JIRSS (2015)
Vol. 14, No. 2, pp 71-92
DOI: 10.7508/jirss.2015.02.005

A Flexible Class of Skew Logistic Distribution

C. Satheesh Kumar[1], L. Manju[2]

[1]Department of Statistics, University of Kerala, Trivandrum, Kerala, India.
[2]Department of Community Medicine, Sree Gokulam Medical College, Trivandrum, Kerala, India.

Abstract. Here we consider a new class of skew logistic distribution as a generalized mixture of the standard logistic and skew logistic distributions, and study some of its important aspects. The tail behaviour of the distribution is compared with that of the skew logistic distribution and a location-scale extension of the distribution is proposed. Further the maximum likelihood estimation of the parameters of the extended class of distribution is attempted. The usefulness of the proposed class of distribution is illustrated with the help of a data set.

Keywords. Density function, Maximum likelihood estimation, Skew logistic distribution, Simulation.

MSC: 60E05, 60E10.

1 Introduction

The logistic distribution (LD) has received much attention in several areas of scientific research especially in areas such as bioassay problems (Finney , 1952), study of income distributions (Fisk , 1961), analysis of survival data (Plackett , 1959) and modelling of the spread of an innovation (Oliver , 1969). A detailed account of the properties and applications of the LD is available in Balakrishnan (1992). The LD is considered as an alternative to the normal distribution in several practical occassions. Analogous to the skew normal distribution of Azzalini (1985), Wahed and Ali (2001) introduced and studied the skew logistic distribution (SLD) through the

C. Satheesh Kumar(✉)(drcsatheeshkumar@gmail.com), Manju. L (manjuhariharan@yahoo.co.in)

following probability density function (p.d.f):

$$f_1(x,\beta) = \frac{2e^{-x}}{(1+e^{-x})^2(1+e^{-\beta x})}, \tag{1}$$

where $x \in R = (-\infty, +\infty)$ and $\beta \in R$. Gupta and Kundu (2010) named the SLD with p.d.f (1) as the "generalised logistic distribution (GLD)" and derived some of its basic properties. Nadarajah (2009) proposed an extended form of this p.d.f as

$$f_2(x,\beta,\lambda) = \frac{2e^{(-\frac{x}{\beta})}}{\beta(1+e^{(-\frac{x}{\beta})})^2(1+e^{(-\frac{\lambda x}{\beta})})}, \tag{2}$$

in which $x \in R$, $\beta > 0$ and $\lambda \in R$. Clearly, when $\lambda = 0$, the p.d.f (2) reduces to the p.d.f of the standard logistic distribution. Chakraborty et al. (2012) considered a new skew logistic distribution (L_S) with the following p.d.f, in which $x \in R$, $\alpha \geqslant 1$, $\lambda \in R$ and $\beta > 0$:

$$f_3(x;\lambda,\alpha,\beta) = \frac{[1+\{\sin(\lambda x/2\beta)\}/\alpha]e^{(-\frac{x}{\beta})}}{\beta(1+e^{(-\frac{x}{\beta})})^2}. \tag{3}$$

Asgharzadeh et al. (2013) proposed a generalized skew logistic distribution (GSL) using the type III generalized logistic distribution through the following p.d.f, in which $x \in R$, $\alpha > 0$ and $\beta \in R$:

$$f_4(x;\alpha,\beta) = 2g_\alpha(x)G_\alpha(\beta x), \tag{4}$$

where

$$g_\alpha(x) = \frac{1}{B(\alpha,\alpha)}\frac{e^{-\alpha x}}{(1+e^{-x})^{2\alpha}}$$

in which $B(.,.)$ is the beta function and

$$G_\alpha(x) = \frac{B_y(\alpha,\alpha)}{B(\alpha,\alpha)},$$

with $y = (1+e^{-x})^{-1}$, and $B_y(.,.)$ is the incomplete beta function.

Hazarika and Chakraborty (2014) considered another skew logistic distribution namely the alpha skew logistic distribution(ASLG), which has the following p.d.f, in which $x \in R$ and $\alpha \in R$:

$$f_5(x;\alpha) = \frac{3\{(1-\alpha x)^2+1\}e^{-x}}{\{6+(\alpha^2\pi^2)\}(1+e^{-x})^2}. \tag{5}$$

A generalised version of the ASLG is also introduced by Hazarika and Chakraborty (2015). But in practice, the data set may become more complex and possess shapes near to SLD, but having relatively more skewed shapes. To tackle such situations, we need more flexible skewed models. So through this paper we consider a more flexible version of the SLD and named it as the modified skew logisitic distribution (MSLD). The advantage of wider skewness range of the proposed model is numerically illustrated in section 2 of the paper immediately after the Corollary 2.3. The rest of the paper is organised as follows. In section 2, we present the definition and some important properties of the MSLD and in section 3, we define the location-scale extension of the MSLD and discuss the maximum likelihood estimation of the parameters of the distribution, along with a numerical data illustration. In section 4, a simulation study is conducted to test the efficiency of the MLEs of MSLD.

We need the following integral representations in the sequel, among them (6) and (7) are from Gradshteyn and Ryzhik (2000, pp. 315 and 340) and (8) is from Prudnikov et al. (1986, pp. 300), in which $2F_1(a,b;c;\theta)$ denotes the Gauss hypergeometric function. For any positive reals u,v and w, and for any positive integer n, we have

$$\int_0^u \frac{x^{\mu-1}dx}{(1+\beta x)^{\nu}} = \frac{u^{\mu}}{\mu} 2F_1(\nu,\mu;1+\mu;-\beta u), |arg(1+\beta\mu)| < \pi; Re(\mu) > 0, \quad (6)$$

$$\int_0^{\infty} x^n e^{-\mu x} dx = \frac{n!}{\mu^{n+1}}, \qquad Re(\mu) > 0 \quad (7)$$

and

$$\int_0^w \frac{x^{\alpha-1}dx}{(x+w)^2} = \frac{w^{\alpha-2}}{2} - (\alpha-1)w^{\alpha-2}\delta(\alpha), \quad Re(\alpha) > 0, \quad (8)$$

in which

$$\delta(\alpha) = \frac{1}{2}[\Psi(\frac{1+\alpha}{2}) - \Psi(\frac{\alpha}{2})] \text{with} \quad \Psi(a) = \frac{d \log \Gamma a}{da}. \quad (9)$$

2 Definition and properties

Here we present the definition of the MSLD and discuss some of its important properties.

Definition 2.1. A random variable X is said to follow the modified skew logistic distribution (MSLD) if its p.d.f is of the following form, in which $x \in R$, $\alpha \geq -1$ and $\beta \in R$.

$$f(x;\alpha,\beta) = \frac{2}{\alpha+2} \frac{e^{-x}}{(1+e^{-x})^2} [1 + \frac{\alpha e^{-\beta x}}{1+e^{-\beta x}}] \quad (10)$$

A distribution with p.d.f (10) hereafter we denoted as the $MSLD(\alpha, \beta)$. Clearly, when $\alpha = 0$ and/or $\beta = 0$ the p.d.f (10) reduces to the p.d.f of the LD. When $\alpha = -1$, the p.d.f (10) reduces to the p.d.f of the SLD as given in (2) with $\beta = 1$ and $\lambda = \beta$. The p.d.f plots of $MSLD(\alpha, \beta)$ for particular choices of α and β are given in figures 1 to 3. From these figures it is seen that the behaviour of skewness depends on the value of α.

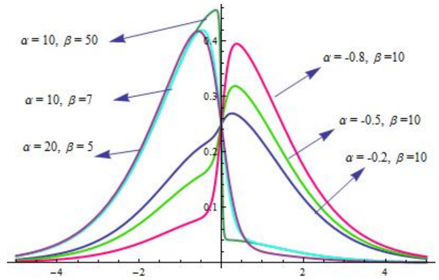

Figure 1: Plots of p.d.f of $MSLD(\alpha, \beta)$ for different values of α and β

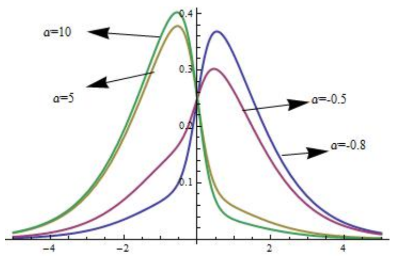

Figure 2: Plots of p.d.f of $MSLD(\alpha, \beta)$ for different values of α with $\beta = 5$

Due to the complexity in the direct integration of (10), we derive certain single and double series representation of the p.d.f, c.d.f, characterestic function and moments of the $MSLD(\alpha, \beta)$ through the following results, those we derived by an analogous procedure employed to obtain similar representations in case of the SLD of Nadarajah (2009).

In order to obtain series representations of the p.d.f (10), we need the following Taylor series expansion.

$$(1 + e^{-\beta x})^{-1} = \begin{cases} e^{\beta x} \sum_{j=0}^{\infty} \binom{-1}{j} e^{\beta j x}, & if \;\; x < 0 \\ \sum_{j=0}^{\infty} \binom{-1}{j} e^{-\beta j x}, & if \;\; x > 0 \end{cases} \tag{11}$$

where,

$$\binom{-x}{y} = \frac{(-1)^y (x + y - 1)!}{y!(x - 1)!},$$

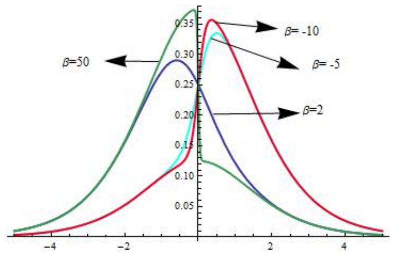

Figure 3: Plots of p.d.f of $MSLD(\alpha,\beta)$ for different values of β with $\alpha = 2$

which can be computed easily with the help of R softwares. In the light of (11), we have the following single series representation of the p.d.f $f(x;\alpha,\beta)$ of the $MSLD(\alpha,\beta)$.

$$f(x;\alpha,\beta) = \begin{cases} \frac{2}{\alpha+2}\left[\frac{e^{-x}}{(1+e^{-x})^2} + \frac{\alpha\sum_{j=0}^{\infty}\binom{-1}{j}e^{(-1+\beta j)x}}{(1+e^{-x})^2}\right], & if\ \ x<0 \\ \frac{2}{\alpha+2}\left[\frac{e^{-x}}{(1+e^{-x})^2} + \frac{\alpha\sum_{j=0}^{\infty}\binom{-1}{j}e^{-(1+\beta+\beta j)x}}{(1+e^{-x})^2}\right], & if\ \ x>0 \end{cases} \tag{12}$$

On expanding the terms $(1+e^{-x})^{-2}$ in (12), we obtain the following double series representation of the p.d.f of the $MSLD(\alpha,\beta)$.

$$f(x;\alpha,\beta) = \begin{cases} \frac{2}{\alpha+2}\left[\sum_{k=0}^{\infty}\binom{-2}{k}e^{(1+k)x} + \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}e^{(1+\beta j+k)x}\right], if\ x<0 \\ \frac{2}{\alpha+2}\left[\sum_{k=0}^{\infty}\binom{-2}{k}e^{-(1+k)x} + \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}e^{-(1+\beta+\beta j+k)x}\right], if\ x>0 \end{cases} \tag{13}$$

Consequently, it is possible to develop a single series as well as double series representations of the cumulative distribution function (c.d.f) of the $MSLD(\alpha,\beta)$, those we present through the following results.

Proposition 2.1. *The c.d.f of the* $MSLD(\alpha,\beta)$ *with p.d.f (10) has the following single series representa-*

tion, for any $x \in R$.

$$F(x)=\begin{cases} \frac{2}{\alpha+2}[\frac{1}{1+e^{-x}}+\alpha\sum_{j=0}^{\infty}\binom{-1}{j}\frac{e^{(1+\beta j)x}}{1+\beta j}{}_2F_1(2,1+\beta j;2+\beta j;-e^{x})], if\ x<0 \\ \frac{2}{\alpha+2}[\frac{1}{2}+\alpha\sum_{j=0}^{\infty}\binom{-1}{j}(\frac{1}{2}-\beta j\ \delta(\beta j+1))], \qquad if\ x=0 \\ F(0)+\frac{2}{\alpha+2}[\frac{1}{1+e^{-x}}-\frac{1}{2} \\ +\alpha\sum_{j=0}^{\infty}\binom{-1}{j}(\frac{1}{2}-(\beta+\beta j)\ \delta(1+\beta+\beta j)-\varphi_j(x;\beta))], \quad if\ x>0 \end{cases} \tag{14}$$

in which

$$\varphi_j(x;\beta)=\frac{e^{-(1+\beta+\beta j)x}}{(1+\beta+\beta j)}{}_2F_1(2,1+\beta+\beta j;2+\beta+\beta j;-e^{-x}).$$

Proof. By definition, for $x < 0$, the c.d.f of the $MSLD(\alpha,\beta)$ has the following form, in the light of (12).

$$F(x)=\frac{2}{\alpha+2}[\int_{-\infty}^{x}\frac{e^{-x}}{(1+e^{-x})^2}dx+\alpha\sum_{j=0}^{\infty}\binom{-1}{j}\int_{-\infty}^{x}\frac{e^{(-1+\beta j)x}}{(1+e^{-x})^2}dx]$$

$$=\frac{2}{\alpha+2}[\frac{1}{1+e^{-x}}+\alpha\sum_{j=0}^{\infty}\binom{-1}{j}\int_{-\infty}^{x}\frac{e^{(-1+\beta j)x}}{(1+e^{-x})^2}dx] \tag{15}$$

On substituting $z=e^x$ in the second term of (15), we get

$$F(x)=\frac{2}{\alpha+2}[\frac{1}{1+e^{-x}}+\alpha\sum_{j=0}^{\infty}\binom{-1}{j}\int_{0}^{e^x}\frac{z^{\beta j}}{(1+z)^2}dz]. \tag{16}$$

Now, by applying (6) in (16), we have

$$F(x)=\frac{2}{(\alpha+2)}[\frac{1}{1+e^{-x}}+\alpha\sum_{j=0}^{\infty}\binom{-1}{j}\frac{e^{x(\beta j+1)}}{\beta j+1}\ {}_2F_1(2,1+\beta j;2+\beta j;-e^{x})] \tag{17}$$

For $x > 0$, the c.d.f of the $MSLD(\alpha,\beta)$ can be written as given below, by using (12).

$$F(x)=F(0)+\frac{2}{\alpha+2}[\int_{0}^{x}\frac{e^{-x}}{(1+e^{-x})^2}dx+\alpha\sum_{j=0}^{\infty}\binom{-1}{j}\int_{0}^{x}\frac{e^{-(1+\beta+\beta j)x}}{(1+e^{-x})^2}dx]$$

$$=F(0)+\frac{2}{\alpha+2}[\frac{1}{1+e^{-x}}-\frac{1}{2}+\alpha\sum_{j=0}^{\infty}\binom{-1}{j}\int_{0}^{x}\frac{e^{-(1+\beta+\beta j)x}}{(1+e^{-x})^2}\ dx] \tag{18}$$

On substituting $z = e^{-x}$ in the second term of (18) to get

$$\begin{aligned} F(x) &= F(0) + \frac{2}{\alpha+2}\Big[\frac{1}{1+e^{-x}} - \frac{1}{2} + \alpha \sum_{j=0}^{\infty} \binom{-1}{j} \int_{e^{-x}}^{1} \frac{z^{\beta+\beta j}}{(1+z)^2} dz\Big] \\ &= F(0) + \frac{2}{\alpha+2}\Big[\frac{1}{1+e^{-x}} - \frac{1}{2} + \alpha \sum_{j=0}^{\infty} \binom{-1}{j} \Big(\int_{0}^{1} \frac{z^{\beta+\beta j}}{(1+z)^2} dz \\ &- \int_{0}^{e^{-x}} \frac{z^{\beta+\beta j}}{(1+z)^2} dz\Big)\Big] \qquad (19) \end{aligned}$$

on splitting the integral. By applying (6) and (8) in (19) we obtain the following.

$$\begin{aligned} F(x) &= F(0) + \frac{2}{\alpha+2}\Big[\frac{1}{1+e^{-x}} - \frac{1}{2} \\ &+ \alpha \sum_{j=0}^{\infty} \binom{-1}{j} \Big(\frac{1}{2} - (\beta+\beta j)\ \delta(1+\beta+\beta j) - \varphi_j(x;\beta)\Big)\Big] \qquad (20) \end{aligned}$$

in which

$$\varphi_j(x;\beta) = \frac{e^{-(1+\beta+\beta j)x}}{(1+\beta+\beta j)}\, {}_2F_1(2, 1+\beta+\beta j;\ 2+\beta+\beta j; -e^{-x}).$$

Repeating the above type of arguments with $x = 0$ yields the following.

$$\begin{aligned} F(0) &= \int_{-\infty}^{0} f(y)dy \\ &= \frac{2}{\alpha+2}\Big[\int_{-\infty}^{0} \frac{e^{-x}}{(1+e^{-x})^2} dx + \alpha \sum_{j=0}^{\infty} \binom{-1}{j} \int_{-\infty}^{0} \frac{e^{-(1-\beta j)x}}{(1+e^{-x})^2} dx\Big] \\ &= \frac{2}{\alpha+2}\Big[\frac{1}{2} + \alpha \sum_{j=0}^{\infty} \binom{-1}{j} \Big[\frac{1}{2} - \beta j\ \delta(1+\beta j)\Big]\Big] \qquad (21) \end{aligned}$$

Hence the proof follows from (17), (20) and (21). □

Proposition 2.2. *The c.d.f of the MSLD(α,β) with p.d.f (10) has the following double series representation, for any $x \in R$.*

$$F(x) = \begin{cases} \frac{2}{\alpha+2}\Big[\sum\limits_{k=0}^{\infty} \binom{-2}{k} \frac{e^{(1+k)x}}{(1+k)} + \alpha \sum\limits_{j=0}^{\infty} \sum\limits_{k=0}^{\infty} \binom{-1}{j}\binom{-2}{k} \frac{e^{(1+\beta j+k)x}}{(1+\beta j+k)}\Big], & if\ x < 0 \\ 1 - \frac{2}{\alpha+2}\Big[\sum\limits_{k=0}^{\infty} \binom{-2}{k} \frac{e^{-(1+k)x}}{(1+k)} + \alpha \sum\limits_{j=0}^{\infty} \sum\limits_{k=0}^{\infty} \binom{-1}{j}\binom{-2}{k} \frac{e^{-(1+\beta+\beta j+k)x}}{(1+\beta+\beta j+k)}\Big], & if\ x \geq 0 \end{cases} \qquad (22)$$

Proof. By definition, the c.d.f of the $MSLD(\alpha, \beta)$ takes the following form, for $x < 0$ in the light of (13).

$$F(x) = \frac{2}{\alpha+2}[\sum_{k=0}^{\infty}\binom{-2}{k}\int_{-\infty}^{x} e^{(1+k)x}dx + \alpha\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\int_{-\infty}^{x} e^{(1+\beta j+k)x}dx]$$
$$= \frac{2}{\alpha+2}[\sum_{k=0}^{\infty}\binom{-2}{k}\frac{e^{(1+k)x}}{(1+k)} + \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\frac{e^{(1+\beta j+k)x}}{(1+\beta j+k)}] \tag{23}$$

In a similar way, by using (13), the c.d.f of the $MSLD(\alpha, \beta)$ can be written as given below, for x > 0

$$\begin{aligned} F(x) &= 1 - \int_{x}^{\infty} f(x)dx \\ &= 1 - \frac{2}{\alpha+2}[\sum_{k=0}^{\infty}\binom{-2}{k}\int_{x}^{\infty} e^{-(1+k)x}dx \\ &+ \alpha\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\int_{x}^{\infty} e^{-(1+\beta+\beta j+k)x}dx] \\ &= 1 - \frac{2}{\alpha+2}[\sum_{k=0}^{\infty}\binom{-2}{k}\frac{e^{-(1+k)x}}{(1+k)} \\ &+ \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\frac{e^{-(1+\beta+\beta j+k)x}}{(1+\beta+\beta j+k)}] \end{aligned} \tag{24}$$

Thus (23) and (24) gives (22). With this c.d f we can easily evaluate the probabilites using software such as mathcad, matlab, R etc. □

Corollary 2.1. When $\alpha = -1$ and $\beta = \lambda$ in (22) we get the c.d.f of SLD with $\beta = 1$.

Proposition 2.3. *The single series representation of the characterestic function $\Phi_X(t)$ of the $MSLD(\alpha, \beta)$ with p.d.f (12) is the the following, for any $t \in R$ and $i = \sqrt{-1}$.*

$$\Phi_X(t) = \frac{2}{\alpha+2}[B(1+it,\ 1-it) + \alpha\sum_{j=0}^{\infty}\binom{-1}{j}\xi(t,\beta)], \tag{25}$$

in which $B(1+it,\ 1-it)$ is the beta function and

$$\xi(t,\beta) = 1 - (\beta j + it)\ \delta(1+\beta j+it) - (\beta+\beta j - it)\ \delta(1+\beta+\beta j - it)$$

with $\delta(a)$ is as defined in (9).

Proof. Let X follows the $MSLD(\alpha, \beta)$ with p.d.f (12). Then by the definition of characterestic function, we have the following, for any $t \in R$ and $i = \sqrt{-1}$.

$$\Phi_x(t) = E(e^{itx})$$
$$= \frac{2}{\alpha+2}[\int_{-\infty}^{0} \frac{e^{(it-1)x}}{(1+e^{-x})^2}dx + \alpha \sum_{j=0}^{\infty} \binom{-1}{j} \int_{-\infty}^{0} \frac{e^{-(1-\beta j-it)x}}{(1+e^{-x})^2}dx + \quad (26)$$
$$\int_{0}^{\infty} \frac{e^{(it-1)x}}{(1+e^{-x})^2}dx + \alpha \sum_{j=0}^{\infty} \binom{-1}{j} \int_{0}^{\infty} \frac{e^{-(1+\beta+\beta j-it)x}}{(1+e^{-x})^2}dx]$$

Combining the first and third integrals (26) reduces to

$$\Phi_x(t) = \frac{2}{\alpha+2}(I_1 + I_2 + I_3), \quad (27)$$

in which

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{(it-1)x}}{(1+e^{-x})^2}dx, \quad (28)$$

$$I_2 = \alpha \sum_{j=0}^{\infty} \binom{-1}{j} \int_{-\infty}^{0} \frac{e^{-(1-\beta j-it)x}}{(1+e^{-x})^2}dx \quad (29)$$

and

$$I_3 = \alpha \sum_{j=0}^{\infty} \binom{-1}{j} \int_{0}^{\infty} \frac{e^{-(1+\beta+\beta j-it)x}}{(1+e^{-x})^2}dx. \quad (30)$$

On substituting $u = (1 + e^{-x})^{-1}$ in (28) we have

$$I_1 = \int_{0}^{1} u^{it}(1-u)^{-it}du$$
$$= B(1 + it, 1 - it) \quad (31)$$

If we put $z = e^x$ in (29) and applying (8) we have

$$I_2 = \alpha \sum_{j=0}^{\infty} \binom{-1}{j} \int_{0}^{1} \frac{z^{\beta j+it}}{(1+z)^2}dz$$
$$= \frac{1}{2} - (\beta j + it)\ \delta(1 + \beta j + it) \quad (32)$$

If we put $v = e^{-x}$ in (30) and applying (8), we get

$$\begin{aligned} I_3 &= \alpha \sum_{j=0}^{\infty} \binom{-1}{j} \int_0^1 \frac{v^{\beta+\beta j-it}}{(1+v)^2} dv \\ &= \frac{1}{2} - (\beta + \beta j - it)\, \delta(1 + \beta + \beta j - it). \end{aligned} \tag{33}$$

Now substituting (31), (32) and (33) in (27) yields (25). □

Proposition 2.4. *The double series representation of the characteristic function $\Phi_X(t)$ of MSLD(α, β) with p.d.f (13) is the the following, for $t \in R$, $i = \sqrt{-1}$, $Re(1 + k - it) > 0$ and $Re(1 + \beta + \beta j + k - it) > 0$.*

$$\begin{aligned} \Phi_X(t) = \frac{2}{\alpha+2}[\sum_{k=0}^{\infty} \binom{-2}{k} (\frac{1}{1+k+it} + \frac{1}{1+k-it}) \\ +\alpha \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{-1}{j} \binom{-2}{k} (\frac{1}{1+\beta j+k+it} + \frac{1}{1+\beta+\beta j+k-it})], \end{aligned} \tag{34}$$

Proof. By using the double series representation (13), we have

$$\begin{aligned} \Phi_X(t) = \frac{2}{\alpha+2}[\sum_{k=0}^{\infty} \binom{-2}{k} (\int_{-\infty}^{0} e^{(1+k+it)x} dx + \int_0^{\infty} e^{-(1+k-it)x} dx) \\ +\alpha \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{-1}{j} \binom{-2}{k} (\int_{-\infty}^{0} e^{(1+\beta j+k+it)x} dx + \int_0^{\infty} e^{-(1+\beta+\beta j+k-it)x} dx)], \end{aligned}$$

which implies (34) by evaluating the integrals. □

Next we obtain an expression for n^{th} raw moments of the MSLD through the following result by utilising the double series representation of the p.d.f (10).

Proposition 2.5. *The n^{th} raw moment μ'_n of the MSLD(α, β) with p.d.f (10) is the following, for $n > 0$*

$$\mu'_n = \begin{cases} \frac{2\alpha n!}{\alpha+2}[\sum_{j=0}^{\infty} (-1)^j \{\Omega(-1, n, \beta) + \Omega^*(-1, n, \beta)\}], \; if \; n \; is \; odd \\ \frac{2n!}{\alpha+2}\left[\begin{array}{l} 2(1-2^{1-n})\zeta(n) + \\ \alpha \sum_{j=0}^{\infty} (-1)^j \{\Omega(-1, n, \beta) + \Omega^*(-1, n, \beta)\}], \; if \; n \; is \; even \end{array} \right. \end{cases} \tag{35}$$

Proof. By definition,

$$
\begin{aligned}
\mu_n' &= \int_{-\infty}^{\infty} x^n f(x)dx \\
&= \frac{2}{\alpha+2}[\sum_{k=0}^{\infty}\binom{-2}{k}\int_{-\infty}^{0} x^n e^{(1+k)x}dx \\
&+ \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\int_{-\infty}^{0} x^n e^{(1+\beta j+k)x}dx \\
&+ \sum_{k=0}^{\infty}\binom{-2}{k}\int_{0}^{\infty} x^n e^{-(1+k)x}dx \\
&+ \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\int_{0}^{\infty} x^n e^{-(1+\beta+\beta j+k)x}dx]
\end{aligned} \tag{36}
$$

which implies the following through evaluating the integrals by applying product rules of integration in the first two terms of (36) and using (7) in its last two terms.

$$
\begin{aligned}
\mu_n' = \frac{2}{\alpha+2}[\sum_{k=0}^{\infty}\binom{-2}{k}\frac{(-1)^n n!}{(1+k)^{n+1}} + \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\frac{(-1)^n n!}{(1+\beta j+k)^{n+1}} \\
+\sum_{k=0}^{\infty}\binom{-2}{k}\frac{n!}{(1+k)^{n+1}} + \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\frac{n!}{(1+\beta+\beta j+k)^{n+1}}]
\end{aligned} \tag{37}
$$

When n is odd, since the sum of the first and third terms of (37) is zero, we get the following.

$$
\begin{aligned}
\mu_n' &= \frac{2\alpha n!}{\alpha+2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}[\frac{1}{(1+\beta+\beta j+k)^{n+1}} + \frac{(-1)^n}{(1+\beta j+k)^{n+1}}] \\
&= \frac{2\alpha n!}{\alpha+2}[\sum_{j=0}^{\infty}(-1)^j\{\Omega(-1,\ n,\ \beta) + \Omega^*(-1,\ n,\ \beta)\}]
\end{aligned} \tag{38}
$$

where

$$\Omega(-1,\ n,\ \beta) = \Phi(-1,\ n+1,\ 1+\beta+\beta j) + (-1)^n\Phi(-1,\ n+1,\ 1+\beta j)$$

and

$$\Omega^*(-1,\ n,\ \beta) = \Phi^*(-1,\ n+1,\ 1+\beta+\beta j) + (-1)^n\Phi^*(-1,\ n+1,\ 1+\beta j),$$

in which $\Phi(z,s,b)$ and $\Phi^{*}(z,s,b)$ are the Lerch function and generalised Lerch functions respectively.
When n is even, we obtain the following by combining the first and third terms of (37).

$$\begin{aligned}
\mu'_n &= \frac{2n!}{\alpha+2}[2\sum_{k=0}^{\infty}\frac{\binom{-2}{k}}{(1+k)^{n+1}}+ \\
&\quad \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}(\frac{1}{(1+\beta+\beta j+k)^{n+1}}+\frac{1}{(1+\beta j+k)^{n+1}})] \\
&= \frac{2n!}{\alpha+2}[2(1-2^{1-n})\zeta(n) \\
&+ \alpha\sum_{j=0}^{\infty}(-1)^{j}(\Omega(-1,n,\beta)+\Omega^{*}(-1,n,\beta))] \qquad (39)
\end{aligned}$$

Thus (38) and (39) together implies (35). □

Corollary 2.2. Differentiation of the characteristic function $\Phi_X(t)$ given in (34) also yields the same expression for the n^{th} order raw moments of MSLD as in (35).

Corollary 2.3. When $\alpha = -1$ and $\beta = \lambda$ in (35) we get the expression for raw moments of the SLD with $\beta = 1$.

By using some mathematical softwares such as MATHCAD, MATHEMATICA and R one can compute the moments of any order. The plots of skewness and kurtosis for varying values of α and β are obtained in Figure 4 to Figure 7. Tables showing the coefficient of skewness and kurtosis for MSLD for particular values of its parameters are included in Appendix B. We have presented the computed values of skewness for positive values of β only, since when β is negative the corresponding values takes opposite sign with same magnitude. Thus, from Table 3 it is evident that the skewness varies from -2.1 to 2.1, which is a wider range of skewness measure compared to those models of Wahed and Ali (2001), Nadarajah (2009) or that of Hazarika and Chakraborty (2014). Thus the proposed skew logistic model can be considered as a more flexible model useful for studying asymmetric data sets, compared to the above mentioned existing models

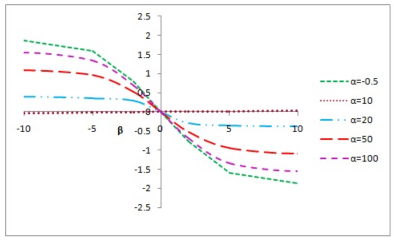

Figure 4: Skewness of $MSLD(\alpha, \beta)$ for different values of α for β =-10,...,10

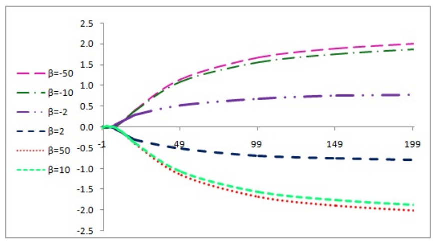

Figure 5: Skewness of $MSLD(\alpha, \beta)$ for different values of β for α =-1,...,200

Proposition 2.6. *If $X_1 \sim MSLD(\alpha, \beta)$ and $X_2 \sim SLD(\lambda, \gamma)$, we have the following.*
i. For $\lambda > 0$ and $0 < \gamma < 1$, X_1 has thicker tail compared to X_2.
ii. For $\lambda > 0$ and $\gamma > 1$, X_1 has thinner tail compared to X_2.
iii. For $\lambda < 0$ and $0 < \gamma < 1$, X_1has thicker tail compared to X_2.
iv. For $\lambda < 0$ and $\gamma > 1$ and $\lambda + \gamma - 1 >(<) 0$, X_1 has thinner tail than X_2.

Proof. The tail behaviour of two distributions can be compared by taking the limiting ratio (LR) of their density (Tse , 2009). Faster the ratio approaches to zero(infinity) thinner(thicker) will be the tail of the numerator density compared to the denominator density. The limiting ratio of densities of random variables $X_1 \sim MSLD(\alpha, \beta)$ and $X_2 \sim SLD(\lambda, \gamma)$ defined as

$$
\begin{aligned}
LR &= \lim_{x\to\infty} \frac{f_{X_1}(x)}{f_{X_2}(x)} \\
&= \lim_{x\to\infty} \frac{\frac{2}{(\alpha+2)}\frac{e^{-x}}{(1+e^{-x})^2}(1+\frac{\alpha e^{-\beta x}}{1+e^{-\beta x}})}{\frac{2e^{-\frac{x}{\gamma}}}{\gamma(1+e^{-\frac{x}{\gamma}})^2(1+e^{-\frac{\lambda x}{\gamma}})}} \\
&= \lim_{x\to\infty} \frac{\gamma}{\alpha+2}[\frac{1+e^{\frac{-x}{\gamma}}}{1+e^{-x}}]^2[\frac{1+(1+\alpha)e^{-\beta x}}{1+e^{-\beta x}}]e^{-x(1-\frac{1}{\gamma})}[1+e^{\frac{-\lambda x}{\gamma}}] .
\end{aligned}
$$

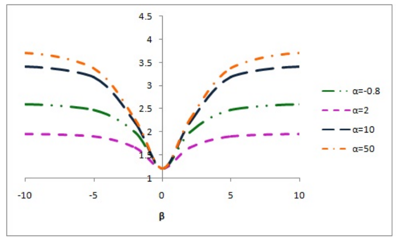

Figure 6: Kurtosis of $MSLD(\alpha, \beta)$ for different values of α for β =-10,...,10

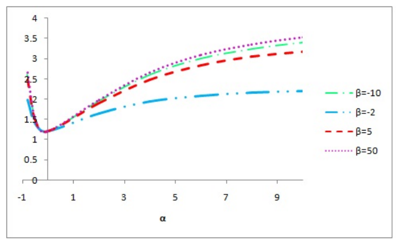

Figure 7: Kurtosis of $MSLD(\alpha, \beta)$ for different values of β for α =-1,...,10

(i) For $\lambda > 0$ and $0 < \gamma < 1$, since

$$\lim_{x\to\infty} e^{-x(1-\frac{1}{\gamma})} = \infty$$

and

$$\lim_{x\to\infty} (1 + e^{-\frac{\lambda x}{\gamma}}) = 1,$$

we have

$$LR = \lim_{x\to\infty} \frac{\gamma}{\alpha+2}[\frac{1+e^{\frac{-x}{\gamma}}}{1+e^{-x}}]^2[\frac{1+(1+\alpha)e^{-\beta x}}{1+e^{-\beta x}}]e^{-x(1-\frac{1}{\gamma})}[1+e^{\frac{-\lambda x}{\gamma}}] \to \infty \quad as\ x \to \infty$$

Thus, for $\lambda > 0$ and $0 < \gamma < 1$, X_1 has thicker tail than X_2.
(ii) For $\lambda > 0$ and $\gamma > 1$, since

$$\lim_{x\to\infty} e^{-x(1-\frac{1}{\gamma})} = 0$$

and

$$\lim_{x\to\infty} (1 + e^{-\frac{\lambda x}{\gamma}}) = 1,$$

we have

$$LR = \lim_{x\to\infty} \frac{\gamma}{\alpha+2} [\frac{1+e^{\frac{-x}{\gamma}}}{1+e^{-x}}]^2 [\frac{1+(1+\alpha)e^{-\beta x}}{1+e^{-\beta x}}] e^{-x(1-\frac{1}{\gamma})}[1+e^{\frac{-\lambda x}{\gamma}}]$$
$$\to 0 \quad as\ x \to \infty$$

.

Thus, for $\lambda > 0$ and $\gamma > 1$, X_1 has thinner tail than X_2.
(iii) For $\lambda < 0$ and $0 < \gamma < 1$, since

$$\lim_{x\to\infty} e^{x(\frac{1}{\gamma}-1)} = \infty$$

and

$$\lim_{x\to\infty} (1+e^{-\frac{\lambda x}{\gamma}}) = \infty,$$

we have

$$LR = \lim_{x\to\infty} \frac{\gamma}{\alpha+2} [\frac{1+e^{\frac{-x}{\gamma}}}{1+e^{-x}}]^2 [\frac{1+(1+\alpha)e^{-\beta x}}{1+e^{-\beta x}}] e^{-x(1-\frac{1}{\gamma})}[1+e^{\frac{-\lambda x}{\gamma}}]$$
$$\to \infty \quad as\ x \to \infty.$$

Thus, for $\lambda < 0$ and $0 < \gamma < 1$, X_1 has thicker tail than X_2.
(iv) For $\lambda < 0$, $\gamma > 1$ and $\lambda + \gamma - 1 > (<)0$, since

$$\lim_{x\to\infty} e^{-x(1-\frac{1}{\gamma})} = 0$$

$$\lim_{x\to\infty} (1+e^{-\frac{\lambda x}{\gamma}}) = \infty$$

and

$$\lim_{x\to\infty} e^{-\frac{x(\lambda+\gamma-1)}{\gamma}} = 0(\infty),$$

we have

$$LR = \lim_{x\to\infty} \frac{\beta}{\alpha+2} [\frac{1+e^{\frac{-x}{\beta}}}{1+e^{-x}}]^2 [\frac{1+(1+\alpha)e^{-\beta x}}{1+e^{-\beta x}}] e^{-x(1-\frac{1}{\beta})}[1+e^{\frac{-\lambda x}{\beta}}]$$
$$\to 0 \quad as\ x \to \infty$$

.

Thus for $\lambda < 0, \beta > 1$ and $\lambda + \gamma - 1 > (<)0$, X_1 has thinner tail than X_2.

□

3 Location scale extension and Maximum likelihood estimation

In this section we define an extended form of the $MSLD(\alpha, \beta)$ by introducing the location parameter μ and scale parameter σ in its p.d.f and discuss the maximum likelihood estimation of the parameter of the location scale extended form of the $MSLD(\alpha, \beta)$. We put the definition of this extended form as follows:

Definition 3.1. Let Z follows the $MSLD(\alpha, \beta)$ with p.d.f (10). Then for any $\mu \in R$ and $\sigma > 0$, the distribution of $X = \mu + \sigma Z$ is called "the extended MSLD" and its p.d.f takes the following form, in which $Z \in R$, $\alpha \geq -1$ and $\beta > 0$.

$$f(x, \mu, \sigma; \alpha, \beta) = \frac{2}{\alpha + 2} \frac{e^{-\frac{(x-\mu)}{\sigma}}}{\sigma(1 + e^{-\frac{(x-\mu)}{\sigma}})^2} [1 + \frac{\alpha e^{\frac{-\beta(x-\mu)}{\sigma}}}{1 + e^{\frac{-\beta(x-\mu)}{\sigma}}}] \tag{40}$$

A distribution with p.d.f (40) hereafter we denoted as $EMSLD(\mu, \sigma, \alpha, \beta)$.

Next, we discuss the maximum likelihood estimation of $EMSLD(\mu, \sigma, \alpha, \beta)$. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population having the $EMSLD(\mu, \sigma, \alpha, \beta)$ with p.d.f (40). The log likelihood fumction $l = ln\ L(\mu, \sigma, \alpha, \beta)$ of the sample is the following,

$$\begin{aligned} l &= n \ln 2 - n \ln(\alpha + 2) - n \ln \sigma - \frac{1}{\sigma} \sum_{i=1}^{n} (x_i - \mu) - 2 \sum_{i=1}^{n} \ln[1 + e^{\frac{-(x_i-\mu)}{\sigma}}] \\ &+ \sum_{i=1}^{n} \ln[1 + \frac{\alpha e^{\frac{-\beta(x_i-\mu)}{\sigma}}}{1 + e^{\frac{-\beta(x_i-\mu)}{\sigma}}}] \end{aligned} \tag{41}$$

On differentiating (41) with respect to the parameters μ, σ, α and β and then equating to zero, we obtain the following likelihood equations, in which $z_i = \frac{x_i - \mu}{\sigma}$ and $\Lambda_{ij}(x; \mu, \sigma; \alpha, \beta) = [1 + (1 + \alpha)^j e^{-\beta z_i}]^{-1}$.

$$n = 2\sum_{i=1}^{n} e^{-z_i} \Lambda_{i0}(x; \mu, \sigma, \alpha, 1) - \alpha\beta \sum_{i=1}^{n} e^{-\beta z_i} \Lambda_{i0}(x; \mu, \sigma, \alpha, \beta) \Lambda_{i1}(x; \mu, \sigma, \alpha, \beta), \tag{42}$$

$$\frac{n}{\alpha + 2} = \sum_{i=1}^{n} e^{-\beta z_i} \Lambda_{i1}(x; \mu, \sigma, \alpha, \beta), \tag{43}$$

$$n\sigma = \sum_{i=1}^{n} (x_i - \mu) - 2 \sum_{i=1}^{n} (x_i - \mu) e^{-z_i} \Lambda_{i0}(x; \mu, \sigma, \alpha, 1)$$

$$+\alpha\beta\sum_{i=1}^{n}(x_i-\mu)\Lambda_{i0}(x;\mu,\sigma,\alpha,\beta)\Lambda_{i1}(x;\mu,\sigma,\alpha,\beta) \tag{44}$$

$$\sum_{i=1}^{n}\frac{(x_i-\mu)e^{-\beta z_i}}{\Lambda_{i0}(x;\mu,\sigma,\alpha,\beta)\Lambda_{i1}(x;\mu,\sigma;\alpha,\beta)}=0, \tag{45}$$

On solving the system of equations (42) - (45) with the help of some mathematical softwares such as MATLAB, MATHCAD, MATHEMATICA, R etc. one can obtain the maximum likelihood estimates (MLE) of the parameters of the $EMSLD(\mu,\sigma,\alpha,\beta)$. Next we obtain the Fisher information matrix based on the likelihood equations as follows,

$$I=\frac{-1}{n}((I_{ij}))_{4\times 4}, \tag{46}$$

in which

$$\begin{aligned}
I_{11}&=E(\frac{\partial^2 l}{\partial\mu^2}),\ I_{12}=E(\frac{\partial^2 l}{\partial\mu\partial\sigma}),\ I_{13}=E(\frac{\partial^2 l}{\partial\mu\partial\alpha}),\ I_{14}=E(\frac{\partial^2 l}{\partial\mu\partial\beta})\\
I_{21}&=E(\frac{\partial^2 l}{\partial\sigma\partial\mu}),\ I_{22}=E(\frac{\partial^2 l}{\partial\sigma^2}),\ I_{23}=E(\frac{\partial^2 l}{\partial\sigma\partial\alpha}),\ I_{24}=E(\frac{\partial^2 l}{\partial\sigma\partial\beta})\\
I_{31}&=E(\frac{\partial^2 l}{\partial\alpha\partial\mu}),\ I_{32}=E(\frac{\partial^2 l}{\partial\alpha\partial\sigma}),\ I_{33}=E(\frac{\partial^2 l}{\partial\alpha^2}),\ I_{34}=E(\frac{\partial^2 l}{\partial\alpha\partial\beta})\\
I_{41}&=E(\frac{\partial^2 l}{\partial\beta\partial\mu}),\ I_{42}=E(\frac{\partial^2 l}{\partial\beta\partial\sigma}),\ I_{43}=E(\frac{\partial^2 l}{\partial\beta\partial\alpha}),\ I_{44}=E(\frac{\partial^2 l}{\partial\beta^2})
\end{aligned}$$

The expressions for I_{ij}'s are included in Appendix A.
For numerical illustration, we consider the variable y_1 of the data set taken from Anthony (2004), Table A.15, page 590. We obtain the MLEs of the parameters of the $EMSLD(\mu,\sigma,\alpha,\beta)$ by using the R software. The initial values are obtained by equating the first two raw moment of the
$EMSLD(\mu,\sigma,\alpha,\beta)$ with the corresponding sample raw moments. Kolmogrov-Smirnov statistic (KSS) value and certain information measures such as Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion ($CAIC$) and Hannan Quinn Information Criterion (HQC) values are computed for comparing the model $EMSLD(\mu,\sigma,\alpha,\beta)$ with the existing models - $LD(\mu,\sigma)$, $SLD(\mu,\sigma,\beta)$ of Nadarajah (2009), $L_S(\mu,\sigma,\beta_1,\beta_2)$ of Chakraborty et al. (2012), $ASLG(\mu,\sigma,\beta)$ of Hazarika and Chakraborty (2014) and $GSL(\mu,\sigma,\alpha,\beta)$ of Asgharzadeh et al. (2013). The numerical results obtained are summarised in Table 1. From Table 1 it is clear that $EMSLD(\mu,\sigma,\alpha,\beta)$ is more appropriate to the data set, compared to the other existing models. The empirical cumulative distribution of the data set is plotted along with the corresponding c.d.fs of each model in figure 8 also support the suitability of the $EMSLD(\mu,\sigma,\alpha,\beta)$ to the data set compared to other models.

Table 1: Estimated values of the parameters with the corresponding KSS, AIC , BIC, CAIC and HQC values.

Distribution:	LD (μ,σ)	SLD (μ,σ,β)	Ls $(\mu,\sigma,\beta_1,\beta_2)$	ASLG (μ,σ,β)	GSL $(\mu,\sigma,\alpha,\beta)$	EMSLD $(\mu,\sigma,\beta,\alpha)$
$\hat{\mu}$	3.152	7.329	4.033	5.102	7.107	7.231
$\hat{\sigma}$	2.494	3.569	2.493	2.232	0.209	3.516
$\hat{\beta}$	-	-8.448	-	0.298	-8.367	102.797
$\hat{\beta_1}$	-	-	-10.518	-	-	-
$\hat{\beta_2}$	-	-	1.968	-	-	-
$\hat{\alpha}$	-	-	-	-	0.045	81.703
KSS	0.153	0.135	0.206	0.141	0.117	0.094
p-value	0.037	0.089	0.002	0.067	0.190	0.439
AIC	992.396	926.494	981.372	980.300	910.507	906.800
BIC	998.667	935.902	993.915	989.707	923.050	919.343
CAIC	1000.667	938.902	997.915	992.707	927.050	923.343
HQC	994.941	930.311	986.462	984.117	915.597	911.889

4 Simulation

In order to assess the efficiency of the MLE of the parameters of EMSLD with p.f.d $f(\cdot)$, in this section, we have conducted a brief simulation study. In order to simulate values of a random variable Y with p.d.f $f(\cdot)$, we adopt the following procedure based on the acceptance/rejection method.

Step 1. Simulate X=x from the p.d.f f_1 of the standard logistic distribution

Step 2. Generate U, an independent uniform random variable on $(0, 1)$ and $\frac{f(x)}{f_1(x)} < c$, for all x.

Step 3. If $U \leq \frac{f(x)}{cf_1(x)}$, then accept $Y = X$ otherwise go to step 1. Here c is is the constant such that $\sup_x \{\frac{f(x)}{f_1(x)}\} \leq c$.

By applying the above procedure we have carried out a simulation study based on the following set of parameters of the EMSLD.

μ=7.231, σ= 3.516, α=81.703 and β =102.797. The computed values of the bias and mean square error(MSE) corresponding to sample sizes 100, 200, 300 and 500 respectively are given in Table 2. From the table it can be seen that both the absolute bias and MSEs in respect of each parameters of the MSLD are in decreasing order as the sample size increases.

Acknowledgements

The authors would like to express their sincere thanks to the Editor and the anonymous Referees for their valuable comments on an early version of the paper which greatly improved the quality and presentation of the manuscript.

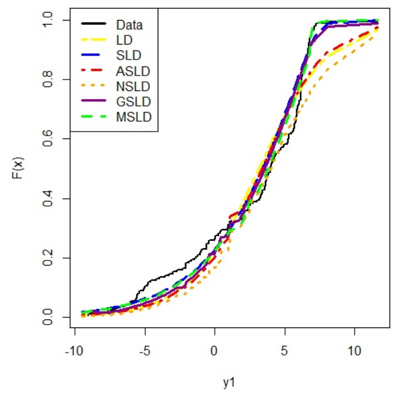

Figure 8: Empirical distribution of the data set along with the fitted c.d.fs

Table 2: Bias and Mean Square Error(MSE) within brackets of the simulated data set.

sample size	μ	σ	α	β
100	0.223	-0.172	0.738	0.974
	(0.653)	(0.031)	(0.654)	(0.949)
200	-0.162	-0.143	-0.164	0.941
	(0.042)	(0.026)	(0.143)	(0.887)
300	0.108	0.073	-0.131	0.720
	(0.034)	(0.010)	(0.065)	(0.519)
500	0.021	-0.005	-0.078	0.206
	(0.008)	(0.007)	(0.041)	(0.247)

References

Anthony, C. (2004), *Exploring multivariate data with the forward search*. NewYork: Springer.

Asgharzadeh, A., Esmaeili, L., Nadarajah, S. and Shih, S. H. (2013), A generalized skew logistic distribution. *REVSTAT*, **11**, 317-338.

Azzalini, A. (1985), A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics*, **12**, 171-178.

Balakrishnan, N. (1992), *Handbook of Logistic Distribution*, New York: Dekker.

Chakraborty, S., Hazarika, P. and Ali, M. M. (2012), A new skew-logistic distribution and its properties. *Pakistan Journal of Statistics*, **28**(4), 513-524.

Finney, D. J. (1952), *Statistical Methods in Biological Assay*. New York: Hafner Publications.

Fisk, P. R. (1961), The graduation of income distributions. *Econometrica*, **29**, 171-185.

Gradshteyn, I. S. and Ryzhik, I. M. (2000), *Tables of Integrals, Series, and Products*, San Diego: Academic Press.

Gupta, R. D. and Kundu, D. (2010), Generalised logistic distributions. *Journal of Applied Statistical Science*, **18**(1), 51-56.

Hazarika, P. and Chakraborty, S. (2014), Alpha-skew-logistic-distribution. *IOSR Journal of Mathematics*, **10**(4), 36-46.

Hazarika, P. and Chakraborty, S. (2015), A note on alpha-skew-generalized logistic distribution. *IOSR Journal of Mathematics*, **11**(1), Ver.IV, 85-88.

Nadarajah, S. (2009), The skew logistic distribution. *AStA Advances in Statistical Analysis*, **93**, 187-203.

Oliver, F. R. (1969), Another generalisation of the logistic growth function. *Econometrica*, **37**, 144-147.

Plackett, R. L. (1959), The analysis of life test data. *Technometrics*, **1**, 9-19.

Prudnikov, A. P., Brychkov, Y. A. and Marichev, O. I. (1986), *Integrals and Series*, New York: Gordon and Breach.

Tse Yiu-Kuen. (2009), *Non-life Acturial Models Theory, Methods and Evaluation*, Cambridge University Press.

Wahed, A. S. and Ali, M. M. (2001), The skew-logistic distribution. *Journal of Statistical Research*, **35**(2), 71-80.

Appendix A

We obtain the elements of the Fisher information matrix (46) as given below.

$$\begin{aligned} I_{11} &= \frac{-2n}{\sigma^2}J_1 + \frac{n\alpha\beta^2}{\sigma^2}J_2 - \frac{n\alpha\beta^2(1+\alpha)}{\sigma^2}J_3, \\ I_{12} &= \frac{-n}{\sigma^2} - \frac{2n}{\sigma^2}(J_4 - J_1 - J_5) + \frac{n\alpha\beta^2}{\sigma^2}J_6 - \frac{n\alpha\beta}{\sigma^2}J_2 \end{aligned}$$

$$
\begin{aligned}
&- \frac{n\alpha\beta(2+\alpha)}{\sigma^2}J_7 - \frac{n\alpha\beta(1+\alpha)}{\sigma^2}J_3 - \frac{n\alpha\beta^2(1+\alpha)}{\sigma^2}J_8,\\
I_{13} &= \frac{n\beta}{\sigma}J_9,\\
I_{14} &= \frac{-n\beta}{\sigma}J_6 + \frac{n\alpha}{\sigma}J_2 + \frac{n\alpha(2+\alpha)}{\sigma}J_7 + \frac{n\alpha(1+\alpha)}{\sigma}J_3 + \frac{n\alpha\beta(1+\alpha)}{\sigma}J_8,\\
I_{22} &= \frac{n}{\sigma^2} - \frac{2nE(Z)}{\sigma^2} + \frac{2n}{\sigma^2}J_{10} + \frac{4n}{\sigma^2}(J_4 + J_{11}) + n\alpha\beta^2 J_{12} + n\alpha\beta^2(1+\alpha)J_{13},\\
I_{23} &= \frac{n\beta}{\sigma}J_{14},\\
I_{24} &= \frac{n\alpha}{\sigma}J_6 - \frac{n\alpha\beta}{\sigma}J_{12} + \frac{n\alpha(2+\alpha)}{\sigma}J_{15} + \frac{n\alpha\beta(1+\alpha)}{\sigma}J_{13} + \frac{n\alpha(1+\alpha)}{\sigma}J_8,\\
I_{33} &= \frac{n}{(2+\alpha)^2} - nJ_{16},\\
I_{34} &= -nJ_{14},\\
I_{44} &= n\alpha J_{12} - n\alpha(1+\alpha)J_{13},
\end{aligned}
$$

in which $Z = \frac{X-\mu}{\sigma}$,

$$
\begin{aligned}
J_1 &= E\left(\frac{e^{-z}}{(1+e^{-z})^2}\right),\ \ J_2 = E\left(\frac{e^{-\beta z}}{(1+(1+\alpha)e^{-\beta z})^2(1+e^{-\beta z})^2}\right),\\
J_3 &= E\left(\frac{e^{-3\beta z}}{(1+(1+\alpha)e^{-\beta z})^2(1+e^{-\beta z})^2}\right),\ \ J_4 = E\left(\frac{ze^{-z}}{(1+e^{-z})^2}\right),\\
J_5 &= E\left(\frac{e^{-2z}}{(1+e^{-z})^2}\right),\ \ J_6 = E\left(\frac{ze^{-\beta z}}{(1+(1+\alpha)e^{-\beta z})^2(1+e^{-\beta z})^2}\right),\\
J_7 &= E\left(\frac{e^{-2\beta z}}{(1+(1+\alpha)e^{-\beta z})^2(1+e^{-\beta z})^2}\right),\ \ J_8 = E\left(\frac{ze^{-3\beta z}}{(1+(1+\alpha)e^{-\beta z})^2(1+e^{-\beta z})^2}\right),\\
J_9 &= E\left(\frac{e^{-\beta z}}{(1+(1+\alpha)e^{-\beta z})^2}\right),\ \ J_{10} = E\left(\frac{z^2e^{-z}}{(1+e^{-z})^2}\right),\\
J_{11} &= E\left(\frac{ze^{-2z}}{(1+e^{-z})^2}\right),\ \ J_{12} = E\left(\frac{z^2e^{-\beta z}}{(1+(1+\alpha)e^{-\beta z})^2(1+e^{-\beta z})^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_{13} &= E\left(\frac{z^2e^{-3\beta z}}{(1+(1+\alpha)e^{-\beta z})^2(1+e^{-\beta z})^2}\right),\ \ J_{14} = E\left(\frac{ze^{-\beta z}}{(1+(1+\alpha)e^{-\beta z})^2}\right),\\
J_{15} &= E\left(\frac{ze^{-2\beta z}}{(1+(1+\alpha)e^{-\beta z})^2(1+e^{-\beta z})^2}\right),\ \ J_{16} = E\left(\frac{e^{-2\beta z}}{(1+(1+\alpha)e^{-\beta z})^2}\right).
\end{aligned}
$$

Note that the expectations can be computed numerically with the help of mathematical softwares such as MATHEMATICA, MATLAB, MATHCAD, R etc.

Appendix B

Table 3: Coefficient of Skewness and Kurtosis of MSLD for varying values of α and β

	Coefficient of Skewness							Coefficient of Kurtosis						
β / α	0	5	10	50	100	150	200	0	5	10	50	100	150	200
-0.8	0.0000	-0.0037	-0.0065	-0.0203	-0.0086	-0.0088	-0.0089	1.1997	2.4819	2.6016	2.6610	2.6747	2.6815	2.6854
-0.5	0.0000	-0.0154	-0.0186	-0.0203	-0.0207	-0.0209	-0.0210	1.1997	1.5001	1.5223	1.5331	1.5356	1.5368	1.5375
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.1997	1.1995	1.1995	1.1995	1.1995	1.1995	1.1995
2	0.0000	0.0194	0.0247	0.0275	0.0281	0.0284	0.0286	1.1997	1.8978	1.9544	1.9821	1.9885	1.9916	1.9934
4	0.0000	0.0037	0.0065	0.0082	0.0086	0.0088	0.0089	1.1997	2.4819	2.6016	2.6610	2.6747	2.6815	2.6854
6	0.0000	-0.0026	-0.0013	-0.0008	-0.0007	-0.0006	-0.0006	1.1997	2.8323	3.0002	3.0844	3.1038	3.1135	3.1191
8	0.0000	-0.0260	-0.0228	-0.0213	-0.0209	-0.0208	-0.0207	1.1997	3.0460	3.2493	3.3520	3.3758	3.3877	3.3945
10	0.0000	-0.0674	-0.0655	-0.0645	-0.0643	-0.0642	-0.0641	1.1997	3.1802	3.4098	3.5264	3.5535	3.5670	3.5748
12	0.0000	-0.1198	-0.1218	-0.1226	-0.1228	-0.1230	-0.1230	1.1997	3.2667	3.5159	3.6431	3.6727	3.6875	3.6959
14	0.0000	-0.1779	-0.1855	-0.1892	-0.1901	-0.1906	-0.1908	1.1997	3.3232	3.5873	3.7227	3.7542	3.7700	3.7790
16	0.0000	-0.2380	-0.2526	-0.2599	-0.2616	-0.2625	-0.2630	1.1997	3.3604	3.6361	3.7778	3.8108	3.8273	3.8368
18	0.0000	-0.2981	-0.3204	-0.3317	-0.3343	-0.3357	-0.3365	1.1997	3.3848	3.6695	3.8162	3.8504	3.8674	3.8773
20	0.0000	-0.3568	-0.3872	-0.4027	-0.4065	-0.4083	-0.4094	1.1997	3.4005	3.6923	3.8430	3.8782	3.8958	3.9059
50	0.0000	-0.9542	-1.0882	-1.1599	-1.1761	-1.1858	-1.1907	1.1997	3.3792	3.7018	3.8711	3.9107	3.9306	3.9420
100	0.0000	-1.3408	-1.5553	-1.6714	-1.6991	-1.7136	-1.7217	1.1997	3.3011	3.6257	3.7971	3.8374	3.8576	3.8690
150	0.0000	-1.5051	-1.7561	-1.8926	-1.9254	-1.9423	-1.9519	1.1997	3.2619	3.5853	3.7564	3.7967	3.8168	3.8283
200	0.0000	-1.5955	-1.8669	-2.0150	-2.0508	-2.0690	-2.0794	1.1997	3.2393	3.5616	3.7323	3.7724	3.7925	3.8041