Area Specific Confidence Intervals for a Small Area Mean under the Fay-Herriot Model

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Abstract. Small area estimates have received much attention from both private and public sectors due to the growing demand for effective planning of health services, apportioning of government funds and policy and decision making. Surveys are generally designed to give representative estimates at national or district level, but estimates of variables of interest are often also needed at lower levels. These cannot be reliably obtained from the survey data as the sample sizes at these levels are too small. Census data are often available, but only give limited information with respect to the variables of interest. This problem is addressed by using small area estimation techniques, which combine the estimates from the survey and census data sets. The main purpose of this paper is obtaining confidence intervals based on the empirical best linear unbiased predictor (EBLUP) estimates. One of the criticism of the mean squared error (MSE) estimators is that it is not area-specific since it does not involve the direct estimator in its expression. However, most of the confidence intervals in the literature are constructed based on those MSEs. In this paper, we propose area specific confidence intervals for small area parameters under the Fay-Herriot model using area specific MSEs. We extend these confidence intervals to the difference between two small area means. The effectiveness of the proposed methods are also investigated via simulation studies and compared with the Cox (1975) and Prasad and Prasad and Rao (1990) methods. Our simulation results show that the proposed methods have higher coverage probabilities. Those methods are applied to the percentage of food expenditure measures in Ethiopia using the 2010/11 Household Consumption Expenditure (HCE) survey and the 2007 census data sets.

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1   Introduction

Small area statistics are needed for policy and decision making, local planning and fund distribution. It is known that direct estimates from a survey are highly unreliable as they have large standard errors, coefficients of variation and wide confidence intervals. The main reason behind this is that a sample survey is designed to provide reliable data for large areas such as national or regional levels, rather than that of small areas (Datta and Ghosh , 2012; Rao , 2003). Combining information from different surveys or from a census and a survey may address the problem (Chatterjee *et al.* , 2008; Rao , 2003). This is a less expensive tool for increasing sample sizes in order to improve the precision of the estimates.

In the Fay-Herriot model it is assumed that the parameter of interest is related to the vector of auxiliary variables through a linear regression model: $\theta_i = \mathbf{x}'_{\mathbf{i}}\boldsymbol{\beta} + v_i, i = 1, ..., m$. Let y_i $(i = 1, ..., m)$ denote the direct small area estimator of θ_i for area i from the survey; $\mathbf{x_i}=(x_{1i}, ..., x_{ki})'$ is a $k\times1$ vector of known covariates for area i related to the target parameters and $\boldsymbol{\beta} = (\beta_1, ..., \beta_k)'$ is a $k \times 1$ vector of regression coefficients. Using this, the Fay-Herriot model (Fay and Herriot , 1979) is defined as

$$y_i = \mathbf{x}'_i\boldsymbol{\beta} + v_i + e_i, i = 1, ..., m, \tag{1.1}$$

where the area specific random effects v_i are assumed to be independent and identically distributed (iid) with $E(v_i) = 0$, $Var(v_i) = A(\geq 0)$ and $e_i \stackrel{iid}{\sim} N[0, \psi_i]$.

Small area means or totals can be expressed as linear combinations of fixed and random effects. The best known method for the prediction of mixed effects is the best linear unbiased predictor (BLUP). It is a weighted combination of the direct estimator y_i and the regression synthetic estimator $\mathbf{x}'_i\hat{\boldsymbol{\beta}}$ (Rao , 2003). The BLUP for θ_i for the Fay-Herriot model is given by

$$\tilde{\theta}_i^B = (1 - \gamma_i)y_i + \gamma_i\mathbf{x}'_i\tilde{\boldsymbol{\beta}}. \tag{1.2}$$

When the unknown parameters (β, A) are replaced by their suitable estimators $(\hat{\beta}, \hat{A})$ obtained from different methods of estimation, for example, the method of moments (PR) used by Prasad and Rao (1990), Maximum Likelihood (ML) and Restricted Maximum Likelihood (REML) used by Datta and Lahiri (2000), the

Fay and Herriot (1979) (FH) estimation method, we obtain the empirical best linear unbiased predictor (EBLUP) of θ_i which is given by

$$\hat{\theta}_i^{EB} = (1 - \hat{\gamma}_i)y_i + \hat{\gamma}_i \mathbf{x'_i}\hat{\beta}, \quad i = 1, ..., m, \tag{1.3}$$

where $\hat{\beta} = (X'\hat{\mathbf{V}}^{-1}X)^{-1}X'\hat{\mathbf{V}}^{-1}y$, $\hat{\gamma}_i = \frac{\psi_i}{\psi_i + \hat{A}}$ and $\hat{\mathbf{V}} = diag(\psi_1, ..., \psi_m) + \hat{A}\mathbf{I}_m$.

Using the EBLUP estimate from a small area model, one can obtain confidence interval (CI), which is the area of interest for this paper. Datta *et al.* (2002) and Diao *et al.* (2013) among others have derived CIs using Taylor series expansion method. Hall and Maiti (2006), Chatterjee *et al.* (2008) and Kubokawa and Nagashima (2012) also addressed it using the parametric bootstrap method. This shows that resampling methods and Taylor series methods are common approaches for estimating MSEs and developing CIs when exact formulas are not available.

Calibration and computational difficulties of bootstrap methods have been discussed by Hall *et al.* (2000) and Nankervis (2005). In addition to this, the coverage accuracy of CIs based on bootstrap method has been improved through calibration (Chatterjee *et al.*, 2008). Chatterjee *et al.* (2008, page 7) state that "however, it is not always clear what property of an interval, that is, length, coverage, end points or some other characteristic, ought to be calibrated". "In addition to the computational difficulties especially in the case of calibration, these methods have other issues, for example, choosing between equal tail or shortest interval quantile points, and so on" (Diao *et al.* (2013, page 498)). There are also questions on the use of pivotal statistics and calibration (Chatterjee *et al.*, 2008). Considering this, it is found that calibration is both time and computational effort consuming and the results often lack straightforward interpretability (Chatterjee *et al.*, 2008). It is also not clear on how to construct CIs for the difference of two small area means using resampling approach (Diao *et al.*, 2013).

However, CIs based on the Taylor series expansion method are easy to implement in practice. This is because the model parameters need to be estimated only once using standard software, and can then be used in the construction of plug-in type CIs (Diao *et al.*, 2013).

One of the criticism of the MSE estimators are that they are not area-specific in the sense that they do not explicitly depend on the direct estimator although the area-specific auxiliary data is involved in the component of the uncertainty due to estimation of the regression coefficient (Rao, 2003). Following Rao (2001), we propose area specific MSE estimators and use them to construct CIs. It is known that most of the CIs in the literature are constructed by using the non area specific MSEs. Having this in mind, the objective of this paper is to propose area specific confidence intervals for a small area mean based on the EBLUP

estimator in the Fay-Herriot model.

It should be noted that when the estimated variance of the random component is negative, Prasad and Rao (1990) set it equal to zero. Those zero estimates will cause problems in the computing procedure and also lead the EBLUP estimator to run into difficulties (Chatterjee and Lahiri , 2002; Li , 2006). In addition, the contribution of the MSE estimate, assuming all parameters are known, becomes zero. This is one of the limitation of CIs based on EBLUP.

In summary, construction of CIs has received little attention when compared to point estimation in the issue of small area estimation methodology. Construction of CIs has also been limited and restricted to individual small area means. Furthermore, interval estimation for two or more population means is still an area to which very little attention has been paid. Most recently, Diao *et al.* (2013) proposed CIs for a small area mean as well as for the difference of two small area means. "We note that, since the small area estimators are not independent, this extension from a single prediction CI to one for differences (or more generally, linear combinations) of predictions is not immediate" (Diao *et al.* , 2013, page 4). In this paper we proposed CIs for a small area mean and for the difference of two small area means based on EBLUP for a coverage accuracy of order $O(m^{-3/2})$ under unequal sampling variances for the FH model.

This paper is organized as follows. In Section 2, we describe the MSE estimation. In Section 3, we present the proposed confidence intervals. In Section 4, we use the 2010/11 HCE survey and the 2007 census data sets to estimate percentage of food expenditure measures. The final Section presents concluding remarks.

2   Mean Squared Error Estimation

The most common practical problem in small area estimation is measuring the variability associated with the EBLUP. The MSE is generally used as a measure of variability under the EBLUP estimator. Methods of approximating and estimating $\text{MSE}(\hat{\theta}_i^{\text{EB}})$ that includes the uncertainty when estimating $\hat{\beta}$ and $\hat{A}$ have been studied extensively in the literature, e.g., in Rao (2003); Chatterjee *et al.* (2008); Datta and Ghosh (2012); Hall and Maiti (2006).

2.1   Estimation of $\text{MSE}(\hat{\theta}_i^{\text{EB}})$

The MSE estimator based on the PR moment estimator is given by

$$\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}}) = g_{1i}(\hat{A}) + g_{2i}(\hat{A}) + 2g_{3i}(\hat{A}), \tag{2.1}$$

where $g_{1i}(\hat{A}) = \hat{A}\psi_i(\hat{A} + \psi_i)^{-1}$, $g_{2i}(\hat{A}) = \frac{\psi_i^2}{(\hat{A}+\psi_i)^2}\mathbf{x}_i'\left\{\sum_{u=1}^{m}(\hat{A} + \psi_u)^{-1}\mathbf{x}_u\mathbf{x}_u'\right\}^{-1}\mathbf{x}_i$, and $g_{3i}(\hat{A}) = 2\psi_i^2(\psi_i + \hat{A})^{-3}m^{-2}\sum_{u=1}^{m}(\psi_u + \hat{A})^2$.

Datta *et al.* (2005) obtained the MSE estimator based on the FH moment estimator which is given by

$$\widehat{\mathrm{MSE}}(\hat{\theta}_i^{\mathrm{EB}}) = g_{1i}(\hat{A}) + g_{2i}(\hat{A}) + 2g_{3i}(\hat{A}) - b_{\hat{A}}(\hat{A})\left\{B_i(\hat{A})\right\}^2, \qquad (2.2)$$

where $g_{3i}(\hat{A}) = 2m\psi_i^2(\psi_i + \hat{A})^{-3}\left\{\sum_{u=1}^{m}(\psi_u + \hat{A})^{-1}\right\}^{-2}$.

For the FH estimator $\hat{A}$, we need the following expression for its bias to terms of order $O(m^{-1})$:

$$b_{\hat{A}}(\hat{A}) = \frac{2\left[m\sum_{i=1}^{m}(\hat{A} + \psi_i)^{-2} - \left\{\sum_{i=1}^{m}(\hat{A} + \psi_i)^{-1}\right\}^2\right]}{\left\{\sum_{i=1}^{m}(\hat{A} + \psi_i)^{-1}\right\}^3}.$$

Note that if $E\left[\widehat{\mathrm{MSE}}(\hat{\theta}_i^{\mathrm{EB}}) - \mathrm{MSE}(\hat{\theta}_i^{\mathrm{EB}})\right] = o(m^{-1})$, then $\widehat{\mathrm{MSE}}(\hat{\theta}_i^{\mathrm{EB}})$ is the second-order unbiased estimator.

2.2 Area Specific MSE Estimation for EBLUP

In this paper, we proposed alternative area specific MSE estimators by extending the Rao (2001) area specific estimators. Thus an area specific estimator of $g_{3i}(A)$ is simply obtained by writing $g_{3i}(A)$ as

$$g_{3i}^*(\hat{A}, y_i) = \psi_i^2(\psi_i + \hat{A})^{-4}E(y_i - \mathbf{x}_i'\hat{\beta})^2V(\hat{A}), \qquad (2.3)$$

where the middle expectation term can be estimated by $(y_i - \mathbf{x}_i'\hat{\beta})^2$, as noted by Rao (2001). The resulting estimator $g_{3i}^*(\hat{A})$ is area-specific since it involves the direct estimator. It is less stable than the Prasad and Rao $g_{3i}(\hat{A})$ even if its instability will not affect the corresponding MSE estimator. As a result, the coefficient of variation (CV) of $\widehat{\mathrm{MSE}}^*(\hat{\theta}_i^{\mathrm{EB}})$ should be comparable to the CV of $\widehat{\mathrm{MSE}}(\hat{\theta}_i^{\mathrm{EB}})$, at least for moderate to large m (Rao , 2003).

(a) An area specific estimator of $g_{3i}(\hat{A})$ according to Rao (2001) is given by

$$Rao\ (2001): \quad g_{3\mathrm{Rao}i}(\hat{A}, y_i) = \frac{\psi_i^2}{(\psi_i + A)^4}(y_i - \mathbf{x}_i'\hat{\beta})^2V(\hat{A}). \qquad (2.4)$$

(b) We propose another area specific estimator of $g_{3i}(\hat{A})$ which is an extension of Rao (2001)'s area specific $g_{3\text{Rao}i}(\hat{A}, y_i)$ and is given by

$$\text{JY}: \quad g_{3\text{JY}i}(\hat{A}, y_i) = \frac{\psi_i^2}{(\psi_i + A)^4} \frac{(y_i - \mathbf{x}_i'\hat{\beta})^2}{\hat{A} + \psi_i - \mathbf{x}_i'(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i} V(\hat{A}), \qquad (2.5)$$

where $Var(y_i - \mathbf{x}_i'\tilde{\beta}) = A + \psi_i - \mathbf{x}_i'(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i$. This proposal standardizes the residuals in order to make them scale free (Ananya , 2007).

(c) We propose another $g_{3\text{JY}1i}(A)$ which includes area dependent auxiliary variables as

$$\text{JY1}: \quad g_{3\text{JY}1i}(A) = g_{3i}(A) - g_{5i}(A), \qquad (2.6)$$

where $g_{5i}(A) = \frac{g_{2i}(A)}{(A+\psi_i)^2} V(\hat{A})$. Note that this estimator does not include y_i, but it includes the area specific auxiliary variables unlike the usual $g_{3i}(A)$.

In this paper, we derive the theorems using Stein identity. Stein (1981) showed that

$$E\left[h_i(y)(y_i - x_i'\beta)\right] = (\psi_i + A)E\left[\frac{\partial h_i(y)}{\partial y_i}\right], \qquad (2.7)$$

where $h_i(y)$ is an absolutely continuous function. This identity is called Stein identity (Datta *et al.* , 2011). The following regularity conditions are used for deriving all the theorems.

- Assumption 1: The quantities $\mathbf{x}_i, A, \psi_i$ are bounded.

- Assumption 2: v_i and e_i are mutually independent with $v_i \overset{ind}{\sim} N[0, A]$ and $e_i \overset{ind}{\sim} N[0, \psi_i], i = 1, ..., m$

- Assumption 3: $\hat{A} = \hat{A}(\mathbf{y})$ is an estimator of A which satisfies
  (i) they are even functions of $\mathbf{y}$, so that $\hat{A}(-\mathbf{y}) = \hat{A}(\mathbf{y})$,
  (ii) they are translation invariant functions, so that $\hat{A}(\mathbf{y} + \mathbf{X}d) = \hat{A}(\mathbf{y})$ for any $d \in R^p$ and for all $\mathbf{y}$.

- Assumption 4: $\max_{1 \le i \le m} \mathbf{x}_i' (\mathbf{X}'\mathbf{X})^{-1} \mathbf{x}_i \to 0$ as $m \to \infty$, or

$$\max_{1 \le i \le m} \mathbf{x}_i'(\mathbf{X}'\mathbf{X})^{-1} \mathbf{x}_i = O(m^{-1}),$$

(Prasad and Rao , 1990) where $\mathbf{x}_i$ is the i-th row of $\mathbf{X}$. All conditions are very often assumed in the small area estimation literature.

Theorem 2.1. *Under the assumptions 1, 2, 3 and 4,*

$$E|g_{3\text{JY}1i}(\hat{A}) - g_{3\text{JY}1i}(A)| = o(m^{-1})$$

The proof of this theorem can be found in a Supplementary material.

2.2.1 Area Specific MSE Estimation Based on the ML or REML Estimators

According to Rao (2003), the two alternative area specific estimators based on the ML Estimators (Datta and Lahiri , 2000) are given by

$$\widehat{\mathrm{MSE}}_1^*(\hat{\theta}_i^{\mathrm{EB}}) \approx g_{1i}(\hat{A}) - b_{\hat{A}}(\hat{A})\nabla g_{1i}(\hat{A}) + g_{2i}(\hat{A}) + 2g_{3i}^*(\hat{A}, y_i), \tag{2.8}$$

and

$$\widehat{\mathrm{MSE}}_2^*(\hat{\theta}_i^{\mathrm{EB}}) \approx g_{1i}(\hat{A}) - b_{\hat{A}}(\hat{A})\nabla g_{1i}(\hat{A}) + g_{2i}(\hat{A}) + g_{3i}(\hat{A}) + g_{3i}^*(\hat{A}, y_i), \tag{2.9}$$

where $b_{\hat{A}}(\hat{A}) = \frac{-tr\left\{[\sum_{i=1}^m (\hat{A}+\psi_i)^{-1}\mathbf{x}_i\mathbf{x}_i']^{-1}[\sum_{i=1}^m (\hat{A}+\psi_i)^{-2}\mathbf{x}_i\mathbf{x}_i']\right\}}{\sum_{i=1}^m (\hat{A}+\psi_i)^{-2}}$.

Similarly, the two alternative area specific estimators based on the REML Estimators (Datta and Lahiri , 2000) are given by

$$\widehat{\mathrm{MSE}}_1^*(\hat{\theta}_i^{\mathrm{EB}}) \approx g_{1i}(\hat{A}) + g_{2i}(\hat{A}) + 2g_{3i}^*(\hat{A}, y_i), \tag{2.10}$$

and

$$\widehat{\mathrm{MSE}}_2^*(\hat{\theta}_i^{\mathrm{EB}}) \approx g_{1i}(\hat{A}) + g_{2i}(\hat{A}) + g_{3i}(\hat{A}) + g_{3i}^*(\hat{A}, y_i). \tag{2.11}$$

2.2.2 Area Specific MSE Estimation Based on the Method of Moments

The two alternative area specific estimators based on the FH Estimator (Datta *et al.* , 2005) are given by

$$\widehat{\mathrm{MSE}}_1^*(\hat{\theta}_i^{\mathrm{EB}}) \approx g_{1i}(\hat{A}) + g_{2i}(\hat{A}) + 2g_{3i}^*(\hat{A}, y_i) - b_{\hat{A}}(\hat{A})\left\{B_i(\hat{A})\right\}^2, \tag{2.12}$$

and

$$\widehat{\mathrm{MSE}}_2^*(\hat{\theta}_i^{\mathrm{EB}}) \approx g_{1i}(\hat{A}) + g_{2i}(\hat{A}) + g_{3i}(\hat{A}) + g_{3i}^*(\hat{A}, y_i) - b_{\hat{A}}(\hat{A})\left\{B_i(\hat{A})\right\}^2. \tag{2.13}$$

Note that $g_{3i}^*(\hat{A}, y_i)$ can be an area specific estimator of $g_{3i}(\hat{A})$. See Rao (2003) for the detailed discussion about the extension of the MSE estimation to an area specific MSE estimation.

3 Area Specific Confidence Intervals under the Fay-Herriot Model

3.1 Area Specific Confidence Intervals for a Small Area Mean

In this paper, we develop a closed form area specific confidence intervals for θ_i based on $\hat{\theta}_i^{\mathrm{EB}}$ under unequal sampling error variances and that coverage probability is correct to $O(m^{-3/2})$. We derive the theorem similarly to Diao *et al.* (2013) using Taylor series expansion.

Theorem 3.1. *For any real $z > 0$,*

$$P\left[\theta_i \, \epsilon \, \hat{\theta}_i^{\mathrm{EB}} \pm z \times \sqrt{\widehat{\mathrm{MSE}}(\hat{\theta}_i^{\mathrm{EB}})}\right] = 2\Phi(z) - 1 - \frac{z\phi(z)}{4}\eta + O(m^{-3/2}),$$

where $\eta = (z^2 + 1)\frac{\psi_i^2}{A^2(A+\psi_i)^2}V(\hat{A})$ will give an EBLUP CI that matches the nominal CP to the order of $O(m^{-3/2})$.

Note that the proof of this theorem can be found in the Supplementary material.

These CIs based on the EBLUP can be written as

$$\hat{\theta}_i^{\mathrm{EB}} \pm t_\alpha^* \sqrt{\widehat{\mathrm{MSE}}(\hat{\theta}_i^{\mathrm{EB}})}, \tag{3.1}$$

where $t_\alpha^* = z_{\alpha/2}(1 + h(\hat{A}))$, $h(\hat{A}) = (z_{\alpha/2}^2 + 1)\frac{\psi_i^2}{8\hat{A}^2(\hat{A}+\psi_i)^2}V(\hat{A})$.

With this in mind, the recent CIs by Diao *et al.* (2013) is given by

$$\text{Diao et al. (2013)}: \quad \hat{\theta}_i^{\mathrm{EB}} \pm t_\alpha \sqrt{\widehat{\mathrm{MSE}}(\hat{\theta}_i^{\mathrm{EB}})}, \tag{3.2}$$

where $t_\alpha = z_{\alpha/2} + \frac{z_{\alpha/2}^3 + z_{\alpha/2}}{8h_i^4}\frac{g_{3i}\psi_i^2}{\hat{A}+\psi_i}$, and $h_i^2 = g_{1i} + g_{2i}$ for the PR and REML estimators.

Both (3.1) and (3.2) are constructed by using the non area specific MSEs which are given by (2.1) and (2.2). In addition, Theorem 3.1 is based on the usual non area specific $g_{3i}(A)$. However, the following corollaries are obtained by using the proposed area specific MSEs which are given by (2.8-2.13). Once Theorem 3.1 is proved, Corollary 3.1, Corollary 3.2 and Corollary 3.3 can be easily obtained through replacing the usual $g_{3i}(A)$ by area specific $g_{3i}(A, y_i)$'s given by (2.4), (2.5) and (2.6), respectively. The corresponding area specific CIs are also given in (3.7), (3.8) and (3.9), respectively.

Corollary 3.1. *The CP of $I^{\mathrm{CEBRao}}(\hat{A})$ based on the Rao (2001) is written as*

$$P[\theta_i \in I^{\mathrm{CEBRao}}(\hat{A})] = 2\Phi(z) - 1 - \frac{z\phi(z)}{4}\eta_{\mathrm{Rao}} + O(m^{-3/2}), \tag{3.3}$$

where $\eta_{\mathrm{Rao}} = (z^2 + 1)\frac{\psi_i^2}{\hat{A}^2(\hat{A}+\psi_i)^3}(y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}})^2 V(\hat{A})$.

Corollary 3.2. *The CP of $I^{\mathrm{CEBJY}}(\hat{A})$ based on JY is written as*

$$P[\theta_i \in I^{\mathrm{CEBJY}}(\hat{A})] = 2\Phi(z) - 1 - \frac{z\phi(z)}{4}\eta_{\mathrm{JY}} + O(m^{-3/2}),$$

where $\eta_{\mathrm{JY}} = (z^2 + 1)\frac{\psi_i^2}{\hat{A}^2(\hat{A}+\psi_i)^3}\frac{(y_i - \mathbf{x}_i'\tilde{\boldsymbol{\beta}})^2}{\hat{A}+\psi_i - \mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i}V(\hat{A})$.

Corollary 3.3. The CP of $I^{\text{CEBJY1}}(\hat{A})$ based on JY1 is written as

$$P[\theta_i \in I^{\text{CEBJY1}}(\hat{A})] = 2\Phi(z) - 1 - \frac{z\phi(z)}{4}\eta_{\text{JY1}} + O(m^{-3/2}),$$

where $\eta_{\text{JY1}} = (z^2 + 1)\frac{\psi_i^2}{\hat{A}^2(\hat{A}+\psi_i)^3}\left(\hat{A} + \psi_i - \mathbf{x}_i'(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i\right)V(\hat{A}).$

3.1.1 Comparison of Confidence Intervals for a Small Area Mean: Simulation Study

We consider the simulation set-up by Datta *et al.* (2005) with minor modifications. The sampling error, e_i, are generated from $N(0, \psi_i)$ for ψ_i specified by the following three different variance patterns: pattern (I) 0.7, 0.6, 0.5, 0.4, 0.3, pattern (II) 2, 0.6, 0.5, 0.4, 0.2 and pattern (III) 4, 0.6, 0.5, 0.4, 0.1. The random effects v_i are generated from two different distributions, namely normal $N(0, 1)$ and Laplace $(0, 1)$ distributions. We generate 10,000 data sets from $y_i = v_i + e_i$, $(i = 1, \cdots, m)$. The small areas are divided into five equal sized groups, and the ψ_i's remain the same in each group. For the FH model with m = 15, 20, 30, 40, 50, 60, 80 and 100 areas and $A = 1$, we have investigated the coverage probabilities and average lengths of the proposed methods. The coverage probability (CP) and average length (AL) of confidence intervals can be expressed as

$$CP_i = \sum_{i=1}^{m} \frac{|\theta_i \in CI_i|}{m}, \quad AL_i = \sum_{i=1}^{m} \frac{|length\ of\ CI_i|}{m}, \tag{3.4}$$

where CI_i, i=1,...,m, is the confidence interval of a certain method.

The main goal of this paper is to develop a closed form improved CIs for a small area mean based on the EBLUP using the Taylor series expansion methods under the FH model. In this section, a simulation study is conducted to investigate the performance of the proposed methods with the Cox (1975) and Prasad and Rao (1990) type CIs. The performance measures are the CP and AL for the 95% CIs. The results are based on 10000 simulations under the three different sampling variance patterns. We wrote an extension of SAS codes written by Li (2007). We also compared the results obtained by PR, FH, ML and REML estimators. Consider the following six different types of CIs.

- Cox (1975) suggested to generate the following CI:

$$\text{Method I (Meth I)}: \hat{\theta}_i^{\text{EB}} \pm z_{\alpha/2}\psi_i^{1/2}(1 - \hat{\gamma}_i)^{1/2}. \tag{3.5}$$

- Prasad and Rao (1990) proposed a CI which is given by

$$\text{Method II (Meth II)}: \hat{\theta}_i^{\text{EB}} \pm z_{\alpha/2}\sqrt{\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}})}. \tag{3.6}$$

• The improved CI based on the EBLUP is given by

$$\text{Method III (Meth III)} : \hat{\theta}_i^{\text{EB}} \pm t_\alpha^* \sqrt{\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}})}, \qquad (3.7)$$

where $t_\alpha^* = z_{\alpha/2}(1 + h(\hat{A}))$, $h(\hat{A}) = (z_{\alpha/2}^2 + 1)\frac{\psi_i^2}{8\hat{A}^2(\hat{A}+\psi_i)^2}V(\hat{A})$.

The corresponding area specific versions of (3.6) are written as

$$\text{CIRao} : \hat{\theta}_i^{\text{EB}} \pm t_{\text{Rao}} \sqrt{\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}})}, \qquad (3.8)$$

where $t_{\text{Rao}} = z_{\alpha/2} + (z_{\alpha/2}^3 + z_{\alpha/2})\frac{(A+\psi_i)g_{3\text{Rao}i}}{8A^2}$, $g_{3\text{Rao}i}(\hat{A}) = \frac{\psi_i^2}{(A+\psi_i)^4}(y_i - x_i'\hat{\beta})^2 V(\hat{A})$ and

$$\text{CIJY} : \hat{\theta}_i^{\text{EB}} \pm t_{\text{JY}} \sqrt{\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}})}, \qquad (3.9)$$

where

$$t_{\text{JY}} = z_{\alpha/2} + (z_{\alpha/2}^3 + z_{\alpha/2})\frac{(A + \psi_i)g_{3\text{JY}i}}{8A^2}$$

and

$$g_{3\text{JY}i}(\hat{A}, y_i) = \frac{\psi_i^2}{(\psi_i + \hat{A})^4}\frac{(y_i - \mathbf{x}_i'\tilde{\beta})^2}{\hat{A} + \psi_i - \mathbf{x}_i'(\mathbf{X'\Sigma^{-1}X})^{-1}\mathbf{x}_i}V(\hat{A}).$$

Another improved CI based on the EBLUP is written as

$$\text{CIJY1} : \hat{\theta}_i^{\text{EB}} \pm t_{\text{JY1}} \sqrt{\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}})}, \qquad (3.10)$$

where

$$t_{\text{JY1}} = z_{\alpha/2} + (z_{\alpha/2}^3 + z_{\alpha/2})\frac{(A + \psi_i)g_{3\text{JY1}i}}{8A^2}$$

and

$$g_{3\text{JY1}i}(\hat{A}) = g_{3i}(A) - \frac{g_{2i}(A)}{(A + \psi_i)^2}V(\hat{A}).$$

In this section, we compare six different CIs given in (3.4), (3.5), (3.6), (3.7), (3.8) and (3.9), which are referred as Method I, Method II, Method III, CIRao, CIJY and CIJY1, respectively. Method I and Method II represents the Cox (1975) and Prasad and Rao (1990) CIs for θ_i. Both of them are naive methods and have a coverage accuracy $O(m^{-1})$. However, Method III, CIRao, CIJY and CIJY1 are improved CIs with coverage accuracy $O(m^{-3/2})$. The CPs and ALs of the proposed methods are comparable with Diao *et al.* (2013)'s CIs. However, CIRao and CIJY are constructed based on the proposed area specific MSEs.

The simulation results in Tables 1-6 shows the estimated CPs and ALs of CIs for θ_i under normal and Laplace random effects distributions for number of small areas, $m = 20$. For example, when $\hat{A} = \hat{A}_{PR}$ and pattern (III), Method I has a CP of 80.2%, 81.8%, 81.8%, 82.0% and 84.1% for G1, G2, G3, G4 and G5 respectively for normal random effects distribution. The CPs of the proposed methods are larger than Method I and Method II for all patterns. Method III, CIRao, CIJY and CIJY1 meet the nominal coverage rate more frequently than Method I and Method II. Thus, Method III, CIRao, CIJY and CIJY1 perform better than Method I and Method II in terms of coverage accuracy. However, the AL of these Methods is a bit larger than Method I and Method II for large sampling variances (G1 and G2). The proposed methods extend the widths of Method I and Method II so as to satisfy the nominal confidence level (Kubokawa , 2011).

We have also considered Laplace distribution to assess the robustness of the methods to possible deviations from the normality assumptions. In terms of CPs the proposed methods perform better than Method I and Method II. Method I has got a series under coverage problem (as low as 66.6%, 66.4%, 66.6%, 66.6% and 67.7% for G1, G2, G3, G4 and G5 respectively, values which are far below the 95% nominal value) for pattern (III) and when $A = \hat{A}_{PR}$ under Laplace distribution.

Figure 2 reports the CPs of CIs over a range of number of small areas m for pattern (II). The CPs and ALs of CIs of all the methods are obtained through the above simulation exercise for areas such as m=15, 20, 30, 40, 50, 60, 80 and 100. As the number of small areas m increases the CPs increase and the ALs decrease for all methods and patterns. When $m \geq 60$, there is very little difference introduced by using any of the methods in terms of ALs and CPs for all patterns. All the methods except Method I perform better any time in terms of CPs and ALs. There is also no clear difference among the proposed methods in terms of CPs and ALs under the normal distribution. The percentage difference between CPs and ALs are negligible.

3.2 CIs for the Difference of Two Small Area Means

In this section we are interested in the difference of two small area means than in the specific values of the small area means themselves. The difference between estimated means (i.e., direct or EBLUP) gives information about the difference between population means. This is also an extension of Diao *et al.* (2013) CIs for $\theta_i - \theta_j$ where $i \neq j$ based on EBLUP estimator. In this paper, we develop a closed form area specific confidence intervals for $\theta_i - \theta_j$ based on $\hat{\theta}_i^{\text{EB}} - \hat{\theta}_j^{\text{EB}}$ under unequal sampling error variances and coverage probability is correct to $O(m^{-3/2})$. We derive this theorem similar to Theorem 3.2. Note that the proof of this theorem can also be found in the Supplementary material.

Theorem 3.2. *For any z > 0, the improved confidence interval is given by*

$$P\left[\theta_i - \theta_j \in I^{\text{CEBD}}(\hat{A})\right] = 2\Phi(z) - 1 - \frac{z\phi(z)}{4}\eta_D + O(m^{-3/2}),$$

where

$$\eta_D = (z^2 + 1)\left(\frac{A\psi_i}{A + \psi_i} + \frac{A\psi_j}{A + \psi_j}\right)^{-2}\left(\frac{\psi_i^2}{(A + \psi_i)^2} + \frac{\psi_j^2}{(A + \psi_j)^2}\right)^2 V(\hat{A}).$$

Consider the following three corollaries that can be easily obtained from Theorem 3.2.

Corollary 3.4. *The CP of $I^{\text{CEBDRao}}(\hat{A})$ based on the Rao (2001) is written as*

$$P[\theta_i - \theta_j \in I^{\text{CEBDRao}}(\hat{A})] = 2\Phi(z) - 1 - \frac{z\phi(z)}{4}\eta_{\text{RaoD}} + O(m^{-3/2}), \qquad (3.11)$$

where

$$\eta_{\text{RaoD}} = \frac{(z_{\alpha/2}^2 + 1)}{8}\left(g_{1i}(A) + g_{1j}(A)\right)^{-2}\frac{\left((A + \psi_i)g_{3\text{Rao}i}(A) + (A + \psi_j)g_{3\text{Rao}j}(A)\right)^2}{V(A)},$$

$g_{3\text{Rao}i}(\hat{A}) = \frac{\psi_i^2}{(\hat{A}+\psi_i)^4}(y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}})^2 V(\hat{A})$ and $g_{3\text{Rao}j}(\hat{A}) = \frac{\psi_j^2}{(\hat{A}+\psi_j)^4}(y_j - \mathbf{x}_j'\hat{\boldsymbol{\beta}})^2 V(\hat{A})$.

Corollary 3.5. *The CP of $I^{\text{CEBDJY}}(\hat{A})$ based on JY is written as*

$$P[\theta_i - \theta_j \in I^{\text{CEBDJY}}(\hat{A})] = 2\Phi(z) - 1 - \frac{z\phi(z)}{4}\eta_{\text{JYD}} + O(m^{-3/2}),$$

where

$$\eta_{\text{JYD}} = \frac{(z_{\alpha/2}^2 + 1)}{8}\left(g_{1i}(A) + g_{1j}(A)\right)^{-2}\frac{\left((A + \psi_i)g_{3\text{JY}i}(A) + (A + \psi_j)g_{3\text{JY}j}(A)\right)^2}{V(A)},$$

$$g_{3\text{JY}i}(\hat{A}) = \frac{\psi_i^2}{(\hat{A} + \psi_i)^4}\frac{(y_i - \mathbf{x}_i'\tilde{\boldsymbol{\beta}})^2}{\hat{A} + \psi_i - \mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i}V(\hat{A})$$

and

$$g_{3\text{JY}j}(\hat{A}) = \frac{\psi_j^2}{(\hat{A} + \psi_j)^4}\frac{(y_j - \mathbf{x}_j'\hat{\boldsymbol{\beta}})^2}{\hat{A} + \psi_j - \mathbf{x}_j'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_j}V(\hat{A}).$$

Corollary 3.6. The CP of $I^{\text{CEBDJY1}}(\hat{A})$ based on JY1 is written as

$$P[\theta_i - \theta_j \in I^{\text{CEBDJY1}}(\hat{A})] = 2\Phi(z) - 1 - \frac{z\phi(z)}{4}\eta_{\text{JY1D}} + O(m^{-3/2}),$$

where

$$\eta_{\text{JY1D}} = \frac{(z_{\alpha/2}^2 + 1)}{8}\left(g_{1i}(A) + g_{1j}(A)\right)^{-2}\frac{\left((A + \psi_i)g_{3\text{JY1}i}(A) + (A + \psi_j)g_{3\text{JY1}j}(A)\right)^2}{V(A)},$$

$$g_{3\text{JY1}i}(A) = g_{3i}(A) - g_{5i}(A), \quad \text{where} \quad g_{5i}(A) = \frac{g_{2i}(A)}{(A + \psi_i)^2}V(\hat{A})$$

and

$$g_{3\text{JY1}j}(A) = g_{3j}(A) - g_{5j}(A), \quad \text{where} \quad g_{5j}(A) = \frac{g_{2j}(A)}{(A + \psi_j)^2}V(\hat{A}).$$

3.2.1 Comparison of Confidence Intervals for the Difference between Two Small Area Means: Simulation Study

The main purpose of this section is to find out which method performs best under normal and non normal random effects distributions. Here we have adopted the simulation set up of Section 3.1.1 with some modifications.

The two kinds of CIs for $\theta_i - \theta_j$ where $i \neq j$ are computed using the following formula:

$$\text{Method ID (MetID)}: \hat{\theta}_i^{\text{EB}} - \hat{\theta}_j^{\text{EB}} \pm z_{\alpha/2}\left(\psi_i^{1/2}(1 - \hat{\gamma}_i)^{1/2} + \psi_j^{1/2}(1 - \hat{\gamma}_j)^{1/2}\right), \quad (3.12)$$

$$\text{Method IID (MetIID)}: \hat{\theta}_i^{\text{EB}} - \hat{\theta}_j^{\text{EB}} \pm z_{\alpha/2}\sqrt{\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}}) - \widehat{\text{MSE}}(\hat{\theta}_j^{\text{EB}})}. \quad (3.13)$$

The improved CIs for $\theta_i - \theta_j$ where $i \neq j$ based on the EBLUP estimator in the FH model under unequal sampling variances consideration are given by

$$\text{Method IIID (MetIIID)}: \quad \hat{\theta}_i^{\text{EB}} - \hat{\theta}_j^{\text{EB}} \pm t_{\alpha_D}\sqrt{\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}}) - \widehat{\text{MSE}}(\hat{\theta}_j^{\text{EB}})}, \quad (3.14)$$

where $t_{\alpha_D} = z_{\alpha/2}\left[\frac{(z_{\alpha/2}^2+1)}{8}\left(g_{1i}(A) + g_{1j}(A)\right)^{-2}\frac{((A+\psi_i)g_{3i}(A)+(A+\psi_j)g_{3j}(A))^2}{V(A)} + 1\right]$.

Similarly, the corresponding area specific versions of (3.12) are written as

$$\text{CIRaoD}: \hat{\theta}_i^{\text{EB}} - \hat{\theta}_j^{\text{EB}} \pm t_{\text{RaoD}}\sqrt{\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}}) - \widehat{\text{MSE}}(\hat{\theta}_j^{\text{EB}})}, \quad (3.15)$$

where and

$$\text{CIJYD}: \hat{\theta}_i^{\text{EB}} - \hat{\theta}_j^{\text{EB}} \pm t_{\text{JYD}}\sqrt{\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}}) - \widehat{\text{MSE}}(\hat{\theta}_j^{\text{EB}})}, \quad (3.16)$$

where $t_{\text{JYD}} = z_{\alpha/2} \left[\frac{(z_{\alpha/2}^2+1)}{8} \left(g_{1i}(A) + g_{1j}(A) \right)^{-2} \frac{\left((A+\psi_i) g_{3\text{JY}i}(A) + (A+\psi_j) g_{3\text{JY}j}(A) \right)^2}{V(A)} + 1 \right]$.

Another improved CIs for $\theta_i - \theta_j$ based on $\hat{\theta}_i^{\text{EB}} - \hat{\theta}_j^{\text{EB}}$ where $i \neq j$ is written as

$$\text{CIJY1D} : \hat{\theta}_i^{\text{EB}} - \hat{\theta}_j^{\text{EB}} \pm t_{\text{JY1D}} \sqrt{\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}}) - \widehat{\text{MSE}}(\hat{\theta}_j^{\text{EB}})} \tag{3.17}$$

where $t_{\text{JY1D}} = z_{\alpha/2} \left[\frac{(z_{\alpha/2}^2+1)}{8} \left(g_{1i}(A) + g_{1j}(A) \right)^{-2} \frac{\left((A+\psi_i) g_{3\text{JY1}i}(A) + (A+\psi_j) g_{3\text{JY1}j}(A) \right)^2}{V(A)} + 1 \right]$.

Similar to the previous case, in this section, we compare six different CIs given in (3.10), (3.11), (3.12), (3.13), (3.14) and (3.15), which are referred as Method ID, Method IID, Method IIID, CIRaoD, CIJYD and CIJY1D, respectively. We see that the CPs of Method IIID, CIRaoD, CIJYD and CIJY1D behaved satisfactorily. They never differed from their 95% nominal value by more than 1.9%, 0.8%, 0.8% and 0.2% for FH, PR, ML and REML estimators, respectively, for pattern (II) and normal random effects distribution. The ALs of the proposed methods are a bit larger than Method ID and Method IID especially for large sampling variances (G1 and G2) and small sample sizes. This is to be expected since Method IIID, CIRaoD, CIJYD and CIJY1D increases the width of the intervals to meet the nominal coverage.

As far as the Laplace distribution is concerned, we see that for $m = 20$ there is under coverage problem for all the methods, but the under coverage problem becomes more series for Method ID. For this distribution, the CPs of the proposed methods never differed from their 95% nominal value by more than 3.9%, 4.2%, 3.2% and 3.5% (slightly below the 95% nominal value) for FH, PR, ML and REML estimators, respectively, for pattern (II). These findings are sensible for this heavy tail distribution. Like in the previous section, Figure 3 report CPs of the methods over a range of number of small areas, m, for pattern (II). As the number of small areas, m, increases the CPs increase for all the methods and patterns. The CPs and ALs for all variance patterns can be found in the Supplementary material.

4 Application to Real Data

In this section, we illustrate the proposed methods to estimate percentage of food expenditure by using the 2010/11 HCE and the 2007 census data sets of Ethiopia. Expenditure refers to total expenditure, including accommodation, food, purchases, travel, leisure activities and miscellaneous expenditure. The proposed CIs are used to estimate the percentage of food expenditure relative to total expenditure in 94 small areas (zones) in Ethiopia.

In 2012, only 6.6% of consumer expenditure go to food consumed at home relative to total consumption expenditures in America. The amount of income

spent on food fell from 17.5% to 9.7% between 1960 and 2007 in America (Morrison , 2014). Countries such as Spain, Norway, Belgium and France spend twice as much on food compared to America. Whereas countries like Mexico, China and Turkey spend three times as much on food compared to America. In contrast, people in poor countries are forced to spend the largest share of income on food (The Economist, 2013). The percentage of food expenditure spend on food in 2012 for some African countries such as Ethiopia, Kenya, Niger, Namibia and South Africa are 39.84%, 47.44%, 44.99%, 24.36% and 26.04%, respectively ("Household consumption expenditure", 2014). Food prices started increasing in 2007 and exerting pressure on household budgets. Low income households spend a greater percentage of income on food (Carr *et al.* , 2014). If you spent very less on food, that leaves more money to things like health care, education, energy, home and saving.

4.1 Generalized Variance Function

We use the generalized variance function (GVF) to smooth out the uncertainty of the design based variance estimate. The factors $exp(\frac{\hat{\sigma}^2}{2}) = 1.29$ are the bias-correction terms in the log-linear analysis for the percentage of food expenditure estimates. Underestimation of the true variances will be occurred when we ignore the correction term in GVF method (Rivest and Belmonte , 2000; Rao , 2001). Figure 2 (top two) illustrates the GVF and variance in dots and stars for the percentage of food expenditure measures. From the figure we can easily observe that how the GVF smooth out the unreliable and noisy design based estimated variances (Esteban *et al.* , 2012; Hawala and Lahiri , 2010).

4.2 Percentage of Food Expenditure Estimates

The direct and model-based CI estimates are computed for the percentage of food expenditure measures. We present some of the CI estimates based on CIRao since almost all the proposed methods perform similarly. Zone3 50.03% (95% CI=47.21% - 52.86%), Borena 51.10% (95% CI=48.26% - 53.94%), Itang 52.68% (95% CI=49.88% - 55.49%), Nuer 52.73% (95% CI=49.78% - 55.68%) and Basketo 59.64% (95% CI=56.74% - 62.54%) spent the highest share of income on food (above 50%). This result matches with other economical indicators, as they are quite deprived provinces where the percentage of food expenditure is important" Militino *et al.* (2012, page 2941). In contrast, zones such as Zone5 29.90% (95% CI=27.21%–32.61%), Konso 30.71% (95% CI=27.86% - 33.56%), Mekelle 32.54% (95% CI=29.73% - 35.36%), Nifas Silk 32.88% (95% CI=30.01% - 35.74%), Bahirdar 33.83% (95% CI=31.04% - 36.61%), Akaki 34.26% (95% CI=31.43% - 37.09%) and

Adama 35.14% (95% CI=32.34% - 37.94%) spent the smallest share of income on food.

4.3 Coverage Diagnostic

This method evaluates the validity of the CIs based on EBLUP estimates. It is used to measure the overlap between the 95% CIs generated by the direct and the model based estimates in small area estimation procedure (Brown et al. , 2001). This test can be done as follows. Let X_1 and X_2 be two independent normal random variables having the same mean and variance $\sigma_{X_1}^2$ and $\sigma_{X_2}^2$ respectively. If $z(\alpha)$ is such that the probability that a standard normal variable takes values greater than $z(\alpha)$ is $\alpha/2$, then a sufficient condition for there to be probability of α that the two intervals $X_1 \pm z(\beta)\sigma_{X_1}$ and $X_2 \pm z(\beta)\sigma_{X_2}$ do not overlap is when

$$
z(\beta) = z(\alpha)\left(1 + \frac{\sigma_{X_1}}{\sigma_{X_2}}\right)^{-1}\sqrt{1 + \frac{\sigma_{X_1}^2}{\sigma_{X_2}^2}}, \tag{4.1}
$$

where σ_{X_1} is the estimated standard error of the EBLUP estimate and σ_{X_2} is the estimated standard error of the direct estimate, $z(\alpha) = 1.96$. In our real data application $z(\beta) \approx 1.39$. This confirms that the overlap proportion is 95%. In addition, Figure 1 shows that the CIs based on the EBLUP estimates lies within the CIs of the direct estimates.

5 Conclusion

The findings of this study indicate that there is no single method which dominates all the other methods in terms of the proposed methods. However, all the proposed methods have been numerically shown to be superior to the

Furthermore, CIs based on resampling approach do not have sensible empirical plug-in which is routinely used in place of the unknown parameters (Chatterjee and Lahiri , 2002). However, the advantages of our proposed methods are computational. Once we estimate the unknown parameters, then plug-in the estimates to MSEs and CIs.

The usefulness has also been shown through the application to the HCE survey data in Ethiopia to estimate percentage of food expenditure measures at the zone level. We derive zonal-level estimates of percentage of food expenditure measures by using small area estimation techniques to link data from the 2010/11 HCE and 2007 population and housing Census.
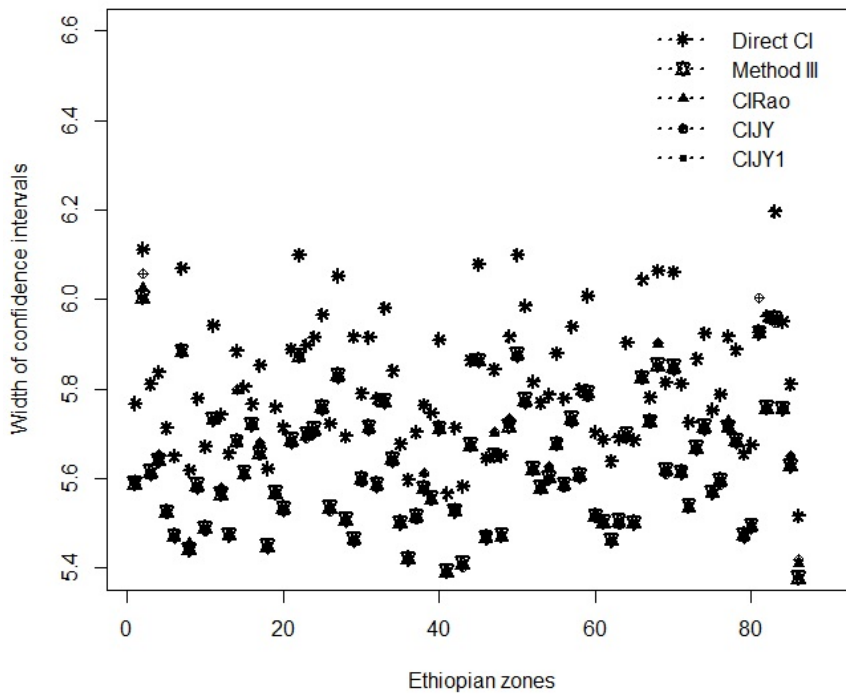
Figure 1: A plot illustrating the width of CIs of percentage of food expenditure estimates versus Ethiopian zones in 2010/11; Direct CI refers the CI based on the direct survey estimates. `Method III`, `CIRao`, `CIJY` and `CIJY1` referring the CIs given by equations (3.7 - 3.10).

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Appendices

Appendix A: Proof of Theorem 2.1

Proof.

$$\tilde{g}_{3JY1i}(\hat{A}) - \tilde{g}_{3JY1i}(A) = (g_{3i}(\hat{A}) - g_{3i}(A)) - (g_{5i}(\hat{A}) - g_{5i}(A))$$
$$= I_1 - I_2.$$

According to Prasad and Rao (1990),

$$E|I_1| = |g_{3i}(\hat{A}) - g_{3i}(A)| = o(m^{-1}).$$

Let us show that $E|I_2| = o(m^{-1})$.

$$g_{5i}(A) = \frac{g_{2i}(A)}{(A + \psi_i)^2} V(\hat{A})$$
$$= \frac{\psi_i^2}{(A + \psi_i)^4} \mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i V(\hat{A})$$

For example, for the PR moment estimator, $V(\hat{A}) = \frac{2}{m^2}\sum_{i=1}^{m}(A + \psi_i)^2$. Substituting reveals that

$$g_{5i}(A) = \frac{2\psi_i^2 \sum_{i=1}^{m}(A + \psi_i)^2}{m^2(A + \psi_i)^4} \mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i,$$

$$I_2 = g_{5i}(\hat{A}) - g_{5i}(A)$$
$$= \frac{2\psi_i^2 \sum_{i=1}^{m}(\hat{A} + \psi_i)^2}{m^2(\hat{A} + \psi_i)^4} \mathbf{x}_i'(\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{X})^{-1}\mathbf{x}_i - \frac{2\psi_i^2 \sum_{i=1}^{m}(A + \psi_i)^2}{m^2(A + \psi_i)^4} \mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i.$$

Rewriting I_2

$$I_2 = \frac{2\psi_i^2 \sum_{i=1}^m (\hat{A} + \psi_i)^2}{m^2 (\hat{A} + \psi_i)^4} \mathbf{x}_i'(\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{X})^{-1}\mathbf{x}_i - \frac{2\psi_i^2 \sum_{i=1}^m (\hat{A} + \psi_i)^2}{m^2 (\hat{A} + \psi_i)^4} \mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i$$

$$+ \frac{2\psi_i^2 \sum_{i=1}^m (\hat{A} + \psi_i)^2}{m^2 (\hat{A} + \psi_i)^4} \mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i - \frac{2\psi_i^2 \sum_{i=1}^m (A + \psi_i)^2}{m^2 (A + \psi_i)^4} \mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i$$

$$= (2/m)\psi_i^2(\hat{A} + \psi_i)^{-4}(\hat{A}^2 + 2\hat{A}\bar{\psi} + \sum \psi_i^2/m)\left[\mathbf{x}_i'(\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{X})^{-1}\mathbf{x}_i - \mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i\right]$$

$$+ \mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i\left[(2/m^2)\psi_i^2(\hat{A} + \psi_i)^{-4}\sum_{i=1}^m (\hat{A} + \psi_i)^2\right.$$

$$\left. - (2/m^2)\psi_i^2(A + \psi_i)^{-4}\sum_{i=1}^m (A + \psi_i)^2\right].$$

Applying Taylor series expansion around A, we get

$$I_3 = \mathbf{x}_i'(\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{X})^{-1}\mathbf{x}_i - \mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i$$

$$= (\hat{A} - A)\left[\partial \mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i/\partial A\right]_{A=A^*}$$

$$= -(\hat{A} - A)\mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{*-1}\mathbf{X})^{-1}(\mathbf{X}'\boldsymbol{\Sigma}^{*-2}\mathbf{X})^{-1}(\mathbf{X}'\boldsymbol{\Sigma}^{*-1}\mathbf{X})^{-1}\mathbf{x}_i$$

and $E|I_3| = o(m^{-1})$ (see Prasad and Rao, 1990).

It can also been shown that

$$\mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i \leq (1/m)(A + \psi_U)(m \, max_i\mathbf{x}_i'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{x}_i) = O(m^{-1})$$

$$I_4 = (2/m^2)\psi_i^2\left[(\hat{A} + \psi_i)^{-4}\sum_{i=1}^m (\hat{A} + \psi_i)^2 - (A + \psi_i)^{-4}\sum_{i=1}^m (A + \psi_i)^2\right]$$

$$= (2/m)\psi_i^2(\hat{A} + \psi_i)^{-4}\left[(\hat{A} - A)^2 + 2(A + \bar{\psi})(\hat{A} - A)\right]$$

$$+ (2/m)\psi_i^2(A^2 + 2A\bar{\psi} + \sum \psi_i^2/m)\left[(\hat{A} + \psi_i)^{-4} - (A + \psi_i)^{-4}\right]$$

$$= I_5 + I_6,$$

where $\bar{\psi} = \sum_{i=1}^m \psi_i/m$.

Using a similar argument to Prasad and Rao (1990), it can be shown that

$$E|I_5| = o(m^{-1})$$

and

$$E|I_6| = o(m^{-1}).$$

Thus, $E|\tilde{g}_{3JY1i}(\hat{A}) - \tilde{g}_{3JY1i}(A)| = o(m^{-1})$. $\qquad\qquad\square$

Appendix B: Proof of Theorem 3.1

Proof. The proof of the theorem about CIs based on the EBLUP estimator is given. The conditional distribution of θ_i given y_i is given by

$$\theta_i \mid y_i \sim N(\hat{\theta}_i^{\mathrm{EB}}, g_{1i}(A)),$$

where $\hat{\theta}_i^{\mathrm{EB}} = (1 - \hat{\gamma}_i)y_i + \hat{\gamma}_i x_i'\hat{\beta}$, $\tilde{\theta}_i^{B} = (1 - \gamma_i)y_i + \gamma_i x_i'\tilde{\beta}$, $g_{1i}(A) = \frac{A\psi_i}{A+\psi_i}$.

Consider the following expression as

$$P\left[\theta_i \le \hat{\theta}_i^{\mathrm{EB}} + z \times \sqrt{\widehat{\mathrm{MSE}}(\hat{\theta}_i^{\mathrm{EB}})}\right] = E[\Phi(z + F(z))]$$

The symbols $\Phi(z)$ and $\phi(z)$ represents the distribution function and the probability density function, respectively, of the $N(0,1)$ distribution. Thus, the CP of $I^{\mathrm{CEB}}(\hat{A})$ is written as

$$P[\theta_i \in I^{\mathrm{CEB}}(\hat{A})] = P[-z + F(-z) < \frac{\theta_i - \hat{\theta}_i^{\mathrm{EB}}}{\sqrt{g_{1i}(A)}} < z + F(z)]$$
$$= E[\Phi(z + F(z)) - \Phi(-z + F(-z))],$$

where $F(z) = \frac{z \times \left(\sqrt{\widehat{\mathrm{MSE}}(\hat{\theta}_i^{\mathrm{EB}})} - \sqrt{g_{1i}(A)}\right) + \hat{\theta}_i^{\mathrm{EB}} - \tilde{\theta}_i^{B}}{\sqrt{g_{1i}(A)}}$.

Using the Taylor expansion with an integral remainder term, $\Phi(z + F(z))$ is evaluated as

$$\Phi(z + F(z)) = \Phi(z) + F(z)\phi(z) + \frac{1}{2}F^2(z)\phi'(z)$$
$$+ \frac{1}{2}\int_z^{z+F(z)} (z + F(z) - x)^2\phi''(x)dx$$
$$= \Phi(z)$$
$$+ [F(z) - \frac{z}{2}F^2(z)]\phi(z) - \frac{1}{2}\int_z^{z+F(z)} (z + F(z) - x)^2(1 - x^2)\phi(x)dx.$$

Taking expectation on both sides reveals that

$$E\left(\Phi(z + F(z))\right) = \Phi(z) + E\left[F(z)\phi(z) - \frac{z}{2}F^2(z)\phi(z)\right.$$
$$\left. - \frac{1}{2}\int_z^{z+F(z)} (z + F(z) - x)^2(1 - x^2)\phi(x)dx.\right.$$

First consider that

$$\hat{\theta}_i^{EB} - \tilde{\theta}_i^B = (1 - \hat{\gamma}_i)y_i + \hat{\gamma}_i x_i'\hat{\beta} - (1 - \gamma_i)y_i - \gamma_i x_i'\tilde{\beta}$$
$$= (\gamma_i - \hat{\gamma}_i)r_i + \gamma_i x_i'(\hat{\beta} - \tilde{\beta}),$$

where $r_i = (y_i - x_i'\hat{\beta})$. Thus $F(z)$ can be decomposed into three components as

$$F(z) = \frac{z \times \left(\sqrt{\widehat{MSE}(\hat{\theta}_i^{EB})} - \sqrt{g_{1i}(A)} \right)}{\sqrt{g_{1i}(A)}} + \frac{(\gamma_i - \hat{\gamma}_i)r_i}{\sqrt{g_{1i}(A)}} + \frac{\gamma_i x_i'(\hat{\beta} - \tilde{\beta})}{\sqrt{g_{1i}(A)}}$$
$$= S_1 + S_2 + S_3 \quad say.$$

Let us evaluate the expected value of each term as

$$EF(z) = E(S_1 + S_2 + S_3)$$
$$= E(S_1) + E(S_2) + E(S_3).$$

Applying the Taylor series expansion and taking expectation on both sides,

$$ES_1 = z\frac{1}{2} (g_{1i}(A))^{-1} E\left(\widehat{MSE}(\hat{\theta}_i^{EB}) - g_{1i}(A) \right)$$
$$- z\frac{1}{8} (g_{1i}(A))^{-2} E\left(\widehat{MSE}(\hat{\theta}_i^{EB}) - g_{1i}(A) \right)^2$$
$$+ z\frac{3}{16} (g_{1i}(A))^{-1/2} E \int_{g_{1i}(A)}^{\widehat{MSE}(\hat{\theta}_i^{EB})} x^{-5/2} \left(\widehat{MSE}(\hat{\theta}_i^{EB}) - g_{1i}(A) \right)^2 dx.$$

Let us compute the expectation of each term as

$$E\left(\widehat{MSE}(\hat{\theta}_i^{EB}) - g_{1i}(A) \right) = E(\widehat{MSE}(\hat{\theta}_i^{EB})) - E(g_{1i}(A)),$$
$$E(\widehat{MSE}(\hat{\theta}_i^{EB})) = MSE(\hat{\theta}_i^{EB}),$$

since $E(\widehat{MSE}(\hat{\theta}_i^{EB}))$ is nearly a second order unbiased estimator of $MSE(\hat{\theta}_i^{EB})$. Thus, using the Taylor expansion, the second term can be expressed as

$$\widehat{MSE}(\hat{\theta}_i^{EB}) = MSE(\hat{\theta}_i^{EB}(A)) + MSE(\hat{\theta}_i^{EB}(A^*))^{(1)}(\hat{A} - A) + O(m^{-3/2}),$$

$$E\left(\widehat{MSE}(\hat{\theta}_i^{EB}) - g_{1i}(A) \right)^2 = (g_{1i}^{(1)}(A^*))^2 E(\hat{A} - A)^2,$$

where $g_{1i}(A) = \frac{A\psi_i}{A+\psi_i}$, $g_{1i}^{(1)}(A) = \frac{\psi_i}{A+\psi_i} - \frac{A\psi_i}{(A+\psi_i)^2} = \frac{\psi_i^2}{(A+\psi_i)^2}$, $(g_{1i}^{(1)}(A))^2 = \frac{\psi_i^4}{(A+\psi_i)^4}$.

Substituting reveals that

$$E\left(\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}}) - g_{1i}(A)\right)^2 = \frac{\psi_i^4}{(A+\psi_i)^4}E(\hat{A}-A)^2 + O(m^{-3/2}),$$

where $E(\hat{A}-A)^2 = V(\hat{A}) + O_p(m^{-1})$, $V(\hat{A})$ can be the asymptotic variance of the FH, PR, ML and REML estimators of A. Substituting reveals that

$$E\left(\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}}) - g_{1i}(A)\right)^2 = \frac{\psi_i^4}{(A+\psi_i)^4}\frac{2}{m^2}\sum_{i=1}^{m}(A+\psi_i)^2 + O(m^{-1}).$$

Let us compute the integral term using the same approach by Datta *et al.* (2002); Chatterjee and Lahiri (2002) and Diao *et al.* (2013) as follows.

$$E\int_{g_{1i}(A)}^{\widehat{\text{MSE}}(\hat{\theta}_i^{EB})} x^{-5/2}\left(\widehat{\text{MSE}}(\hat{\theta}_i^{EB}) - g_{1i}(A)\right)^2 dx = EI_1 + EI_2$$

$$EI_1 = \left[E\int_{g_{1i}(A)}^{\widehat{\text{MSE}}(\hat{\theta}_i^{EB})} x^{-5/2}\left(\widehat{\text{MSE}}(\hat{\theta}_i^{EB}) - g_{1i}(A)\right)^2 dx\right]I_{\{\widehat{\text{MSE}}(\hat{\theta}_i^{EB})\geq g_{1i}(A)\}}$$

and

$$EI_2 = \left[E\int_{g_{1i}(A)}^{\widehat{\text{MSE}}(\hat{\theta}_i^{EB})} x^{-5/2}\left(\widehat{\text{MSE}}(\hat{\theta}_i^{EB}) - g_{1i}(A)\right)^2 dx\right]I_{\{\widehat{\text{MSE}}(\hat{\theta}_i^{EB})\leq g_{1i}(A)\}}$$

But

$$
\begin{aligned}
EI_1 &= \left[E\int_{g_{1i}(A)}^{\widehat{\text{MSE}}(\hat{\theta}_i^{EB})} x^{-5/2}\left(\widehat{\text{MSE}}(\hat{\theta}_i^{EB}) - g_{1i}(A)\right)^2 dx\right]I_{\{\widehat{\text{MSE}}(\hat{\theta}_i^{EB})\geq g_{1i}(A)\}}\\
&\leq \left[E\int_{g_{1i}(A)}^{\widehat{\text{MSE}}(\hat{\theta}_i^{EB})} (g_{1i}(A))^{-5/2}\left(\widehat{\text{MSE}}(\hat{\theta}_i^{EB}) - g_{1i}(A)\right)^2 dx\right]I_{\{\widehat{\text{MSE}}(\hat{\theta}_i^{EB})\geq g_{1i}(A)\}}\\
&\leq \left[\frac{1}{3}(g_{1i}(A))^{-5/2}E\left(\widehat{\text{MSE}}(\hat{\theta}_i^{EB}) - g_{1i}(A)\right)^3\right]\\
&= O(m^{\frac{-3}{2}})
\end{aligned}
$$

since using Taylor series expansion $E\left(\widehat{\text{MSE}}(\hat{\theta}_i^{EB}) - g_{1i}(A)\right)^3 = O(m^{\frac{-3}{2}})$.

In order to evaluate EI_2, we choose $\epsilon_m = m^{-\alpha}$ $(0 < \alpha < 1/2)$ (Datta *et al.*, 2002, see). According to Diao *et al.* (2013) we can rewrite $I_{\{\widehat{\text{MSE}}(\hat{\theta}_i^{EB})\leq g_{1i}(A)\}}$ as

$$I_{\{\widehat{\text{MSE}}(\hat{\theta}_i^{EB})\leq g_{1i}(A)\}} = I_{\{\widehat{\text{MSE}}(\hat{\theta}_i^{EB})\leq g_{1i}(A)-\epsilon_m\}} + I_{\{g_{1i}(A)-\epsilon_m\leq\widehat{\text{MSE}}(\hat{\theta}_i^{EB})\leq g_{1i}(A)\}}$$

Then,

$$\left[E \int_0^{g_{1i}(A) - \widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB})} x^2 \left(\widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) + x \right)^{-5/2} dx \right] I_{\{\widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) \leq g_{1i}(A) - \epsilon_m\}}$$

$$\leq \left[E \int_0^{g_{1i}(A) - \widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB})} x^{4/2 - 5/2} dx \right] I_{\{\widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) \leq g_{1i}(A) - \epsilon_m\}}$$

$$= E \left[2 \left\{ g_{1i}(A) - \widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) \right\}^{1/2} I_{\{\widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) \leq g_{1i}(A) - \epsilon_m\}} \right]$$

$$\leq 2 \sqrt{E|g_{1i}(A) - \widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB})|} \sqrt{P \left(g_{1i}(A) - \widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) \geq \epsilon_m \right)}$$

$$\leq O(m^{-1/2}) E \sqrt{\frac{\left(g_{1i}(A) - \widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) \right)^3}{\epsilon_m^3}}$$

$$= O(m^{\frac{-3}{2}}),$$

since using Taylor series expansion $E \left(\widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) - g_{1i}(A) \right)^3 = O(m^{\frac{-3}{2}})$ and also

$$E \left[\int_{\widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB})}^{g_{1i}(A)} x^{-5/2} \left(x - \widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) \right)^2 dx I_{\{g_{1i}(A) - \epsilon_m \leq \widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) \leq g_{1i}(A)\}} \right]$$

$$\leq (g_{1i}(A) - \epsilon_m)^{-5/2} E \left[\int_{\widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB})}^{g_{1i}(A)} \left(x - \widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) \right)^2 dx I_{\{g_{1i}(A) - \epsilon_m \leq \widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) \leq g_{1i}(A)\}} \right]$$

$$\leq \frac{1}{3} (g_{1i}(A) - \epsilon_m)^{-5/2} E \left(g_{1i}(A) - \widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) \right)^3$$

$$= O(m^{\frac{-3}{2}}),$$

since using Taylor series expansion $E \left(\widehat{\mathrm{MSE}}(\hat{\theta}_i^{EB}) - g_{1i}(A) \right)^3 = O(m^{\frac{-3}{2}})$. Thus, $EI_1 + EI_2 = O(m^{\frac{-3}{2}})$.

Let us compute S_2 as

$$S_2 = \frac{(\gamma_i - \hat{\gamma}_i)(y_i - x_i' \hat{\beta})}{\sqrt{g_{1i}(A)}}.$$

By the Taylor expansion, $\gamma_i(\hat{A}) = \gamma_i(A) + \gamma_i^{(1)}(A^*)(\hat{A} - A)$ and $\hat{\beta} = \tilde{\beta}(A) + \tilde{\beta}^{(1)}(A^*)(\hat{A} - A)$, where $|A^* - A| \leq |\hat{A} - A|$ and $\gamma_i^{(1)}(A^*)$ is the derivative of $\tilde{\beta}(A)$ evaluated at $A = A^*$, similarly $\tilde{\beta}^{(1)}(A^*)$ is the derivative of $\gamma_i(A)$ evaluated at $A = A^*$. Thus, $\gamma_i(\hat{A}) - \gamma_i(A) = \gamma_i^{(1)}(\hat{A} - A)$ and $\hat{\beta} - \tilde{\beta}(A) = \tilde{\beta}^{(1)}(\hat{A} - A)$. Substituting reveals that

$$S_2 = -g_{1i}^{-1/2} \gamma_i^{(1)} \left((\hat{A} - A)(y_i - x_i' \tilde{\beta}(A)) - (\hat{A} - A) x_i' \tilde{\beta}^{(1)}(\hat{A} - A) \right).$$

Taking expectation on both sides,

$$ES_2 = -g_{1i}^{-1/2} E\left[(\hat{\gamma}_i - \gamma_i)(y_i - x_i'\hat{\beta})\right]$$

$$= -g_{1i}^{-1/2}\gamma_i^{(1)} E\left[(\hat{A} - A)(y_i - x_i'\tilde{\beta}(A))\right] + g_{1i}^{-1/2}\gamma_i^{(1)} E\left[(\hat{A} - A)x_i'\tilde{\beta}^{(1)}(\hat{A} - A)\right].$$

Let the first part can be written as

$$S_2^* = -g_{1i}^{-1/2}\gamma_i^{(1)} E\left[(\hat{A} - A)(y_i - x_i'\tilde{\beta}(A))\right]$$

$$= -g_{1i}^{-1/2}\gamma_i^{(1)} E\left[(\hat{A} - A)(y_i - x_i'\beta)\right] + o(m^{-1}).$$

Using Stein identity (see equation (8)), $ES_2^* = -g_{1i}^{-1/2}\gamma_i^{(1)}(A + \psi_i)E(\frac{\partial \hat{A}}{\partial y_i}) = 0$, since $\frac{\partial \hat{A}}{\partial y_i} = O_P(m^{-1})$. The expected value of S_3 can be evaluated as

$$S_3 = g_{1i}(A)^{-1/2}\gamma_i x_i'\left(\sum_{i=1}^{m}\frac{x_i x_i'}{A + \psi_i}\right)^{-1}\sum_{i=1}^{m}\frac{x_i}{A + \psi_i}e_i,$$

since $\tilde{\beta} = \left(\sum_{i=1}^{m}\frac{x_i x_i'}{A+\psi_i}\right)^{-1}\sum_{i=1}^{m}\frac{x_i}{A+\psi_i}y_i$,

$$E(S_3) = 0.$$

Combination of the above results, we have

$$EF(z) = E(S_1) + E(S_2) + E(S_3)$$

$$= E(S_1) + 0 + 0$$

$$= \frac{z}{2}\left(g_{1i}(A)\right)^{-1}\left(g_{2i}(A) + g_{3i}(A)\right) - \frac{z}{8}\left(g_{1i}(A)\right)^{-2}\frac{\psi_i^4}{(A + \psi_i)^4}V(\hat{A}) + O(m^{-3/2})$$

$$= \frac{z}{2}\left(\frac{A\psi_i}{A + \psi_i}\right)^{-1}\left(\frac{\psi_i^2}{(A + \psi_i)^2}x_i'(\sum_{i=1}^{m}\frac{x_i x_i'}{A + \psi_i})^{-1}x_i + \frac{\psi_i^2}{(A + \psi_i)^3}V(\hat{A})\right)$$

$$- \frac{z}{8}\left(\frac{A\psi_i}{A + \psi_i}\right)^{-2}\frac{\psi_i^4}{(A + \psi_i)^4}V(\hat{A}) + O(m^{-3/2}).$$

$$EF(z)^2 = E(S_1 + S_2 + S_3)^2$$

$$= E(S_1^2) + E(S_2^2) + E(S_3^2) + 2E(S_1 S_2) + 2E(S_1 S_3) + 2E(S_2 S_3).$$

We compute all the expectation values turn by turn as follows.

Let us evaluate $S_1 S_2$ using the Taylor expansion as

$$S_1 S_2 = \frac{z \times \left(\sqrt{\widehat{MSE}(\hat{\theta}_i^{EB})} - \sqrt{g_{1i}(A)}\right)(\gamma_i - \hat{\gamma}_i)(y_i - x_i'\hat{\beta})}{g_{1i}(A)}$$

$$= g_{1i}^{-1}(A)\frac{-z}{2}\left(\widehat{MSE}(\hat{\theta}_i^{EB}) - g_{1i}(A)\right)(\gamma_i - \hat{\gamma}_i)(y_i - x_i'\beta)$$

$$= -g_{1i}^{-1}(A)\frac{z}{2}g_{1i}^{(1)}(A)\gamma_i^{(1)}(A)\left((\hat{A} - A)^2(y_i - x_i'\tilde{\beta}(A)) - (\hat{A} - A)^3 x_i'\tilde{\beta}^{(1)}(A)\right).$$

Letting $c = -g_{1i}^{-1}(A)\frac{z}{2}g_{1i}^{(1)}(A)\gamma_i^{(1)}(A)$ and taking expectation on both sides gives that

$$ES_1S_2 = cE\left((\hat{A} - A)^2(y_i - x_i'\tilde{\beta}(A))\right) - cE\left((\hat{A} - A)^3 x_i'\tilde{\beta}^{(1)}(A)\right) = r_{11} + r_{12},$$

say,

$$Er_{11} = cE\left((\hat{A} - A)^2(y_i - x_i'\tilde{\beta}(A))\right)$$
$$= cE\left((\hat{A} - A)^2(y_i - x_i'\beta)\right) + o(m^{-1}).$$

Using Stein identity (see equation (8)),

$$Er_{11} = c(A + \psi_i)E\left(\frac{\partial}{\partial y_i}(\hat{A} - A)^2\right)$$
$$= c(A + \psi_i)E\left(2(\hat{A} - A)\frac{\partial \hat{A}}{\partial y_i}\right),$$

but $\frac{\partial \hat{A}}{\partial y_i} = O_P(m^{-1})$.

$$Er_{12} = cE\left((\hat{A} - A)^3 x_i'\tilde{\beta}^{(1)}(A)\right)$$
$$= cx_i'\tilde{\beta}^{(1)}(A)E(\hat{A} - A)^3.$$

Since $E(\hat{A} - A)^{2S} = O(m^{-S})$, (Prasad and Rao, 1990), thus $E(\hat{A} - A)^3 = O(m^{-3/2})$.
 Let us evaluate S_2S_3 as using the Taylor series expansion as

$$S_2S_3 = g_{1i}^{-1}(y_i - x_i'\hat{\beta})\gamma_i x_i'(\hat{\beta} - \tilde{\beta})(\gamma_i - \hat{\gamma}_i)$$
$$= -g_{1i}^{-1}(y_i - x_i'\tilde{\beta}(A))\gamma_i x_i'\tilde{\beta}^{(1)}(A)(\hat{A} - A)\gamma_i^{(1)}(A)(\hat{A} - A)$$
$$+ g_{1i}^{-1}x_i'\tilde{\beta}^{(1)}(A)(\hat{A} - A)\gamma_i x_i'\tilde{\beta}^{(1)}(A)(\hat{A} - A)\gamma_i^{(1)}(A)(\hat{A} - A)$$
$$= r_{21} + r_{22}.$$

Taking expectation on both sides gives

$$Er_{21} = -g_{1i}^{-1}\gamma_i^{(1)}(A)\gamma_i E\left((y_i - x_i'\tilde{\beta}(A))x_i'\tilde{\beta}^{(1)}(A)(\hat{A} - A)^2\right)$$
$$= -g_{1i}^{-1}\gamma_i^{(1)}(A)\gamma_i E\left((y_i - x_i'\beta)x_i'\tilde{\beta}^{(1)}(A)(\hat{A} - A)^2\right) + o(m^{-1}).$$

Using Stein identity (see equation (8)),

$$Er_{21} = -g_{1i}^{-1}\gamma_i^{(1)}(A)\gamma_i(A + \psi_i)E\left(\frac{\partial}{\partial y_i}\left\{x_i'\tilde{\beta}^{(1)}(A)(\hat{A} - A)^2\right\}\right) + o(m^{-1}).$$

Note that

$$
\frac{\partial}{\partial y_i} \left\{ x_i' \tilde{\beta}^{(1)}(A)(\hat{A} - A)^2 \right\}
$$

$$
= \frac{\partial}{\partial y_i} \left\{ x_i' \tilde{\beta}^{(1)}(A) \right\} (\hat{A} - A)^2 + 2 x_i' \tilde{\beta}^{(1)}(A)(\hat{A} - A) \frac{\partial \hat{A}}{\partial y_i},
$$

where

$$
\frac{\partial}{\partial y_i} \left\{ x_i' \tilde{\beta}^{(1)}(A) \right\} = \frac{\partial}{\partial A} x_i' \left\{ \frac{\partial}{\partial y_i} \tilde{\beta}(A) \right\}
$$

$$
= \frac{\partial}{\partial A} \left\{ \frac{\tilde{h}_{ii}}{A + \psi_i} \right\} + O(m^{-1})
$$

in which $\tilde{h}_{ii} = x_i' \left\{ \sum_{j=1}^{m} (A + \psi_j)^{-1} x_j x_j' \right\}^{-1}$ is $O(m^{-1})$, $\frac{\partial \hat{A}}{\partial y_i} = O_P(m^{-1})$ and $\frac{\partial}{\partial y_i}(\hat{\beta}(\hat{A})) = O_P(m^{-1})$ (see Kubokawa (2011, page 14) and Datta *et al.* (2011, page 20)).

$$
Er_{22} = g_{1i}^{-1} x_i' \tilde{\beta}^{(1)}(A)(\hat{A} - A) \gamma_i x_i' \tilde{\beta}^{(1)}(A)(\hat{A} - A) \gamma_i^{(1)}(A)(\hat{A} - A)
$$

$$
= g_{1i}^{-1} \gamma_i \gamma_i^{(1)}(A) \left\{ x_i' \tilde{\beta}^{(1)}(A) \right\}^2 E(\hat{A} - A)^3 = O(m^{-3/2}).
$$

Thus, $E(S_2 S_3) = O(m^{-3/2})$.

Let us evaluate $E(S_1 S_3)$ as

$$
S_1 S_3 = \frac{z \times \left(\sqrt{\widehat{\mathrm{MSE}}(\hat{\theta}_i^{\mathrm{EB}})} - \sqrt{g_{1i}(A)} \right) \gamma_i x_i'(\hat{\beta} - \tilde{\beta})}{g_{1i}(A)}.
$$

Using the Taylor expansion we have $= g_{1i}^{-1} z \gamma_i g_{1i}^{(1)} x_i' \tilde{\beta}^{(1)}(\hat{A} - A)^2$. Then,

$$
E(S_1 S_3) = g_{1i}^{-1} z \gamma_i g_{1i}^{(1)} x_i' \tilde{\beta}^{(1)} E(\hat{A} - A)^2
$$

$$
= g_{1i}^{-1} z \gamma_i g_{1i}^{(1)} x_i' \tilde{\beta}^{(1)} \mathrm{Var}(\hat{A}).
$$

We know that

$$
S_1^2 = \left(\frac{z \left(\sqrt{\widehat{\mathrm{MSE}}(\hat{\theta}_i^{\mathrm{EB}})} - \sqrt{g_{1i}(A)} \right)}{\sqrt{g_{1i}(A)}} \right)^2
$$

$$
= \frac{z^2}{4} \left(g_{1i}^{-1}(A)(\widehat{\mathrm{MSE}}(\hat{\theta}_i^{\mathrm{EB}}) - g_{1i}(A)) \right)^2.
$$

Using the Taylor expansion, we have

$$S_1^2 = \frac{z^2}{4} \left(g_{1i}^{-1}(A)(\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}}) - g_{1i}(A)) \right)^2.$$

Taking expectation on both sides,

$$
\begin{aligned}
ES_1^2 &= \frac{z^2}{4} E \left(g_{1i}^{-1}(A)(g_{1i}^1(A)(\hat{A} - A)) \right)^2 + O(m^{-3/2}) \\
&= \frac{z^2}{4} g_{1i}^{-2}(A)(g_{1i}^1(A))^2 E(\hat{A} - A)^2 + O(m^{-3/2}) \\
&= \frac{z^2 \psi_i^2}{4A^2(A + \psi_i)^2} V(\hat{A}) + O(m^{-3/2}).
\end{aligned}
$$

Let us evaluate $S_2^2 = g_{1i}^{-1}(\gamma_i - \hat{\gamma}_i)^2(y_i - x_i'\hat{\beta})^2$. By the Taylor expansion,

$$
\begin{aligned}
S_2^2 &= g_{1i}^{-1}(\gamma_i - \hat{\gamma}_i)^2(y_i - x_i'\hat{\beta})^2 \\
&= g_{1i}^{-1}(\gamma_i^{(1)}(A))^2(y_i - x_i'\tilde{\beta}(A))^2(\hat{A} - A)^2 \\
&\quad - 2g_{1i}^{-1}(\gamma_i^{(1)}(A))^2 x_i'\tilde{\beta}^{(1)}(A)(y_i - x_i'\tilde{\beta}(A))(\hat{A} - A)^3 \\
&\quad + (x_i'\tilde{\beta}^{(1)}(A))^2(\hat{A} - A)^4 \\
&= r_{31} + r_{32} + r_{33}.
\end{aligned}
$$

Taking expectation on both sides,

$$
\begin{aligned}
Er_{31} &= g_{1i}^{-1}(\gamma_i^{(1)}(A))^2 E \left\{ (y_i - x_i'\tilde{\beta}(A))^2(\hat{A} - A)^2 \right\} \\
&= g_{1i}^{-1}(\gamma_i^{(1)}(A))^2 E \left\{ (y_i - x_i'\beta)^2(\hat{A} - A)^2 \right\} + o(m^{-1}).
\end{aligned}
$$

Using Stein identity (see equation (8)),

$$
\begin{aligned}
E[(\hat{A} - A)^2(y_i - x_i'\beta)^2] &= (A + \psi_i)E \left[\frac{\partial}{\partial y_i} \left\{ (\hat{A} - A)^2(y_i - x_i'\beta) \right\} \right] \\
&= (A + \psi_i) \\
&\quad \times E \left[2(\hat{A} - A)(y_i - x_i'\beta)\frac{\partial \hat{A}}{\partial y_i} + (\hat{A} - A)^2 \frac{\partial}{\partial y_i} \left\{ (y_i - x_i'\beta) \right\} \right] \\
&\quad + o(m^{-1}) \\
&= (A + \psi_i)E(\hat{A} - A)^2 + o(m^{-1}),
\end{aligned}
$$

$$
Er_{31} = g_{1i}^{-1}(\gamma_i^{(1)}(A))^2(A + \psi_i)E(\hat{A} - A)^2 + o(m^{-1}),
$$

$$
\begin{aligned}
Er_{32} &= 2g_{1i}^{-1}(\gamma_i^{(1)}(A))^2 x_i'\tilde{\beta}^{(1)}(A)E \left\{ (y_i - x_i'\beta)(\hat{A} - A)^3 \right\} + o(m^{-1}) \\
&= 6g_{1i}^{-1}(\gamma_i^{(1)}(A))^2 x_i'\tilde{\beta}^{(1)}(A)(A + \psi_i)E(\hat{A} - A)^2 \frac{\partial \hat{A}}{\partial y_i} + o(m^{-1}),
\end{aligned}
$$

$$
Er_{33} = (x_i'\tilde{\beta}^{(1)}(A))^2 E(\hat{A} - A)^4 = O(m^{-2}),
$$

since $E(\hat{A} - A)^{2S} = O(m^{-S})$. Substituting reveals that

$$
\begin{aligned}
ES_2^2 &= Er_{31} + Er_{32} + Er_{33} \\
&= g_{1i}^{-1}(\gamma_i^{(1)})^2(A + \psi_i)E(\hat{A} - A)^2 + O(m^{-2}) \\
&= \frac{\psi_i}{A(A + \psi_i)^2}V(\hat{A}) + O(m^{-2})
\end{aligned}
$$

since $(\gamma_i^{(1)})^2 = \frac{\psi_i^2}{(A+\psi_i)^4}$ and $E(\hat{A} - A)^2 = V(\hat{A}) + o_p(m^{-1})$. Using this fact, we obtain

$$
ES_2^2 = \frac{\psi_i}{A(A + \psi_i)^2}V(\hat{A}) + O(m^{-2}).
$$

Similarly, we have

$$
E(S_3^2) = \frac{\psi_i}{A(A + \psi_i)}x_i'\left(\sum_{j=1}^{m}(A + \psi_j)^{-1}x_jx_j'\right)^{-1}x_i
$$

$$
E(S_3^3) = O(m^{-2}),
$$

where $\text{Var}(e_i) = \psi_i$. The remainder term $\int_z^{z+F(z)}(z + F(z) - x)^2(x^2 - 1)\phi(x)dx$ can be simplified using the approach by Datta *et al.* (2002), Chatterjee and Lahiri (2002) and Diao *et al.* (2013) as follows. Since within the limits of the integral $0 < |z + F(z) - x| < |F(z)|$ and $|(x^2 - 1)\phi(x)| < 2\phi(\sqrt{3})$, it follows that

$$
\begin{aligned}
&E|\int_z^{z+F(z)}(x^2 - 1)(z + F(z) - x)^2\phi(x)dx| \\
&\leq E|F(z)|^2|\int_z^{z+F(z)}2\phi(\sqrt{3})dx| \\
&= 2E|F(z)|^3\phi(\sqrt{3}) \\
&= O(m^{-3/2}).
\end{aligned}
$$

Finally, combination and simplification of all the above expressions give the CP of $I^{\text{CEB}}(\hat{A})$ as

$$
P[\theta_i \in I^{\text{CEB}}(\hat{A})] = 2\Phi(z) - 1 - \frac{z\phi(z)}{4}\eta + O(m^{-3/2}),
$$

where

$$
\eta = (z^2 + 1)\frac{\psi_i^2}{A^2(A + \psi_i)^2}V(\hat{A}).
$$

$\square$

Appendix C: Proof of Theorem 3.2

Proof. The CP of $I^{\text{CEB}}(\hat{A})$ is written as

$$P\left[\theta_i - \theta_j \in I^{\text{CEB}}(\hat{A})\right] = P\left[-z + F(-z) < \frac{\theta_i - \theta_j - (\hat{\theta}_i^{\text{EB}} - \hat{\theta}_j^{\text{EB}})}{\sqrt{g_{1i}(A) + g_{2j}(A}} < z + F(z)\right]$$

$$= E[\Phi(z + F(z)) - \Phi(-z + F(-z))],$$

where

$$F(z) = \frac{z \times \left(\sqrt{\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}} - \hat{\theta}_j^{\text{EB}})} - \sqrt{g_{1i}(A) + g_{1j}(A)}\right) + \hat{\theta}_i^{\text{EB}} - \tilde{\theta}_i^B - (\hat{\theta}_j^{\text{EB}} - \tilde{\theta}_j^B)}{\sqrt{g_{1i}(A) + g_{1j}(A)}},$$

$g_{1i}(A) = \frac{A\psi_i}{A+\psi_i}$, $g_{1j}(A) = \frac{A\psi_j}{A+\psi_j}$. Using the Taylor expansion with an integral remainder term, $\Phi(z + F(z))$ is evaluated as

$$\Phi(z + F(z)) = \Phi(z) + F(z)\phi(z) + \frac{1}{2}F^2(z)\phi'(z) + \frac{1}{2}\int_z^{z+F(z)} (z + F(z) - x)^2 \phi''(x)dx$$

$$= \Phi(z) + [F(z) - \frac{z}{2}F^2(z)]\phi(z) - \frac{1}{2}\int_z^{z+F(z)} (z + F(z) - x)^2(1 - x^2)\phi(x)dx.$$

Now observe that

$$E\left(\Phi(z + F(z))\right) = \Phi(z) + E\left[F(z)\phi(z) - \frac{z}{2}F^2(z)\phi(z)\right.$$

$$\left. - \frac{1}{2}\int_z^{z+F(z)} (z + F(z) - x)^2(1 - x^2)\phi(x)dx\right].$$

In a similar manner to a one population case, we have

$$\hat{\theta}_i^{\text{EB}} - \tilde{\theta}_i^B - \left\{\hat{\theta}_j^{\text{EB}} - \tilde{\theta}_j^B\right\}$$

$$= (1 - \hat{\gamma}_i)y_i + \hat{\gamma}_i x_i'\hat{\beta} - (1 - \gamma_i)y_i - \gamma_i x_i'\tilde{\beta} -$$

$$\left\{(1 - \hat{\gamma}_j)y_j + \hat{\gamma}_j x_j'\hat{\beta} - (1 - \gamma_j)y_j - \gamma_j x_j'\tilde{\beta}\right\}$$

$$= (\gamma_i - \hat{\gamma}_i)r_i + \gamma_i x_i'(\hat{\beta} - \tilde{\beta}) - (\gamma_j - \hat{\gamma}_j)r_j - \gamma_j x_j'(\hat{\beta} - \tilde{\beta})$$

where $r_i = (y_i - x_i'\hat{\beta})$ and $r_j = (y_j - x_j'\hat{\beta})$.

Thus $F(z)$ can be decomposed into three components as

$$F(z) = \frac{z \times \left(\sqrt{\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}} - \hat{\theta}_j^{\text{EB}})} - \sqrt{g_{1i}(A) + g_{1j}(A)} \right)}{\sqrt{g_{1i}(A) + g_{1j}(A)}}$$

$$+ \frac{(\gamma_i - \hat{\gamma}_i)r_i - (\gamma_j - \hat{\gamma}_j)r_j}{\sqrt{g_{1i}(A) + g_{1j}(A)}} + \frac{\gamma_i x_i'(\hat{\beta} - \tilde{\beta}) - \gamma_j x_j'(\hat{\beta} - \tilde{\beta})}{\sqrt{g_{1i}(A) + g_{1j}(A)}}$$

$$= Q_1 + Q_2 + Q_3, \quad \text{say.}$$

We obtained the expected values of Q_1 and Q_1^2 given above

$$EQ_1 = (g_{1i} + g_{1j})^{-1}\frac{z}{2} \left(g_{2i}(A) + g_{3i}(A) + g_{2j}(A) + g_{3j}(A) \right)$$

$$- (g_{1i} + g_{1j})^{-1}\frac{z}{2} \left(\frac{2\psi_i\psi_j}{(\hat{A} + \psi_i)(\hat{A} + \psi_j)} x_i' \left\{ \sum_{i=1}^{m} \frac{x_i x_i'}{\hat{A} + \psi_i} \right\}^{-1} x_j \right)$$

$$- (g_{1i} + g_{1j})^{-2}\frac{z^2}{8} \left(\frac{\psi_i^2}{(A + \psi_i)^2} + \frac{\psi_j^2}{(A + \psi_j)^2} \right)^2 V(\hat{A}) + O(m^{-3/2}),$$

$$E(Q_1^2) = (g_{1i} + g_{1j})^{-2}z^2 E \left(\widehat{\text{MSE}}(\hat{\theta}_i^{\text{EB}} - \hat{\theta}_j^{\text{EB}}) - (g_{1i}(A) + g_{1j}(A)) \right)^2$$

$$= (\frac{A\psi_i}{(A + \psi_i)} + \frac{A\psi_j}{(A + \psi_j)})^{-2}z^2 \left(\frac{\psi_i^2}{(A + \psi_i)^2} + \frac{\psi_j^2}{(A + \psi_j)^2} \right)^2 V(\hat{A})$$

$$+ O(m^{-3/2}).$$

Let us now evaluate Q_2^2 as

$$Q_2^2 = \left(\frac{(\gamma_i - \hat{\gamma}_i)r_i - (\gamma_j - \hat{\gamma}_j)r_j}{\sqrt{g_{1i}(A) + g_{1j}(A)}} \right)^2$$

$$= (g_{1i} + g_{1j})^{-1} \left((\gamma_i - \hat{\gamma}_i)r_i - (\gamma_j - \hat{\gamma}_j)r_j \right)^2$$

$$= (g_{1i} + g_{1j})^{-1} \left\{ ((\gamma_i - \hat{\gamma}_i)r_i)^2 + \left((\gamma_j - \hat{\gamma}_j)r_j \right)^2 - 2(\gamma_i - \hat{\gamma}_i)r_i(\gamma_j - \hat{\gamma}_j)r_j \right\}$$

$$= Q_{11} + Q_{12} + Q_{13}, \quad \text{Say.}$$

Let us evaluate each term turn by turn as

$$Q_{11} = (g_{1i} + g_{1j})^{-1}(\gamma_i - \hat{\gamma}_i)^2(y_i - x_i'\hat{\beta})^2.$$

Using the Taylor expansion,

$$
\begin{aligned}
Q_{11} &= (g_{1i} + g_{1j})^{-1}(\gamma_i^{(1)}(A))^2(\hat{A} - A)^2(y_i - x_i'\tilde{\beta} - x_i'\tilde{\beta}^{(1)}(A)(\hat{A} - A))^2 \\
&= (g_{1i} + g_{1j})^{-1}(\gamma_i^{(1)}(A))^2(\hat{A} - A)^2(y_i - x_i'\tilde{\beta})^2 \\
&\quad - 2(g_{1i} + g_{1j})^{-1}(\gamma_i^{(1)}(A))^2(\hat{A} - A)^2 x_i'\tilde{\beta}^{(1)}(A)(\hat{A} - A)(y_i - x_i'\tilde{\beta}) \\
&\quad + (g_{1i} + g_{1j})^{-1}(\gamma_i^{(1)}(A))^2(\hat{A} - A)^2 \left(x_i'\tilde{\beta}^{(1)}(A)(\hat{A} - A) \right)^2 .
\end{aligned}
$$

We evaluated this part for the case of one population mean, thus

$$
\begin{aligned}
EQ_{11} &= (g_{1i} + g_{1j})^{-1}(\gamma_i^{(1)})^2(A + \psi_i)E(\hat{A} - A)^2, \\
EQ_{11} &= (g_{1i} + g_{1j})^{-1}(\gamma_i^{(1)})^2(A + \psi_i)V(\hat{A}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
EQ_{12} &= (g_{1i} + g_{1j})^{-1}(\gamma_j^{(1)})^2(A + \psi_j)V(\hat{A}), \\
Q_{13} &= -2(g_{1i} + g_{1j})^{-1}(\gamma_i - \hat{\gamma}_i)(y_i - x_i'\hat{\beta})(\gamma_j - \hat{\gamma}_j)(r_j - x_j'\hat{\beta}).
\end{aligned}
$$

Using the Taylor expansion,

$$
\begin{aligned}
Q_{13} &= -2(g_{1i} + g_{1j})^{-1}(\gamma_i^{(1)}(A)(\hat{A} - A)(y_i - x_i'\tilde{\beta} - x_i'\tilde{\beta}^{(1)}(A)(\hat{A} - A)) \\
&\quad \times (\gamma_j^{(1)}(A)(\hat{A} - A)(y_j - x_j'\tilde{\beta} - x_j'\tilde{\beta}^{(1)}(A)(\hat{A} - A)).
\end{aligned}
$$

Using Stein identity (see equation (8)),

$$
Q_{13} = -2(g_{1i} + g_{1j})^{-1}(A + \psi_i)\gamma_i^{(1)}\frac{\partial \hat{A}}{\partial y_i}(A + \psi_j)\gamma_j^{(1)}\frac{\partial \hat{A}}{\partial y_j},
$$

since $\frac{\partial \hat{A}}{\partial y_i} = O_P(m^{-1})$. Therefore,

$$
\begin{aligned}
Q_2^2 &= Q_{11} + Q_{12} + Q_{13} \\
&= (g_{1i} + g_{1j})^{-1}\left((\gamma_i^{(1)})^2(A + \psi_i) + (\gamma_j^{(1)})^2(A + \psi_j) \right)V(\hat{A}) \\
&= \left(\frac{A\psi_i}{(A + \psi_i)} + \frac{A\psi_j}{(A + \psi_j)} \right)^{-1}\left(\frac{\psi_i^2}{(A + \psi_i)^3} + \frac{\psi_j^2}{(A + \psi_j)^3} \right)V(\hat{A}).
\end{aligned}
$$

The expected value of Q_3^2 can be given as

$$EQ_3^2 = E\left(\frac{\gamma_i x_i'(\hat{\beta} - \tilde{\beta}) - \gamma_j x_j'(\hat{\beta} - \tilde{\beta})}{\sqrt{g_{1i}(A) + g_{1j}(A)}}\right)^2$$

$$= (g_{1i} + g_{1j})^{-1}\left[\gamma_i^2 x_i'\left\{\sum_{u=1}^{m}(A + \psi_u)^{-1}x_u x_u'\right\}^{-1}x_i + \gamma_j^2 x_i'\left\{\sum_{u=1}^{m}(A + \psi_u)^{-1}x_u x_u'\right\}^{-1}x_i\right]$$

$$- 2(g_{1i} + g_{1j})^{-1}\gamma_i \gamma_j x_i'\left\{\sum_{u=1}^{m}(A + \psi_u)^{-1}x_u x_u'\right\}^{-1}x_j$$

$$= \left(\frac{A\psi_i}{A + \psi_i} + \frac{A\psi_j}{A + \psi_j}\right)^{-1}[\frac{\psi_i^2}{(A + \psi_i)^2}x_i'\left\{\sum_{u=1}^{m}(A + \psi_u)^{-1}x_u x_u'\right\}^{-1}x_i$$

$$+ \frac{\psi_j^2}{(A + \psi_j)^2}x_i'\left\{\sum_{u=1}^{m}(A + \psi_u)^{-1}x_u x_u'\right\}^{-1}x_i$$

$$- 2\frac{\psi_i}{(A + \psi_i)}\frac{\psi_j}{(A + \psi_j)}x_i'\left\{\sum_{u=1}^{m}(A + \psi_u)^{-1}x_u x_u'\right\}^{-1}x_j].$$

The integral and remainder terms can be easily simplified similar to Theorem 1. Combination and simplification of all the above results gives as

$$P\left[\theta_i - \theta_j \in I^{\text{CEB}}(\hat{A})\right] = 2\Phi(z) - 1 - \frac{z\phi(z)}{4}\eta + O(m^{-3/2}),$$

where

$$\eta = (z^2 + 1)(\frac{A\psi_i}{A + \psi_i} + \frac{A\psi_j}{A + \psi_j})^{-2}\left(\frac{\psi_i^2}{(A + \psi_i)^2} + \frac{\psi_j^2}{(A + \psi_j)^2}\right)^2 V(\hat{A}).$$

$\square$

Appendix D: Graphs and Tables
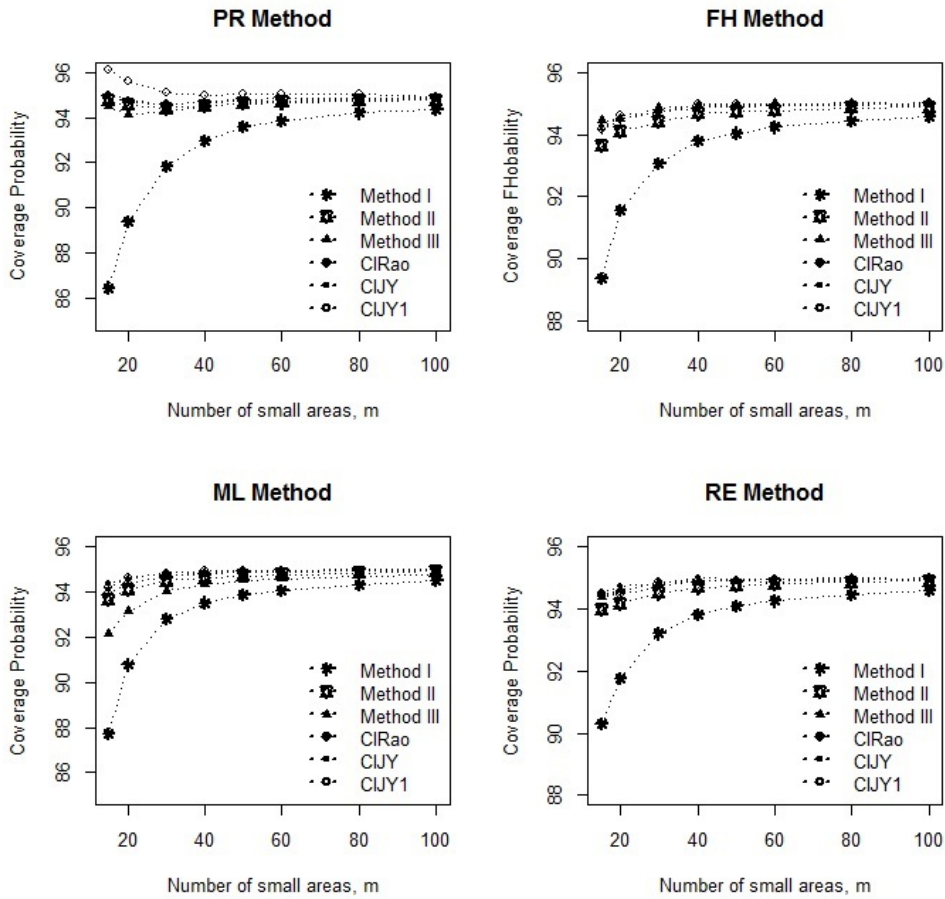
Figure 2: Simulated values of the CPs of Method I, Method II, Method III, CIRao, CIJY and CIJY1 for nominal 95% CIs for ψ_i pattern II.[2]

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[2]`Meth I`, `Meth II`, `Meth III`, `CIRao`, `CIJY` and `CIJY1` referring the CIs given by equations (3.4 - 3.9). Methods of estimating A: PR, Prasad and Rao (1990); ML, maximum likelihood; REML, restricted maximum likelihood; FH, Fay and Herriot (1979).

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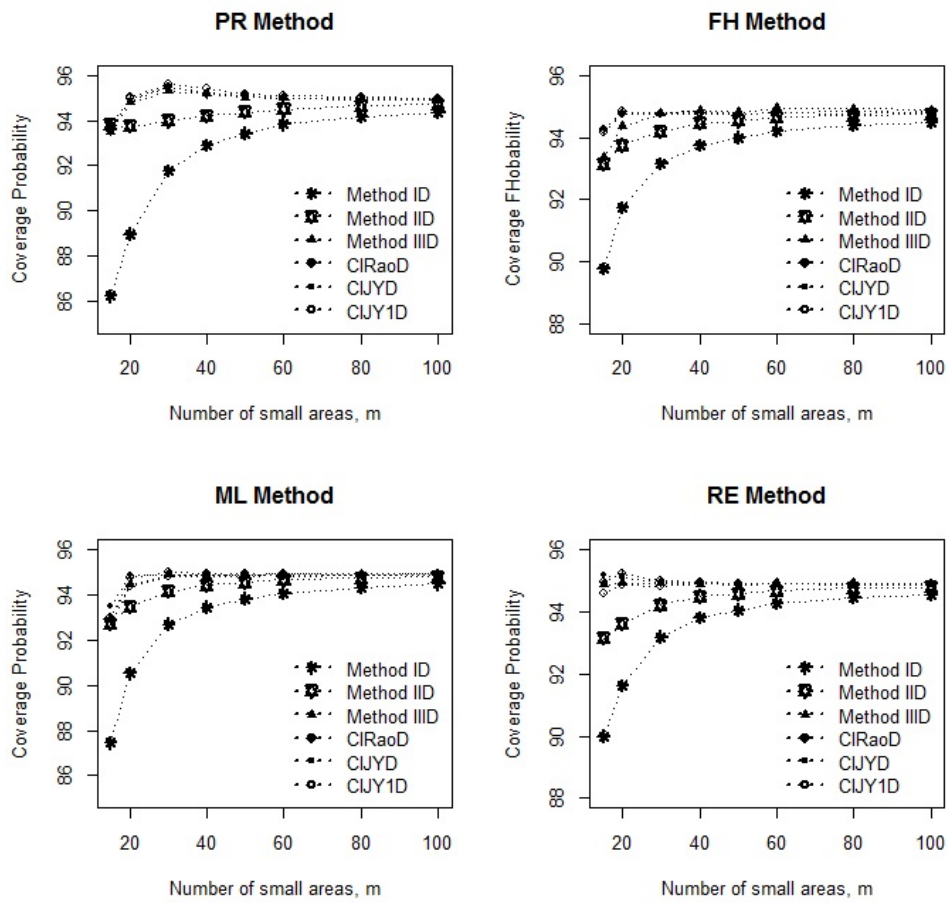
Figure 3: Simulated values of the CPs of Method ID, Method IID, Method IIID, CIRaoD, CIJYD and CIJY1D for nominal 95% CIs for ψ_i pattern II[4].

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[4]Note that: `Method ID`, `Method IID`, `Method IIID`, `CIRaoD`, `CIJYD` and `CIJY1D` referring the CIs given by equations (3.10 - 3.15). Methods of estimating A: PR, Prasad and Rao (1990); ML, maximum likelihood; REML, restricted maximum likelihood; FH, Fay and Herriot (1979).
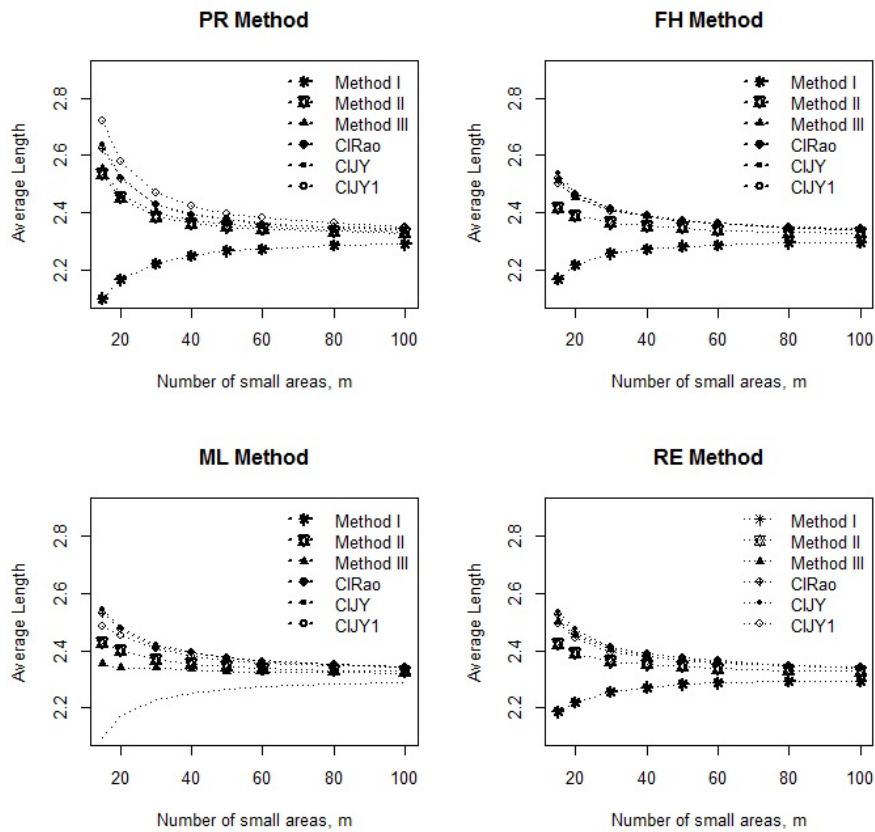
Figure 4: Simulated values of the CPs and ALs of Method I, Method II, Method III, CIRao, CIJY and CIJY1 for nominal 95% CIs for ψ_i pattern II.[6]

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[6]`Meth I, Meth II, Meth III, CIRao, CIJY` and `CIJY1` referring the CIs given by equations (3.4 - 3.9). Methods of estimating A: PR, Prasad and Rao (1990); ML, maximum likelihood; REML, restricted maximum likelihood; FH, Fay and Herriot (1979).

[8]Note that:

- `Method ID, Method IID, Method IIID, CIRaoD, CIJYD` and `CIJY1D` referring the CIs given by equations (3.10 - 3.15).

- Methods of estimating A: PR, Prasad and Rao (1990); ML, maximum likelihood; REML, restricted maximum likelihood; FH, Fay and Herriot (1979).

- Groups: G1, areas 1-3; G2, areas 4-6; G3, areas 7-9; G4, areas 10-12, G5, areas 13-15.

Table 1: CPs for nominal 95% CIs for ψ_i pattern (II) under normal random effects distributions and $m = 20$. G1, areas 1-3; G2, areas 4-6; G3, areas 7-9; G4, areas 10-12, G5, areas 13-15.

Groups	CP												
	Normal Distribution												
	PR						REML					
	Meth I	Meth II	Meth III	CIRao	CIJY	CIJY1	Meth I	Meth II	Meth III	CIRao	CIJY	CIJY1
Pattern I												
G1	91.4	93.1	93.7	94.6	94.7	94.5	91.7	94.1	94.6	94.9	94.7	94.5
G2	91.9	93.6	94.0	94.7	94.7	94.7	92.0	94.3	94.8	94.9	94.9	94.7
G3	92.0	93.6	94.0	94.7	94.9	94.7	92.3	94.5	94.9	94.7	94.7	94.7
G4	92.5	94.0	94.3	95.0	95.0	95.0	92.5	94.7	94.9	95.1	95.2	95.0
G5	92.9	94.3	94.5	95.1	95.2	95.0	92.9	95.0	95.1	95.1	95.1	95.0
Pattern II												
G1	87.6	91.5	93.0	92.8	92.5	94.3	90.6	93.1	93.8	94.0	93.7	93.8
G2	89.1	94.4	93.9	94.9	94.8	95.6	91.8	94.2	94.5	94.6	94.5	94.7
G3	89.3	94.7	94.3	94.8	95.1	95.8	91.9	94.4	94.8	94.7	94.7	94.7
G4	89.3	95.0	94.2	95.3	95.2	96.0	92.1	94.6	94.8	94.8	94.6	95.0
G5	90.3	95.8	95.1	95.8	95.9	96.1	93.1	95.0	95.2	95.1	95.3	95.0
Pattern III												
G1	80.2	87.7	92.2	89.1	89.5	94.6	89.9	92.0	93.5	93.5	93.7	93.7
G2	81.8	95.1	96.7	96.2	96.2	98.1	91.9	94.0	94.4	94.4	94.5	94.4
G3	81.8	95.2	96.7	96.3	96.7	98.1	92.2	94.3	94.6	94.7	94.8	94.4
G4	82.0	95.9	96.9	96.9	96.9	97.9	92.4	94.4	94.6	94.7	94.9	94.7
G5	84.1	96.4	96.5	96.6	96.7	97.0	93.9	95.1	95.2	95.2	95.2	95.1

Table 2: CPs for nominal 95% CIs for ψ_i pattern (II) under Laplace random effects distributions and $m = 20$. G1, areas 1-3; G2, areas 4-6; G3, areas 7-9; G4, areas 10-12, G5, areas 13-15.

	CP											
	Laplace Distribution											
	PR						REML					
Groups	Meth I	Meth II	Meth III	CIRao	CIJY	CIJY1	Meth I	Meth II	Meth III	CIRao	CIJY	CIJY1
Pattern I												
G1	1.89	2.12	2.16	2.26	2.27	2.27	1.90	2.21	2.25	3.10	3.12	3.68
G2	1.82	2.05	2.09	2.19	2.20	2.21	1.83	2.14	2.18	2.82	2.83	3.70
G3	1.75	1.98	2.01	2.11	2.11	2.12	1.76	2.07	2.09	2.82	2.82	3.72
G4	1.65	1.88	1.90	2.01	2.02	2.02	1.66	1.96	1.99	2.83	2.84	3.76
G5	1.52	1.74	1.76	1.87	1.88	1.87	1.52	1.83	1.84	3.26	3.28	4.39
Pattern II												
G1	2.11	2.55	2.68	2.76	2.76	2.86	2.23	2.51	2.60	2.61	2.62	2.60
G2	1.69	2.29	2.27	2.38	2.38	2.54	1.82	2.13	2.15	2.16	2.17	2.15
G3	1.62	2.24	2.21	2.31	2.32	2.48	1.74	2.05	2.07	2.08	2.08	2.06
G4	1.52	2.18	2.12	2.24	2.25	2.40	1.64	1.95	1.96	1.97	1.98	1.95
G5	1.22	1.99	1.89	2.04	2.05	2.19	1.33	1.61	1.61	1.62	1.62	1.60
Pattern III												
G1	2.10	2.69	3.36	2.25	2.25	2.24	2.37	2.58	2.72	2.73	2.73	2.72
G2	1.51	2.71	3.04	2.18	2.18	2.18	1.82	2.07	2.10	2.12	2.12	2.09
G3	1.44	2.72	3.03	2.09	2.10	2.09	1.75	1.98	2.01	2.03	2.03	2.00
G4	1.34	2.73	3.04	1.98	1.99	1.98	1.65	1.88	1.90	1.92	1.93	1.89
G5	0.81	3.05	3.47	1.84	1.84	1.83	1.04	1.20	1.20	1.22	1.23	1.19

Table 3: ALs for nominal 95% CIs for ψ_i pattern (II) under Normal random effects distributions and $m = 20$. G1, areas 1-3; G2, areas 4-6; G3, areas 7-9; G4, areas 10-12, G5, areas 13-15.

	PR						REML					
	AL											
	Normal Distribution											
Groups	Meth I	Meth II	Meth III	CIRao	CIJY	CIJY1	Meth I	Meth II	Meth III	CIRao	CIJY	CIJY1
Pattern I												
G1	2.42	2.54	2.60	2.71	2.72	2.71	2.42	2.60	2.66	2.71	2.73	2.70
G2	2.31	2.42	2.47	2.58	2.59	2.58	2.31	2.48	2.53	2.58	2.59	2.57
G3	2.18	2.29	2.32	2.42	2.44	2.42	2.18	2.34	2.38	2.42	2.43	2.41
G4	2.02	2.12	2.14	2.24	2.25	2.24	2.02	2.17	2.19	2.23	2.24	2.22
G5	1.83	1.90	1.92	2.02	2.02	2.00	1.83	1.95	1.96	2.00	2.01	1.99
Pattern II												
G1	2.98	3.28	3.49	3.54	3.54	3.63	3.06	3.30	3.51	3.53	3.54	3.50
G2	2.24	2.52	2.52	2.58	2.59	2.65	2.30	2.49	2.53	2.54	2.54	2.53
G3	2.12	2.39	2.37	2.44	2.44	2.50	2.17	2.35	2.38	2.39	2.39	2.37
G4	1.97	2.23	2.20	2.26	2.27	2.32	2.02	2.17	2.19	2.20	2.20	2.19
G5	1.52	1.75	1.71	1.77	1.77	1.80	1.56	1.65	1.66	1.66	1.66	1.65
Pattern III												
G1	3.06	3.45	4.39	4.46	4.53	5.09	3.36	3.58	3.93	4.17	4.20	4.08
G2	2.07	2.67	2.89	3.00	3.01	3.69	2.30	2.47	2.52	2.58	2.59	2.55
G3	1.95	2.57	2.76	2.91	2.92	3.59	2.18	2.33	2.37	2.42	2.43	2.39
G4	1.81	2.45	2.62	2.81	2.81	3.47	2.02	2.16	2.18	2.23	2.24	2.20
G5	1.05	1.87	2.01	2.54	2.51	3.15	1.16	1.21	1.21	1.24	1.25	1.22

Pattern I / Pattern II / Pattern III labels appear between the PR and REML column groups.

Table 4: ALs for nominal 95% CIs for ψ_i pattern (II) under Laplace random effects distributions and $m = 20$. G1, areas 1-3; G2, areas 4-6; G3, areas 7-9; G4, areas 10-12, G5, areas 13-15.

	AL											
	Laplace Distribution											
	PR						REML					
Groups	Meth I	Meth II	Meth III	CIRao	CIJY	CIJY1	Meth I	Meth II	Meth III	CIRao	CIJY	CIJY1
						Pattern I						
G1	1.89	2.12	2.16	2.26	2.27	2.27	1.90	2.21	2.25	3.10	3.12	3.68
G2	1.82	2.05	2.09	2.19	2.20	2.21	1.83	2.14	2.18	2.82	2.83	3.70
G3	1.75	1.98	2.01	2.11	2.11	2.12	1.76	2.07	2.09	2.82	2.82	3.72
G4	1.65	1.88	1.90	2.01	2.02	2.02	1.66	1.96	1.99	2.83	2.84	3.76
G5	1.52	1.74	1.76	1.87	1.88	1.87	1.52	1.83	1.84	3.26	3.28	4.39
						Pattern II						
G1	2.11	2.55	2.68	2.76	2.76	2.86	2.23	2.51	2.60	2.61	2.62	2.60
G2	1.69	2.29	2.27	2.38	2.38	2.54	1.82	2.13	2.15	2.16	2.17	2.15
G3	1.62	2.24	2.21	2.31	2.32	2.48	1.74	2.05	2.07	2.08	2.08	2.06
G4	1.52	2.18	2.12	2.24	2.25	2.40	1.64	1.95	1.96	1.97	1.98	1.95
G5	1.22	1.99	1.89	2.04	2.05	2.19	1.33	1.61	1.61	1.62	1.62	1.60
						Pattern III						
G1	2.10	2.69	3.36	2.25	2.25	2.24	2.37	2.58	2.72	2.73	2.73	2.72
G2	1.51	2.71	3.04	2.18	2.18	2.18	1.82	2.07	2.10	2.12	2.12	2.09
G3	1.44	2.72	3.03	2.09	2.10	2.09	1.75	1.98	2.01	2.03	2.03	2.00
G4	1.34	2.73	3.04	1.98	1.99	1.98	1.65	1.88	1.90	1.92	1.93	1.89
G5	0.81	3.05	3.47	1.84	1.84	1.83	1.04	1.20	1.20	1.22	1.23	1.19

Table 5: CPs for nominal 95% CIs for ψ_i pattern (II). Results obtained for the normal and Laplace (double exponential) random effects distributions and $m = 20$.

Groups		CP — FH						CP — PR					
		MetID	MetIID	MetIIID	CIRaoD	CIJYD	CIJY1D	MetID	MetIID	MetIIID	CIRaoD	CIJYD	CIJY1D
Normal Distribution													
G1	REML	90.5	92.1	94.9	95.2	95.1	94.8	92.5	93.9	94.4	95.9	96.0	96.0
G2	REML	91.5	93.7	94.9	95.2	95.2	94.9	91.8	93.8	94.4	95.1	95.1	95.0
G3	REML	91.9	94.2	94.9	95.4	95.2	94.9	92.2	94.3	94.4	95.2	94.9	95.0
G4	REML	92.1	94.3	95.1	95.2	95.0	94.8	92.2	94.5	94.4	95.1	94.8	94.8
G5	REML	92.7	95.0	94.8	95.2	95.1	94.8	90.7	93.1	94.2	93.1	93.1	93.1
G1	ML	89.1	91.7	94.6	94.6	94.8	94.5	87.7	90.7	95.6	95.6	95.6	95.8
G2	ML	90.2	93.6	94.5	94.8	94.8	94.4	88.7	93.5	95.1	95.1	94.8	95.2
G3	ML	90.8	94.0	94.6	94.9	94.9	94.2	92.2	94.0	94.7	95.0	94.8	95.0
G4	ML	91.0	94.2	94.5	94.8	95.1	94.3	94.5	94.5	94.6	94.7	94.9	94.8
G5	ML	91.9	95.1	94.2	95.2	94.8	94.4	90.7	95.7	94.2	94.5	94.3	94.5
Laplace Distribution													
G1	REML	82.8	86.1	94.2	91.6	92.6	92.2	88.5	91.0	89.6	91.3	91.2	90.9
G2	REML	83.9	90.1	94.4	91.8	92.8	92.2	87.9	91.9	90.7	93.6	93.8	93.3
G3	REML	83.9	90.7	94.5	91.8	92.8	92.1	88.1	92.6	91.1	94.4	94.5	94.0
G4	REML	84.1	91.5	94.4	91.9	92.7	92.3	88.3	93.3	92.1	95.1	95.1	94.8
G5	REML	85.4	94.0	94.2	91.5	92.5	91.8	87.3	93.5	93.8	95.8	95.9	95.6
G1	ML	85.4	89.3	91.7	92.0	91.9	92.3	81.9	88.0	91.1	91.1	90.8	90.8
G2	ML	86.3	91.9	91.9	92.3	92.5	92.5	82.4	90.7	92.3	92.9	92.8	94.1
G3	ML	86.3	92.3	92.0	92.3	92.3	92.5	82.6	90.7	93.1	93.8	93.7	94.9
G4	ML	86.8	93.1	92.2	93.1	93.2	92.5	82.7	90.4	93.8	94.6	94.6	95.5
G5	ML	87.7	95.2	93.9	95.2	95.0	93.7	83.3	90.1	95.5	96.5	96.6	96.5

Table 6: ALs for nominal 95% CIs for ψ_i pattern (II). Results obtained for the normal and Laplace (double exponential) random effects distributions and $m = 20$.[8]

Dist.	Method	Groups	AL MetID	AL MetIID	AL MetIIID	AL CIRaoD	AL CIJYD	AL CIJY1D	FH MetID	FH MetIID	FH MetIIID	FH CIRaoD	FH CIJYD	FH CIJY1D
Normal Distribution	REML / FH	G1	4.32	4.54	6.94	5.40	5.79	4.52	4.32	4.54	4.53	4.32	4.72	4.69
		G2	3.25	3.46	4.04	3.72	3.95	3.45	3.25	3.46	3.48	3.59	3.67	3.73
		G3	3.07	3.26	3.71	3.46	3.62	3.25	3.07	3.27	3.28	3.43	3.65	3.48
		G4	2.85	3.02	3.34	3.26	3.26	3.02	2.85	3.03	3.04	3.14	3.47	3.19
		G5	2.20	2.31	2.41	2.36	2.37	2.31	2.19	2.32	2.32	2.56	2.62	2.60
Normal Distribution	ML / PR	G1	4.19	4.53	4.36	4.93	3.91	5.25	4.22	4.53	5.30	5.11	5.54	5.25
		G2	3.19	3.46	3.43	3.57	3.95	3.72	3.17	3.50	3.73	3.64	3.82	3.75
		G3	3.02	3.27	3.25	3.34	3.69	3.47	2.99	3.32	3.49	3.40	3.55	3.50
		G4	2.81	3.03	3.03	3.29	3.25	3.18	2.78	3.09	3.19	3.39	3.26	3.21
		G5	2.17	2.33	2.32	2.38	2.37	2.36	2.15	2.42	2.37	3.11	2.37	2.38
Laplace Distribution	REML / FH	G1	3.27	3.42	4.06	4.89	4.47	4.10	3.04	3.20	4.30	3.93	4.29	3.98
		G2	2.68	2.89	3.24	3.59	4.12	3.31	2.68	2.92	3.35	3.28	3.36	3.52
		G3	2.57	2.78	3.08	3.37	2.85	3.15	2.57	2.81	3.17	2.98	3.18	3.34
		G4	2.43	2.63	2.87	3.10	3.14	2.94	2.42	2.67	2.94	2.88	2.93	3.16
		G5	1.96	2.15	2.24	2.33	2.57	2.28	2.09	2.37	2.27	2.83	2.61	2.82
Laplace Distribution	ML / PR	G1	3.38	3.11	4.47	3.83	4.91	3.84	3.40	3.09	4.45	4.63	4.93	4.54
		G2	2.90	2.57	3.42	2.93	3.87	3.15	3.02	2.51	3.46	3.55	3.63	3.59
		G3	2.80	2.47	3.23	2.79	3.06	3.01	2.93	2.40	3.27	3.52	3.61	3.39
		G4	2.66	2.33	2.99	2.30	2.35	2.82	2.83	2.27	3.04	3.25	3.32	3.15
		G5	2.22	1.89	2.28	2.26	2.30	2.22	2.49	1.83	2.32	2.37	3.05	2.40