	2.1429	1.6832	1.2833	1.1168	1.0447

Table 3: Relative efficiency between  $\hat{\theta}_{st}^2$  and  $\hat{\theta}$ 

$m$	$\alpha$	$\delta$							
		0.4	0.6	0.8	1	1.2	1.4	1.6	1.8
2	0.01	1.1746	2.0241	3.2406	3.9156	3.4683	2.6536	1.9906	1.5306
	0.05	1.0485	1.4521	1.9018	2.1255	2.0150	1.7336	1.4436	1.2039
	0.1	0.9961	1.2505	1.5152	1.6546	1.6156	1.4636	1.2828	1.1177
3	0.01	0.9347	1.6301	3.0443	4.2106	3.4423	2.2572	1.5057	1.0722
	0.05	0.9270	1.3160	1.9052	2.4223	1.9965	1.5405	1.1681	0.9123
	0.1	0.9264	1.1847	1.5316	1.7121	1.5669	1.3203	1.0727	0.8859
4	0.01	0.8282	1.3752	2.7274	4.1298	3.1758	1.8840	1.1811	0.8131
	0.05	0.8656	1.1955	1.8315	2.2541	1.9050	1.3568	0.9736	0.7386
	0.1	0.8888	1.1157	1.5046	1.7223	1.5284	1.1928	0.9262	0.7483
5	0.01	0.7780	1.2138	2.4719	4.0056	2.9194	1.6067	0.9682	0.6548
	0.05	0.8383	1.1070	1.7526	2.2424	1.8090	1.2098	0.8382	0.6287
	0.1	0.8735	1.0605	1.4696	1.7205	1.4688	1.0896	0.8227	0.6609
6	0.01	0.7555	1.1044	2.2729	3.8907	2.6992	1.3986	0.8201	0.5497
	0.05	0.8298	1.0421	1.6804	2.2255	1.7201	1.0929	0.7405	0.5548
	0.1	0.8712	1.0178	1.4344	1.7151	1.4134	1.0061	0.7472	0.6026

Table 4: Relative efficiency between  $\hat{\theta}_{st}^3$  and  $\hat{\theta}$ 

$m$	$\alpha$	$\delta$							
		0.4	0.6	0.8	1	1.2	1.4	1.6	1.8
2	0.01	1.0514	1.9485	3.4426	4.4869	4.0296	3.0500	2.2714	1.7455
	0.05	0.9428	1.3763	1.9253	2.2715	2.2163	1.9312	1.6225	1.3654
	0.1	0.9033	1.1792	1.5033	1.7199	1.6752	1.6047	1.4276	1.2602
3	0.01	0.7892	1.5215	3.3571	5.3943	4.3195	2.6897	1.7611	1.2548
	0.05	0.7940	1.2049	1.9417	2.5063	2.2993	1.7775	1.3541	1.06914
	0.1	0.8082	1.0818	1.5141	1.8253	1.7642	1.4952	1.2309	1.0309
4	0.01	0.6675	1.2417	3.0452	5.6974	4.0988	2.2536	1.3911	0.9657
	0.05	0.7183	1.0618	1.8650	2.5990	2.2471	1.5877	1.1453	0.8821
	0.1	0.7589	0.9928	1.4789	1.8695	1.7344	1.3724	1.0793	0.8860
5	0.01	0.6042	1.0603	2.7584	5.8224	3.7963	1.9167	1.1471	0.7891
	0.05	0.6817	0.9569	1.7748	2.6454	2.1621	1.4254	0.9963	0.7621
	0.1	0.7377	0.9234	1.4336	1.8928	1.6888	1.2691	0.9685	0.7922
6	0.01	0.5704	0.9346	2.5178	5.8818	3.5044	1.6636	0.9786	0.6723
	0.05	0.6671	0.8792	1.6876	2.6723	2.0709	1.2935	0.8880	0.6813
	0.1	0.7334	0.8701	1.3875	1.9070	1.6394	1.1752	0.8865	0.7293

computed based on complete data. The results are applied for type-II censored data and record samples. In addition, comparisons are made between these test estimators and a minimum risk-unbiased estimator based on records. Our numerical results show that the proposed test estimators are more efficient when the experimenter has a point guess  $\theta_0$  close to  $\theta$ . Selecting the best shrinkage preliminary test estimator depends on the guess value  $\theta_0$  and shrinkage factor  $k$ . In some situation, it may suffice to fix the parameter  $k$  at some given value. Another choice is it to chose the parameter  $k$  in a data-driven fashion by explicitly minimizing the risk of the shrinkage test estimator  $S$  given in (1.2). Our computations in Sections 4 and 5 show that the test estimator  $\hat{\theta}_{st}^1$ , constructed by the corresponding shrinkage factor  $k = k_1$  and obtained by minimizing the risk of shrinkage estimator  $S$ , performs better than other shrinkage test estimators when  $\delta$  is close to 1.

In a Bayesian perspective, a Bayes estimator is derived by employing a flexible prior distribution for the parameter of interest. Naghizadeh Qomi (2016) considered the problem of Bayesian shrinkage estimation in Rayleigh distribution under the squared log error loss. Construction of Bayes shrinkage estimators under the loss (1.5) is currently under investigation.

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